Theorem regarding Orthogonal Conics.

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Definition. When two conics intersect each other at two points in such a manner that the tangents and normals of the one become the normals and tangents of the other, they may be said to cut each other orthogonally.

Theorem. 1st, A given conic can be cut at every point on it by two conics which are orthogonal to it; 2nd, every conic orthogonal to a given conic passes through two fixed points on the axis of the given conic.

1. Let F and S, Figure 4, be the foci of the given conic, Xx and Yy the directrices, and let R equal the radius of director circle, centre F, and let P be any point on the curve. Draw PSP_1 a focal chord; then Sx, at right angles to PSP_1, cuts the X directrix in x, the centre of the orthogonal circle to F which touches PSP_1 at S, and, since PSP_1, FP and FP_1 are all tangents to this circle, it is therefore the in-circle of triangle PFP_1. Now the normals at P and P_1 bisect the exterior angles of PFP_1 at P and P_1 and the bisectors meet on Fx since Fx bisects angle PFP_1. Calling this point H, we observe that it is the centre of an ex-circle to triangle PFP_1 which touches PSP_1 in S_1 and FP in f^1 so that, if Ff^1 be taken as the radius of the director circle and S_1 as a focus, we get P as a point on an ellipse whose foci are F and S_1 and whose tangent and normal at P are the normal and tangent to the given conic at P; for $Pf^1 = PS_1$ and therefore $FP + S_1P = Ff^1 = R_1$. Similarly $S_1P_1 + FP_1 = Ff^1 = R_1$: therefore the given conic is cut orthogonally at P and P_1 by the ellipse whose foci are F and S_1 and whose directrix is HX_1, a line through H at right angles to FS_1.

The second conic is the Hyperbola whose focal chord in the given conic is FP which cuts the conic in Q and P. Determining Q and taking the normals at Q and P, we can see that they will meet at a point H_1 on the bisector of the exterior angle at S which is the line y_2S, the point y_2 being the centre of the circle of the Y system which was used to determine Q, and S being the focus through which the focal chord QP does not pass. Using the ex-circle H_1 as the orthogonal circle, its points of contact give us a second focus and the length of the radius of the director circle. The focus in this case, being on the chord QP, is therefore F_1 and the radius of director circle is Ss. We can now construct an orthogonal conic which passes through P and Q; thus there are at any point P on a given conic two intersecting conics orthogonal to the given conic.

2. We have seen in the first part that S and S_1 are the points of contact of the in- and an ex-circle of triangle FPP_1; therefore $S_1P = SP_1$ and $S_1P_1 = SP$, and the semi-latus rectum of the given conic, being an harmonic mean to SP and SP_1, is therefore also an harmonic mean to S_1P_1 and S_1P, and the latus rectum of the orthogonal conic is therefore equal to the latus rectum of the given conic.

Now, for any orthogonal conic, one of the original foci must remain a focus, and therefore FS is always a focal chord: then as in first part we observe that, in triangles S_1II_1 and F_1II_1, the original foci are the points of contact of the in- and an ex-circle to the triangles and that therefore $FI_1 = SI$ and $FI = SI_1$; and, the semi latera recta being equal, and each the harmonic mean to FI_1 and FI and at the same time to SI and SI_1, the points I and I_1 are therefore fixed for all conics orthogonal to the given conic.

NOTES.

(1) As proved in the second part it is to be noticed that the latera recta of all orthogonal conics are equal and that to the latus rectum of given conic.

(2) Since the foci are necessarily internal to both the given conic and the orthogonal conic, these two conics must therefore intersect in four points.

The second pair of cuts are determined by OFO_1 at right angles to HF or $O^1SO^1_1$ at right angles to H_1S, since FO, for instance, is the common polar of F to the touching orthogonals at y_2 and H^1 which determine O and O_1 for both curves.

(3) The normals at P and P_1 are parallel to Sf and Sf_1, and the chord FO is parallel to ff_1, being respectively at right angles to the same line.

(4) The second pair of intersections cannot be orthogonal for FOS and FOS_1 cannot have the same bisector.

(5) Conics to be orthogonal must have two foci on a common focal chord, the remaining two coinciding, as for instance in the given conic and the orthogonal ellipse S_1 and S lie on a common focal chord while F is the double focus.

(6) Let R be the radius of director circle R_1 and R_2 the radii of the director circles of the orthogonal conics then the following simple relation exists for them :—

$R_1 = Ff^1 = R + fP + Pf^1 = R + SP + SP_1 = R + PP_1$ for orthogonal ellipse, while from $SQ - FQ = R$ (1) $F_1Q - SQ = R_2$, (2) $FP - SP = R$ (3) and $SP - F_1P = R_2$ (4) by adding (1) and (2), (3) and (4), and then adding, we get $QP = R + R_2$, or $R_2 = QP - R$ in the case of the orthogonal Hyperbola.

(7) When the focal chord common to both is at right angles to FS, then S_1 coincides with S and (6) becomes $R_1 = R + LL_1$, LL_1 being the latus rectum. Halving this becomes $CI = CA + SL$; but $CI = CA + AI$, therefore $AI = SL$, and, if J be the point where the axis of the ellipse cuts the given conic, then by above $JA_1 = AI = SL$. So that if the ellipse be regarded as the given conic, J and J_1 are the fixed points through which all orthogonals pass.

(8) CC_1 is parallel to PP_1; CC_2 is parallel to QP.