12
Charge and current

The idea of charge intuitively relates to that of fields and forces. Charge is that quality or attribute of matter which determines how it will respond to a particular kind of force. It is thus a label which distinguishes forces from one another. The familiar charges are: electric charge, which occurs in Maxwell's equations; the mass, which occurs both in the laws of gravitation and inertia; the colour charge, which attaches to the strong force; and a variety of other labels, such as strangeness, charm, intrinsic spin, chirality, and so on. These attributes are referred to collectively as 'quantum numbers', though a better name might be 'group numbers'.

Charge plays the role of a quantity conjugate to the forces which it labels. Like all variables which are conjugate to a parameter (energy, momentum etc.) charge is a book-keeping parameter which keeps track of a closure or conservation principle. It is a currency for the property it represents. This indicates that the existence of charge ought to be related to a symmetry or conservation law, and indeed this turns out to be the case. An important application of symmetry transformations is the identification of conserved 'charges', and vice versa.

12.1 Conserved current and Noether's theorem

As seen in section 11.3, the spacetime variation of the action reveals a structure which leads to conservation equations in a closed system. The conservation equations have the generic form

$$\partial_t \rho + \vec{\nabla} \cdot \mathbf{J} = \partial_\mu J^\mu = 0, \tag{12.1}$$

for some 'current' J^μ. These are continuity conditions, which follow from the action principle (section 2.2.1). One can derive several different, but equally valid, continuity equations from the action principle by varying the action with respect to appropriate parameters. This is the essence of what is known as *Noether's theorem.*

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

In practice, one identifies the conservation law $\partial_\mu J^\mu = 0$ for current J_μ by varying the action with respect to a parameter, conjugate to its charge. This leads to two terms upon integration by parts: a main term, which vanishes (either with the help of the field equations, or by straightforward cancellation), and a surface term, which must vanish independently for stationary action $\delta S = 0$. The surface term can be written in the form

$$\delta S = \int (\mathrm{d}x)(\partial_\mu J^\mu)\delta\lambda = 0 \tag{12.2}$$

for some J_μ; then we say that we have discovered a conservation law for the current J_μ and parameter λ.

This is most easily illustrated with the aid of examples. As a first example, we shall use this method to prove that the electric current is conserved for a scalar field. We shall set $c = \hbar = 1$ for simplicity here. The gauged action for a complex scalar field is

$$S = \int (\mathrm{d}x) \left\{\hbar^2 c^2 (D^\mu\phi)^*(D_\mu\phi) + m^2c^4\phi^*\phi\right\}. \tag{12.3}$$

Consider now a gauge transformation in which $\phi \to \mathrm{e}^{\mathrm{i}es}\phi$, and vary the action with respect to $\delta s(x)$:

$$\delta S = \int (\mathrm{d}x)\hbar^2c^2\Big\{(D^\mu(-\mathrm{i}e\delta s)\mathrm{e}^{-\mathrm{i}es}\phi)^*(D_\mu \mathrm{e}^{\mathrm{i}es}\phi) \\ +(D^\mu \mathrm{e}^{-\mathrm{i}es}\phi)(D^\mu(\mathrm{i}e\delta s)\mathrm{e}^{\mathrm{i}es}\phi)\Big\}. \tag{12.4}$$

Now using the property (10.41) of the gauge-covariant derivative that the phase commutes through it, we have

$$\delta S = \int (\mathrm{d}x) \left\{(D^\mu(-\mathrm{i}e\delta s)\phi)^*(D_\mu\phi) + (D^\mu\phi)(D^\mu(\mathrm{i}e\delta s)\phi)\right\}. \tag{12.5}$$

We now integrate by parts to remove the derivative from δs and use the equations of motion $(-D^2 + m^2)\phi = 0$ and $-(D^{*2} + m^2)\phi^* = 0$, which leaves only the surface (total derivative) term

$$\delta S = \int (\mathrm{d}x)\delta s(\partial_\mu J^\nu), \tag{12.6}$$

where

$$J^\mu = \mathrm{i}e\hbar^2c^2(\phi^*(D^\mu\phi) - (D^\mu\phi)^*\phi). \tag{12.7}$$

Eqn. (12.2) can be written

$$\frac{1}{c}\int \mathrm{d}\sigma^\mu J_\mu = \text{const}. \tag{12.8}$$

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

In other words, this quantity is a constant of the motion. Choosing the canonical spacelike hyper-surface for σ, eqn. (12.8) has the interpretation

$$\frac{1}{c}\int d\sigma J_0 = \int dx^1 \dots dx^n \rho = Q, \tag{12.9}$$

where ρ is the charge density and Q is therefore the total charge. In other words, Noether's theorem tells us that the total charge is conserved by the dynamical evolution of the field.

As a second example, let us consider dynamical variations of the field $\delta\phi$. Anticipating the discussion of the energy–momentum tensor, we can write eqn. (11.43) in the form

$$\delta S = \int (dx)(\partial_\mu J^\mu) = 0, \tag{12.10}$$

where we have defined the 'current' as

$$J_\mu \delta\lambda \sim \Pi_\mu \delta q - \theta_{\mu\nu}\delta x^\nu. \tag{12.11}$$

This is composed of a piece expressed in terms of the canonical field variables, implying that canonical momentum is conserved for field dynamics,

$$\partial^\mu \Pi_\mu = 0, \tag{12.12}$$

and there is another piece for the mechanical energy–momentum tensor, the parameter is the spacetime displacement δx^μ. This argument is usually used to infer that the canonical momentum and the energy–momentum tensor,

$$\partial^\mu \theta_{\mu\nu} = 0, \tag{12.13}$$

are conserved; i.e. the conservation of mechanical energy and momentum.

If the action is complete, each variation of the action leads to a form which can be interpreted as a conservation law. If the action is incomplete, so that conservation cannot be maintained with the number of degrees of freedom given, then this equation appears as a constraint which restricts the system. In a conservative system, the meaning of this equation is that 'what goes in goes out' of any region of space. Put another way, in a conservative system, the essence of the field cannot simply disappear, it must move around by flowing from one place to another.

Given a conservation law, we can interpret it as a law of conservation of an abstract charge. Integrating the conservation law over spacetime,

$$\int (dx)\ \partial_\mu J^\mu = \int d\sigma^\mu J_\mu = \text{const.} \tag{12.14}$$

Table 12.1. Conjugates and generators.

	Q	v
Translation	p_i	x^i
Time development	$-H$	t
Electric phase	e	$\theta = \int A_\mu \mathrm{d}x^\mu$
Non-Abelian phase	gT^a	$\theta^a = \int A^a_\mu \mathrm{d}x^\mu$

If we choose $\mathrm{d}\sigma^\mu$, i.e. $\mu = 0$, to be a spacelike hyper-surface (i.e. a surface of covariantly constant time), then this defines the total charge of the system:

$$Q(t) = \int \mathrm{d}\sigma \rho(x) = \int \mathrm{d}^n\mathbf{x}\, \rho(x). \tag{12.15}$$

Combining eqns. (12.14) and (12.15), we can write

$$\int \mathrm{d}^n\mathbf{x}\, \partial_\mu J^\mu = -\partial_t \int \mathrm{d}\sigma\rho + \int \mathrm{d}\sigma^i J_i = 0. \tag{12.16}$$

The integral over J_i vanishes since the system is closed, i.e. no current flows in or out of the total system. Thus we have (actually by assumption of closure)

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = 0. \tag{12.17}$$

This equation is well known in many forms. For the conservation of electric charge, it expresses the basic assumption of electromagnetism that charge is conserved. In mechanics, we have the equation for conservation of momentum

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}\sigma\; \theta^i_0 = 0. \tag{12.18}$$

The conserved charge is formally the generator of the symmetry which leads to the conservation rule, i.e. it is the conjugate variable in the group transformation. In a group transformation, we always have an object of the form:

$$\mathrm{e}^{\mathrm{i}Qv}, \tag{12.19}$$

where Q is the generator of the symmetry and v is the conjugate variable which parametrizes the symmetry (see table 12.1). Noether's theorem is an expression of symmetry. It tells us that – if there is a symmetry under variations of a parameter in the action – then there is a divergenceless current associated with that symmetry and a corresponding conserved charge. The formal statement is:

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

The invariance of the Lagrangian under a one-parameter family of transformations implies the existence of a divergenceless current and associated conserved 'charge'.

Noether's theorem is not the only approach to finding conserved currents, but it is the most well known and widely used [2]. The physical importance of conservation laws for dynamics is that

A local excess of a conserved quantity cannot simply disappear –it can only relax by spreading slowly over the entire system.

12.2 Electric current J_μ for point charges

Electric current is the rate of flow of charge

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{12.20}$$

Current density (current per unit area, in three spatial dimensions) is a vector, proportional to the velocity $\mathbf{v}$ of charges and their density ρ:

$$J^i = \rho_e v^i. \tag{12.21}$$

By adding a zeroth component $J^0 = \rho c$, we may write the spacetime-covariant form of the current as

$$J^\mu = \rho_e \beta^\mu, \tag{12.22}$$

where $\beta^\mu = (c, \mathbf{v})$. For a point particle at position $x_0(t)$, we may write the charge density using a delta function. The n-dimensional spatial delta function has the dimensions of density and the charge of the particle is q. The current per unit area J^i is simply q multiplied by the velocity of the charge:

$$\begin{aligned} J^0/c &= \rho(x) = q\ \delta^n(\mathbf{x} - \mathbf{x}_p(t)) \\ J^i &= \rho(x)\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}. \end{aligned} \tag{12.23}$$

Relativistically, it is useful to express the current in terms of the velocity vectors β^μ and U^μ. For a general charge distribution the expressions are

$$\begin{aligned} J^\mu(x) &= \rho c \beta^\mu \\ &= \rho c \gamma^{-1} U^\mu. \end{aligned} \tag{12.24}$$

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

Table 12.2. Currents for various fields.

Field	Current
Point charges, velocity $\mathbf{v}$	$J^0 = e\rho c$
	$\mathbf{J} = e\rho\mathbf{v}$
Schrödinger field	$J^0 = e\psi^*\psi$
	$\mathbf{J} = \mathrm{i}\frac{e\hbar}{2m}(\psi^*(\mathbf{D}\psi) - (\mathbf{D}\psi)^*\psi)$
Klein–Gordon field	$J_\mu = \mathrm{i}e\hbar c^2(\phi^*(D_\mu\phi) - (D_\mu)^*\phi)$
Dirac field	$J_\mu = \mathrm{i}ec\overline{\psi}\gamma_\mu\psi$

Thus, for a point particle,

$$\begin{aligned} J^\mu &= qc\beta^\mu\, \delta^n(\mathbf{x} - \mathbf{x}_p(t)) \\ &= qc\int \mathrm{d}t\, \delta^{n+1}(x - x_p(t))\beta^\mu \\ &= q\int \mathrm{d}\tau\, \delta^{n+1}(x - x_p(\tau))U^\mu. \end{aligned} \tag{12.25}$$

12.3 Electric current for fields

The form of the electric current in terms of field variables is different for each of the field types, but in each case we may define the current by

$$J^\mu = \frac{\delta S_\mathrm{M}}{\delta A_\mu} \tag{12.26}$$

where S_M is the action for matter fields, including their gauge-invariant coupling to the Maxwell field A_μ, but not including the Maxwell action (eqn. (21.1)) itself. The action must be one consisting of complex fields, since the gauge symmetry demands invariance under arbitrary complex phase transformations. A single-component, non-complex field does not give rise to an electric current. The current density for quanta with charge e may be summarized in terms of the major fields as seen in table 12.2. The action principle displays the form of these currents in a straightforward way and also clarifies the interpretation of the source as a current. For example, consider the complex Klein–Gordon field in the presence of a source:

$$S = S_\mathrm{M} + S_J = \int (\mathrm{d}x)\left\{\hbar^2c^2(D^\mu\phi)^*(D_\mu\phi) - J^\mu A_\mu\right\}, \tag{12.27}$$

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

where terms independent of A_μ have been omitted for simplicity. Using eqn. (12.26), and assuming that J_μ is independent of A_μ, one obtains

$$\frac{\delta S_{\rm M}}{\delta A_\mu} = {\rm i}e\hbar c^2 \left[\phi^*(D^\mu\phi) - (D^\mu\phi)^*\phi\right] = J^\mu. \tag{12.28}$$

Note carefully here: although the left and right hand sides are numerically equal, they are not formally identical, since J_μ was assumed to be independent of A_μ under the variation, whereas the left hand side is explicitly dependent on A_μ through the covariant derivative. Sometimes these are confused in the literature leading to the following error.

It is often stated that the coupling for the electromagnetic field to matter can be expressed in the form:

$$S_{\rm M} = S_{\rm M}[A_\mu = 0] + \int ({\rm d}x) J^\mu A_\mu. \tag{12.29}$$

In other words, the total action can be written as a sum of a matter action (omitting A_μ, or with partial derivatives instead of covariant derivatives), plus a linear source term (which is supposed to make up for the gauge parts in the covariant derivatives) plus the Maxwell action. This is incorrect because, for any matter action which has quadratic derivatives (all fields except the Dirac field), one cannot write the original action as the current multiplying the current, just as

$$x^2 \neq \left(\frac{\rm d}{{\rm d}x}x^2\right)x. \tag{12.30}$$

In our case,

$$\frac{\delta S}{\delta A_\mu}A^\mu \neq S. \tag{12.31}$$

The Dirac field does not suffer from this problem. Given the action plus source term,

$$S = S_{\rm M} + S_J = \int ({\rm d}x) \left\{ -\frac{1}{2}{\rm i}\hbar c\overline{\psi}(\gamma^\mu \overrightarrow{D}_\mu - \gamma^\mu \overleftarrow{D}_\mu^{\dagger})\psi \right\}, \tag{12.32}$$

the variation of the action equals

$$\frac{\delta S_{\rm M}}{\delta A_\mu} = {\rm i}qc\overline{\psi}\gamma^\mu\psi = J^\mu. \tag{12.33}$$

In this unique instance the source and current are formally and numerically identical, and we may write

$$S_{\rm M} = S_{\rm M}[A_\mu = 0] + J^\mu A_\mu. \tag{12.34}$$

12.4 Requirements for a conserved probability

According to quantum theory, the probability of finding a particle at a position **x** at time t is derived from an invariant product of the fields. Probabilities must be conserved if we are to have a particle theory which makes sense. For the Schrödinger wavefunction, this is simply $\psi^*\psi$, but this is only true because this combination happens to be a conserved density $N(x)$ for the Schrödinger action.

In order to establish a probability interpretation for other fields, one may use Noether's theorem. In fact, we have already done this. A conserved current is known from the previous section: namely the electric current, but there seems to be no good reason to require the existence of electric charge in order to be able to speak of probabilities. We would therefore like to abstract the invariant structure of the conserved quantity without referring specifically to electric charge – after all, particles may have several charges, nuclear, electromagnetic etc – any one of these should do for counting particle probabilities.

Rather than looking at local gauge transformations, we therefore turn to global phase transformations[1] and remove the reference in the argument of the phase exponential to the electric charge. Consider first the Schrödinger field, described by the action

$$S = \int \mathrm{d}\sigma \mathrm{d}t \left\{ -\frac{\hbar^2}{2m}(\partial^i \psi)^\dagger(\partial_i \psi) - V\psi^*\psi + \frac{\mathrm{i}}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) \right\}. \tag{12.35}$$

The variation of the action with respect to constant δs under a phase transformation $\psi \to \mathrm{e}^{\mathrm{i}s}\psi$ is given by

$$\delta S = \int (\mathrm{d}x) \Bigg\{ -\frac{\hbar^2}{2m}\left[-\mathrm{i}\delta s(\partial^i\psi^*)(\partial_i\psi) + (\partial^i\psi^*)\mathrm{i}\delta s(\partial_i\psi)\right] + \mathrm{i}\left[-\mathrm{i}\delta s\psi^*\partial_t\psi + \mathrm{i}\delta s\psi^*\partial_t\psi\right] \Bigg\}. \tag{12.36}$$

Note that the variation δs need not vanish simply because it is independent of x, (see comment at end of section). Integrating by parts and using the equation of motion,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V = \mathrm{i}\frac{\partial\psi}{\partial t}, \tag{12.37}$$

we obtain the expression for the continuity equation:

$$\delta S = \int (\mathrm{d}x)\delta s \left(\partial_t J^t + \partial_i J^i\right) = 0, \tag{12.38}$$

[1] Global gauge transformations are also called *rigid* since they are fixed over all space.

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

where

$$J^t = \psi^* \psi = \rho$$
$$J^i = \frac{i\hbar^2}{2m} \left[\psi^* (\partial^i \psi) - (\partial^i \psi^*) \psi \right], \tag{12.39}$$

which can be compared to the current conservation equation eqn. (12.1). ρ is the probability density and J^i is the probability current. The conserved probability, by Noether's theorem, is therefore

$$P = \int d\sigma\, \psi^*(x) \psi(x), \tag{12.40}$$

and this can be used to define the notion of an inner product between two wavefunctions, given by the overlap integral

$$(\psi_1, \psi_2) = \int d\sigma\, \psi_1^*(x) \psi_2(x). \tag{12.41}$$

Thus we see how the notion of an invariance of the action leads to the identification of a conserved probability for the Schrödinger field.

Consider next the Klein–Gordon field. Here we are effectively doing the same thing as before in eqn. (12.4), but keeping s independent of x and setting $D_\mu \to \partial_\mu$ and $e \to 1$:

$$S = \int (dx) \hbar^2 c^2 \left\{ (\partial^\mu e^{-is} \phi^*)(\partial_\mu e^{is} \phi) \right\}$$
$$\delta S = \int (dx) \hbar^2 c^2 \left[(\partial^\mu \phi^* (-i\delta s) e^{-is})(\partial_\mu \phi e^{is}) + c.c. \right]$$
$$= \int (dx) \delta s (\partial_\mu J^\mu), \tag{12.42}$$

where

$$J^\mu = -i\hbar^2 c^2 (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*). \tag{12.43}$$

The conserved 'charge' of this symmetry can now be used as the definition of the inner product between fields:

$$(\phi_1, \phi_2) = i\hbar c \int d\sigma^\sigma (\phi_1^* \partial_\sigma \phi_2 - (\partial_\sigma \phi_1)^* \phi_2), \tag{12.44}$$

or, in non-covariant form,

$$(\phi_1, \phi_2) = i\hbar c \int d\sigma (\phi_1^* \partial_0 \phi_2 - (\partial_0 \phi_1)^* \phi_2). \tag{12.45}$$

This is now our notion of probability.

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

Here we have shown that a conserved probability can be attributed to any complex field as a result of symmetry under rigid (global) phase transformations. One should be somewhat wary of the physical meaning of rigid gauge transformations, since this implies a notion of correlation over arbitrary distances and times (a fact which apparently contradicts the finite speed of communication imposed by relativity). Global transformations should probably be regarded as an idealized case. In general, one requires the notion of a charge and associated gauge field, but not necessarily the electromagnetic gauge field. An additional point is: does it make physical sense to vary an object which does not depend on any dynamical variables $\mathbf{x}$, t? How should it vary without any explicit freedom to do so? These points could make one view rigid (global) gauge transformations with a certain skepticism.

12.5 Real fields

A cursory glance at the expressions for the electric current show that J_μ vanishes for real fields. Formally this is because the gauge (phase) symmetry cannot exist for real fields, since the phase is always fixed at zero. Consequently, there is no conserved current for real fields (though the energy–momentum tensor is still conserved). In the second-quantized theory of real fields (which includes the photon field), this has the additional effect that the number of particles with a given momentum is not conserved.

The problem is usually resolved in the second-quantized theory by distinguishing between excitations of the field (particles) with positive energy and those with negative energy. Since the relativistic energy equation $E^2 = p^2c^2 + m^2c^4$ admits both possibilities. We do this by writing the real field as a sum of two parts:

$$\phi = \phi^{(+)} + \phi^{(-)}, \tag{12.46}$$

where $\phi^{(+)*} = \phi^{(-)}$. $\phi^{(+)}$ is a complex quantity, but the sum $\phi^{(+)} + \phi^{(-)}$ is clearly real. What this means is that it is possible to define a conserved current and therefore an inner product on the manifold of positive energy solutions $\phi^{(+)}$,

$$(\phi_1^{(+)}, \phi_2^{(+)}) = \mathrm{i}\hbar c \int \mathrm{d}\sigma^\mu (\phi_1^{(+)*}\partial_\mu \phi_2^{(+)} - (\partial_\mu \phi_1^{(+)})^* \phi_2^{(+)}), \tag{12.47}$$

and another on the manifold of negative energy solutions $\phi^{(-)}$. Thus there is local conservation of probability (though charge still does not make any sense) of particles and anti-particles separately.

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

12.6 Super-conductivity

Consider a charged particle in a uniform electric field E_i. The force on the particle leads to an acceleration:

$$q\,E^i = m\ddot{x}^i. \tag{12.48}$$

Assuming that the particle starts initially from rest, and is free of other influences, at time t it has the velocity

$$\dot{x}^i(t) = \frac{q}{m}\int_0^t E^i\;\mathrm{d}t'. \tag{12.49}$$

This movement of charge represents a current (charge multiplied by velocity). If one considers N such identical charges, then the current is

$$J^i(t) = Nq\dot{x}^i = \frac{Nq^2}{m}\int_0^t E^i\;\mathrm{d}t'. \tag{12.50}$$

Assuming, for simplicity, that the electric field is constant, at time t one has

$$\begin{aligned} J^i(t) &= \frac{Nq^2 t}{m}\;E^i \\ &\equiv \sigma E^i. \end{aligned} \tag{12.51}$$

The last line is Ohm's law, $V = IR$, re-written in terms of the current density J^i and the reciprocal resistance, or conductivity $\sigma = 1/R$. This shows that a free charge has an ohmic conductivity which is proportional to time. It tends to infinity. Free charges are super-conducting.

The classical theory of ohmic resistance assumes that charges are scattered by the lattice through which they pass. After a mean free time of τ, which is a constant for a given material under a given set of thermodynamical conditions, the conductivity is $\sigma = Nq^2\tau/m$. This relation assumes hidden dissipation, and thus can never emerge naturally from a fundamental formulation, without modelling the effect of collisions as a transport problem. Fundamentally, all charges super-conduct, unless they are scattered by some impedance. The methods of linear response theory may be used for this.

If one chooses a gauge in which the electric field may be written

$$E^i = -\partial_t A^i, \tag{12.52}$$

then substitution into eqn. (12.50) gives

$$J^i = -\Lambda\;A^i, \tag{12.53}$$

where $\Lambda = Nq^2/m$. This is known as London's equation, and was originally written down as a phenomenological description of super-conductivity.

The classical model of super-conductivity seems naive in a modern, quantum age. However, the quantum version is scarcely more sophisticated. As noted in ref. [135], the appearance of super-conductivity is a result only of symmetry properties of super-conducting materials at low temperature, not of the detailed mechanism which gives rise to those symmetry properties.

Super-conductivity arises because of an ordered state of the field in which the inhomogeneities of scattering centres of the super-conducting material become invisible to the average state. Consider such a state in a scalar field. The super-conducting state is one of great uniformity, characterized by

$$\partial_\mu \langle \phi(x) \rangle = \langle A_0 \rangle = 0. \tag{12.54}$$

The average value of the field is thus locked in a special gauge. In this state, the average value of the current is given by

$$\langle J_i \rangle = \langle \mathrm{i}e\hbar c^2 (\phi^* (D_\mu \phi) - (D_\mu)^* \phi) \rangle. \tag{12.55}$$

The time derivative of this is:

$$\begin{aligned} \partial_t \langle J_i \rangle &= -e^2 c^2 \partial_t \langle A_i \rangle \\ &= e^2 c^2 \langle E_i \rangle. \end{aligned} \tag{12.56}$$

This is the same equation found for the classical case above. For constant external electric field, it leads to a current which increases linearly with time, i.e. it becomes infinite for infinite time. This corresponds to infinite conductivity. Observe that the result applies to statistical averages of the fields, in the same way that spontaneous symmetry breaking applies to statistical averages of the field, not individual fluctuations (see section 10.7). The individual fluctuations about the ground state continue to probe all aspects of the theory, but these are only jitterings about an energetically favourable super-conducting mean field. The details of how the uniform state becomes energetically favourable require, of course, a microscopic theory for their explanation. This is given by the BCS theory of super-conductivity [5] for conventional super-conductors. More recently, unusual materials have given rise to super-conductivity at unusually high temperatures, where an alternative explanation is required.

12.7 Duality, point charges and monopoles

In covariant notation, Maxwell's equations are written in the form

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \\ \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} &= 0. \end{aligned} \tag{12.57}$$

If one defines the dual F^* of a tensor F by one-half its product with the anti-symmetric tensor, one may write

$$F^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{12.58}$$

and Maxwell's equations become

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \\ \partial_\mu F^{*\mu\nu} &= 0. \end{aligned} \tag{12.59}$$

The similarity between these two equations has prompted some to speculate as to whether a dual current, J_m^μ, could not exist:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \\ \partial_\mu F^{*\mu\nu} &= -\mu_0 J_m^\nu. \end{aligned} \tag{12.60}$$

This would imply an equation of the form

$$\nabla \cdot \mathbf{B} = (\partial_i B^i) = \mu_0 \rho_m \tag{12.61}$$

and the existence of magnetic monopoles. The right hand side of these equations is usually thought of as a source term, or forcing term, for the differential terms on the left hand side. The existence of pointlike singularities is an interesting issue, since it touches the limits of the smooth differential formalism used to express the theory of electromagnetism and drives home the reasoning behind the model of pointlike charges which physicists have adopted.

Consider a Coulomb field surrounding a point. Up to a factor of $4\pi\epsilon_0$, the electric field has the vectorial form

$$E_i = \frac{x_i}{|\mathbf{x}|^m}, \tag{12.62}$$

in n dimensions. When $n = 3$ we have $m = 3$ for the Coulomb field, i.e. a $1/r^2$ force law. The derivative of this field is

$$\begin{aligned} \partial_i E_j &= \partial_i \left(\frac{x_j}{\sqrt{(x^k x_k)}} \right) \\ &= \left(\delta_{ij} - m \, \frac{x_i x_j}{x^k x_k} \right). \end{aligned} \tag{12.63}$$

From this, we have that

$$\begin{aligned} \vec{\nabla} \cdot \mathbf{E} = \partial^i E_i &= \frac{(n-m)}{|\mathbf{x}|^m}, \\ (\vec{\nabla} \times \mathbf{E})_k &= \epsilon_{ijk} \, \partial_i E_j = 0. \end{aligned} \tag{12.64}$$

The last result follows entirely from the symmetry on the indices: the product of a symmetric matrix and an anti-symmetric matrix is zero. What we see is that, in $n > 2$ dimensions, we can find a solution $n = m$ where the field satisfies the equation of motion identically, except at the singularity $x_i = 0$,

https://doi.org/10.1017/9781009289887.015 Published online by Cambridge University Press

where the solution does not exist. In other words, a field can exist without a source, everywhere except at the singular point.

In fact, this is an illusion of the differential formulation of Maxwell's equations; it highlights a conceptual difficulty. The core difficulty is that the equations are really non-local, in the sense that they relate a field at one point to a source at another. This requires an integration over the intermediate points to be well defined differentially. The differential form of Maxwell's equations is really a shorthand for the integral procedure.

At the singular point, the derivative does not exist, and Maxwell's equation becomes meaningless. We can assign a formal meaning to the differential form and do slightly better, as it turns out, by using the potential A_μ, since this can be regularized choosing variables in which the singularity disappears. In that way we can assign a formal meaning to the field around a point and justify the introduction of a source for the field surrounding the singularity using an integral formulation. The formulation we are looking for is in terms of Green functions. Green functions are, in a sense, a regularization scheme for defining the meaning of an ambiguous, irregular (infinite) expression. This is also the first in a long litany of cases where it is necessary to *regularize*, or re-formulate infinite, badly defined expressions in the physics of fields, which result from assumptions about pointlike structure and Green functions.

In terms of the vector potential A_μ, choosing the so-called Coulomb gauge $\partial_i A^i = 0$, we have

$$E_i/c = -\partial_0 A_i - \partial_i A_0, \tag{12.65}$$

so that the divergence of the electric field is

$$\partial_i E^i = -\nabla^2 \phi = \rho. \tag{12.66}$$

Note that we set $\epsilon_0 = 1$ for the purpose of this schematic. The charge density for a point particle with charge q at the origin is written as

$$\begin{aligned}\rho &= q\delta^3(x) \\ &= q\delta(x)\delta(y)\delta(z) \\ &= \frac{q}{4\pi r^2}\delta(r).\end{aligned} \tag{12.67}$$

Thus, in polar coordinates, about the origin,

$$-\nabla^2 \phi(r) = \frac{q\delta(r)}{4\pi\epsilon_0 r}. \tag{12.68}$$

The Green function $G(x, x')$ is defined as the object which satisfies the equation

$$-\nabla^2 G(x, x') = \delta(x - x'). \tag{12.69}$$

If we compare this definition to the Poisson equation for the potential $\phi(x)$ in eqn. (12.68), we see that $G(x, x')$ can interpreted as the scalar potential for a delta-function source at $x = x'$, with unit charge. Without repeating the content of chapter 5, we can simply note the steps in understanding the singularity at the origin. In the case of the Coulomb potential in three dimensions, the answer is well known:

$$\phi(r) = \frac{1}{4\pi r}. \tag{12.70}$$

We can use this to verify the consistency of the Green function definition of the field, in lieu of a more proper treatment later. By multiplying the Poisson equation by the Green function, one has

$$\int d^3\mathbf{x}' \, (-\nabla^2 \phi(x)) G(x, x') = \int d^3\mathbf{x}' \, \rho(x') G(x, x'). \tag{12.71}$$

Integrating by parts, and using the definition of $G(x, x')$,

$$\phi(x) = \int d^3\mathbf{x}' \, \rho(x') G(x, x'). \tag{12.72}$$

Substituting the polar coordinate forms for $\phi(r)$ and using the fact that $G(r, r')$ is just $\phi(r - r')$ in this instance, we have

$$\phi(r) = \frac{1}{4\pi r} = \int \frac{1}{4\pi(r - r')} \frac{\delta(r')}{4\pi r'^2} \, 4\pi r'^2 dr. \tag{12.73}$$

This equation is self-consistent and avoids the singular nature of the r' integration by virtue of cancellations with the integration measure $\int d^3\mathbf{x}' = 4\pi r'^2 dr$. We note that both the potential and the field are still singular at the origin. What we have achieved here, however, is to show that the singularity is related to a delta-function source (well defined under integration). Without the delta-function source ρ, the only consistent solution is $\phi =$ const. in the equation above. Thus we do, in fact, need the source to explain the central Coulomb field.

In fact, the singular structure noted here is a general feature of *central fields*, or conservative fields, whose curl vanishes. A non-vanishing curl, incidentally, indicates the presence of a magnetic field, and thus requires a source for the magnetism, or a magnetic monopole.

The argument for magnetic monopoles is based on the symmetry of the differential formulation of Maxwell's equations. We should pay attention to the singular nature of pointlike sources when considering this point. If we view everything in terms of singularities, then a magnetic monopole exists trivially: it is the Lorentz boost of a point charge, i.e. a string of current. The existence of other monopoles can be inferred from other *topological* singularities in the spacetime occupied by the field.