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INVARIANT SUBMANIFOLDS IN FLOW GEOMETRY

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Abstract

We begin a study of invariant isometric immersions into Riemannian manifolds (M, g) equipped with a Riemannian flow generated by a unit Killing vector field ξ. We focus our attention on those (M, g) where ξ is complete and such that the reflections with respect to the flow lines are global isometries (that is, (M, g) is a Killing-transversally symmetric space) and on the subclass of normal flow space forms. General results are derived and several examples are provided.

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0. Introduction

Symmetric and Hermitian symmetric spaces play in important role in real and complex geometry. The analogs of the last class for contact geometry are the φ-symmetric spaces. They form a subclass of the Riemannian spaces equipped with a complete unit Killing vector field such that the reflections with respect to its integral curves are global isometries. These last spaces have been introduced and studied from the global and local viewpoint in [5, 6, 8] and related papers, and are called globally Killing-transversally symmetric spaces. For the different cases, the corresponding space forms, that is, real, complex, Sasakian and normal flow space forms [7], provide classes of particularly interesting examples.

Isometric immersions and embeddings in real, complex and Sasakian manifolds and their associated space forms have been studied extensively. In this paper we begin a study of isometric immersions into manifolds equipped with a Riemannian

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flow generated by a unit Killing vector field, thereby concentrating mainly on Killing-transversally symmetric spaces and normal flow space forms.

In Section 1 we give some preliminary definitions, formulas and results, and in Section 2 we introduce and derive some results concerning invariant immersions of manifolds with respect to flows such that the image is tangent to the flow lines. In Section 3 we consider the case of Riemannian manifolds such that the flow gives rise to a fibration and study the relation between the total space M and the base space M', in particular with respect to congruence theorems for the immersions into M and the corresponding transversal immersions into M'. Section 4 is devoted to the case of a normal space form M. Several general results are derived and examples are provided for the two cases of normal flow space forms considered in flow geometry [7].

1. Isometric flows

Let (M, g) be a n-dimensional, smooth Riemannian manifold with $n \geq 2$, which is supposed to be connected where necessary. Furthermore, let ∇ denote the Levi-Civita connection of (M, g) and R the corresponding Riemannian curvature tensor with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for $U, V \in \mathscr{X}(M)$, the Lie algebra of smooth vector fields on M.

A tangentially oriented foliation of dimension one on (M, g) is called a *flow*. The leaves of this foliation are the integral curves of a non-singular vector field on M and hence, after normalization, a flow is also given by a unit vector field. In particular, a non-singular Killing vector field defines a *Riemannian flow* and such a flow is said to be an *isometric* flow. See [25] for more information.

In this paper we consider and denote by $\mathscr{F}_\xi$ an isometric flow generated by a *unit* Killing vector field ξ. The flow lines of $\mathscr{F}_\xi$ are geodesics and moreover, a geodesic which is orthogonal to ξ at one of its points, is orthogonal to it at all of its points. Such geodesics are called *transversal* or *horizontal* geodesics.

$\mathscr{F}_\xi$ determines locally a Riemannian submersion. For each $m \in (M, g)$, let $\mathscr{U}$ be a small open neighborhood of m such that ξ is regular on $\mathscr{U}$. Then the mapping $\pi: \mathscr{U} \to \mathscr{U}' = \mathscr{U}/\xi$ is a submersion. Furthermore, let g' denote the induced metric on $\mathscr{U}'$ given by $g'(X', Y') = g(X'^*, Y'^*)$ for $X', Y' \in \mathscr{X}(\mathscr{U}')$ and where X'^*, Y'^* denote the horizontal lifts of X', Y' with respect to the $(n-1)$-dimensional horizontal distribution on $\mathscr{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then $\pi: (\mathscr{U}, g_{|\mathscr{U}}) \to (\mathscr{U}', g')$ is a Riemannian submersion and we may use the tensors A and T, introduced by O'Neill in [14] (see also [1, 23, 25]), in our treatment. Since the leaves are geodesics, $T = 0$. Furthermore, for the integrability tensor A we have

$$A_U\xi = \nabla_U\xi, \qquad A_\xi U = 0,$$
$$A_XY = (\nabla_XY)^v = -A_YX, \qquad g(A_XY, \xi) = -g(A_X\xi, Y)$$

for $U \in \mathscr{X}(M)$, and for horizontal vector fields X, Y. Here v denotes the vertical component.

Next, put

$$HU = -A_U\xi \tag{1.1}$$

and define the $(0, 2)$-tensor field h by $h(U, V) = g(HU, V)$, $U, V \in \mathscr{X}(M)$. Then h is skew-symmetric because ξ is a Killing vector field. Moreover, we have at once

$$A_XY = h(X, Y)\xi = \frac{1}{2}\eta([X, Y])\xi.$$

So we have

$$h = -d\eta. \tag{1.2}$$

Note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable. In this case, since $T = 0$, (M, g) is locally a product of an $(n-1)$-dimensional manifold and a line. Furthermore, the Levi-Civita connection ∇' of g' is determined by

$$\nabla_{X'^*}Y'^* = (\nabla'_{X'}Y')^* + h(X'^*, Y'^*)\xi \tag{1.3}$$

for $X', Y' \in \mathscr{X}(\mathscr{U}')$.

By straightforward computations these formulas yield

LEMMA 1.1 ([5]). *We have*

$$(\nabla_\xi h)(X, Y) = g((\nabla_\xi A)_XY, \xi) = 0, \tag{1.4}$$
$$R(X, Y, Z, \xi) = (\nabla_Z h)(X, Y), \tag{1.5}$$
$$R(X, \xi, Y, \xi) = g(HX, HY) = -g(H^2X, Y) \tag{1.6}$$

for horizontal X, Y, Z and where $R(X, Y, Z, W) = g(R_{XY}Z, W)$.

This lemma yields that the *ξ-sectional curvature* $K(X, \xi)$ of the 2-plane spanned by X and ξ is non-negative for all horizontal X and since $H\xi = 0$, $K(X, \xi) = 0$ for all horizontal X if and only if the skew-symmetric endomorphism H is of maximal rank $n - 1$. In this case, n is necessarily odd and from (1.2) we see that η is then a *contact form* on M. This leads to

DEFINITION 1.1. $\mathcal{F}_\xi$ is said to be a *contact flow* if η is a contact form, that is, if H is of maximal rank.

In the rest of the paper we will need extensively another special type of isometric flow $\mathcal{F}_\xi$ introduced in a natural way in [5]. We recall its definition.

DEFINITION 1.2. $\mathcal{F}_\xi$ is said to be *normal* if $R(X, Y, X, \xi) = 0$ for all horizontal vector fields X, Y.

Here we note that a Sasakian manifold is a Riemannian manifold with a normal flow $\mathcal{F}_\xi$ such that $K(X, \xi) = 1$ for all horizontal X (see [2, 29] for more details).

From Lemma 1.1 we then get the following useful criterion.

PROPOSITION 1.1. *$\mathcal{F}_\xi$ is normal if and only if*

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2U \tag{1.7}$$

for all $U, V \in \mathcal{X}(M)$.

Furthermore, for a normal flow the curvature tensor satisfies the following identities:

$$R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V, \tag{1.8}$$

$$R_{U\xi}V = g(HU, HV)\xi + \eta(V)H^2U, \tag{1.9}$$

and for the Ricci tensor ρ of type $(0, 2)$ we have $\rho(X, \xi) = 0$ for each horizontal X. Moreover, $\rho(\xi, \xi)$ is a non-negative global constant on M.

Next, a Riemannian manifold (M, g) equipped with an isometric flow $\mathcal{F}_\xi$ is said to be *η-Einstein* if ρ is of the form

$$\rho(U, V) = ag(U, V) + b\eta(U)\eta(V)$$

where a and b are constants. Then (M, g) is an *Einstein space* if $b = 0$.

Using (1.3) we obtain the following relation between the curvature tensors associated to ∇ and ∇':

$$\begin{aligned}(R'_{X'Y'}Z')^* = R_{X'^*Y'^*}Z'^* &- g(HY'^*, Z'^*)HX'^* \\ &+ g(HX'^*, Z'^*)HY'^* + 2g(HX'^*, Y'^*)HZ'^*\end{aligned} \tag{1.10}$$

for all $X', Y', Z' \in \mathcal{X}(\mathcal{U}')$. From this we obtain for the corresponding Ricci tensors ρ, ρ':

$$(\rho'(X', Y'))^* = \rho(X'^*, Y'^*) + 2g(HX'^*, HY'^*), \tag{1.11}$$

and for the scalar curvatures τ, τ' we get

$$\tau'^* = \tau + \rho(\xi, \xi). \tag{1.12}$$

The sectional curvatures are related by

$$K'_{m'}(u', w') = K_m(u'^*, w'^*) + 3(h_m(u'^*, w'^*))^2 \tag{1.13}$$

where (u', w') is an orthonormal pair of $T_{m'}\mathscr{U}'$, $m' = \pi(m)$. These equations should be compared with those of O'Neill [14].

Next, we recall the definitions of locally and globally Killing-tranversally symmetric spaces. Therefore, consider an (M, g) equipped with a flow $\mathscr{F}_\xi$ and let $m \in M$. By σ we denote the flow line through m. A local diffeomorphism s_m of M defined in a neighborhood $\mathscr{U}$ of m is said to be a *(local) reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathscr{U}$ and σ, we have $(s_m \circ \gamma)(s) = \gamma(-s)$ for all s with $\gamma(\pm s) \in \mathscr{U}$ and where s denotes the arc length. Then $S_m = s_{m_*}(m)$ defines a linear isometry on $T_m M$ given by $S_m = (-I + 2\eta \otimes \xi)(m)$. Furthermore, since ξ is a Killing vector field, s_m satisfies $s_m = \exp_m \circ S_m \circ \exp_m^{-1}$.

DEFINITION 1.3. A *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) is a Riemannian manifold equipped with an isometric flow $\mathscr{F}_\xi$ such that the local reflection s_m with respect to the flow line through m is an isometry for all $m \in M$.

The two following propositions provide useful characterizations for locally KTS-spaces.

PROPOSITION 1.2 ([5]). *$(M, g, \mathscr{F}_\xi)$ is a locally* KTS-*space if and only if $\mathscr{F}_\xi$ is normal, and moreover we have $(\nabla_X R)(X, Y, X, Y) = 0$ for all horizontal X, Y.*

PROPOSITION 1.3 ([5]). *Let $\mathscr{F}_\xi$ be a normal flow on (M, g). Then $(M, g, \mathscr{F}_\xi)$ is a locally* KTS-*space if and only if each base space $\mathscr{U}'$ of a local Riemannian submersion $\pi: \mathscr{U} \to \mathscr{U}' = \mathscr{U}/\xi$ is a locally symmetric space.*

So, according to the terminology used in [26], $(M, g, \mathscr{F}_\xi)$ is a locally KTS-space if and only if $\mathscr{F}_\xi$ is a *normal transversally symmetric foliation*.

Now, we return to the global case.

DEFINITION 1.4. Let (M, g) be a Riemannian manifold and ξ a non-vanishing complete Killing vector field on it. Then $(M, g, \mathscr{F}_\xi)$ is said to be a *(globally) Killing-transversally symmetric space* (briefly, a KTS-*space*) if and only if for each $m \in M$ there exists a (unique) global isometry $s_m: M \to M$ such that $s_{m*}(m) = (-I + 2\eta \otimes \xi)_m$ on $T_m M$.

The isometry s_m is called the *reflection* of M at m with respect to the flow line of ξ through m. Since it reverses the transversal geodesics through m, s_m is the unique extension of the local reflection at m to the whole of M. Hence, it follows that a KTS-space is a locally KTS-space. In [6] it is proved that, conversely, a complete simply connected locally KTS-space is a KTS-space.

Furthermore, consider an $(M, g, \mathscr{F}_\xi)$ and let $A(M)$ denote the group of all isometries of (M, g) which leave ξ invariant. Then $\mathscr{F}_\xi$ is called a *homogeneous flow* if $A(M)$ acts transitively on M. In [6, 8] it is proved that the flow $\mathscr{F}_\xi$ on a KTS-space is always homogeneous and moreover, in the simply connected case, the manifold $(M, g, \mathscr{F}_\xi)$ is a naturally reductive space.

Finally we give the following definition (see [22] for the details and terminology).

DEFINITION 1.5. An isometric flow $\mathscr{F}_\xi$ on (M, g) is said to be *fibrable* if the following conditions are satisfied:

(i) ξ is a complete, strictly regular vector field;
(ii) the quotient topology for the orbit space $M' = M/\xi$ is Hausdorff;
(iii) if the flow lines are closed, then they have the same length.

Here 'strictly' means that all integral curves are homeomorphic and 'closed' means periodic. Furthermore, note that when M is compact, then (i) and (ii) reduce to the regularity of ξ (see [22, p. 22, Corollary 5]).

It follows from (i) and (ii) that the orbit space $M' = M/\xi$ admits a unique structure of differentiable manifold such that the natural projection $\pi: M \to M'$ is a submersion [22, p. 19, Theorem VIII and p. 28, Theorem XIV]. Moreover, M is a principal G^1-bundle over M', where G^1 denotes the one-parameter subgroup of global isometries ψ_t generated by ξ [15]. G^1 is isomorphic to either the circle group S^1 or to $\mathbb{R}$ depending on whether the integral curves of ξ are closed or not. Here, we identify S^1 with the set $\{e^{2\pi i t},\ t \in \mathbb{R}\}$. If G^1 is a circle (which occurs when M is compact), then the right action of S^1 on M is given by

$$m \circ e^{2\pi i t} = \psi_{lt}(m) \tag{1.14}$$

for each $m \in M$ and where l denotes the length of the integral curves of ξ. When G^1 is isomorphic to $\mathbb{R}$, we identify the action of $t \in \mathbb{R}$ on M with that of $\psi_t \in G^1$. For $G^1 \approx S^1$ the corresponding fundamental vector field ς generated by d/dt is given by

$$\varsigma(m) = \frac{d}{dt}\,|_{t=0}\left(m \circ e^{2\pi i t}\right) = l\xi_m \tag{1.15}$$

and in that case $l^{-1}\eta$ defines a connection form on M. For the case $G^1 \approx \mathbb{R}$, $\varsigma = \xi$ and then η is a connection form. Moreover, in the first case, using (1.2) and the fact that S^1 is Abelian, the usual structure equation takes the form $\Omega = l^{-1}d\eta = -l^{-1}h$,

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where Ω is the curvature form. Now, let h' be the $(0, 2)$-tensor field on M' defined by $h'(X', Y') = g'(H'X', Y')$ for all $X', Y' \in \mathscr{X}(M')$ and where H' is the tensor field determined by $H'X' = \pi_*(HX'^*)$. Then we see that H' is skew-symmetric with respect to g' and $h = \pi^* h'$. The characteristic class $e_{M'}(M) \in H^2(M', \mathbb{Z})$ of M over M' (see [10]) satisfies

$$e_{M'}(M) = \left[-\frac{1}{l}h'\right]. \tag{1.16}$$

In the next sections we shall consider $(M, g, \mathscr{F}_\xi)$ where $\mathscr{F}_\xi$ is fibrable as a principal bundle over $M' = M/\xi$ with the description given above. Our main motivation for this is the consideration of (globally) KTS-spaces $(M, g, \mathscr{F}_\xi)$. Indeed, such spaces are fibrable. To prove this we first recall that $\mathscr{F}_\xi$ is homogeneous. So, if ξ is regular, then it is strictly regular. Now, it is clear that ξ is always regular on a KTS-space because if not, it is impossible to extend the isometric local reflection s_m to a globally isometric one since every point on the flow line σ through m has to be fixed for the corresponding global isometric reflection. Furthermore, the regularity implies that each flow line is a closed submanifold of (M, g) [22, Theorem VII]. Finally, $M' = M/\xi$ is Hausdorff because otherwise there exist, since ξ is also a Killing vector field, two 'different' flow lines on $(M, g, \mathscr{F}_\xi)$ such that the distance between them is everywhere smaller than each $\varepsilon > 0$ and this is impossible because the flow lines are closed submanifolds.

Note that when $\mathscr{F}_\xi$ is a contact flow, then this result also follows from the well-known Boothby-Wang theory for homogeneous contact manifolds [3]. In this context it is worthwile to mention that it follows from [6] that a KTS-space $(M, g, \mathscr{F}_\xi)$ has a contact flow $\mathscr{F}_\xi$ if and only if it is locally irreducible. Moreover, a locally reducible KTS-space is locally isometric to a product of a contact KTS-space and a symmetric space.

We also recall that when $(M, g, \mathscr{F}_\xi)$ is a KTS-space, then (M', g') is a symmetric space [6].

2. Invariant immersions with respect to a flow

Let f be an isometric immersion of a n-dimensional Riemannian manifold (M, g) into an $\bar{n}$-dimensional Riemannian manifold $(\bar{M}, \bar{g})$. In what follows, and if the argument is local, we shall always identify M with its image under f to simplify the notation. Next, let ∇ and $\bar{\nabla}$ denote the Levi-Civita connection of (M, g) and $(\bar{M}, \bar{g})$, respectively, and let $R, \bar{R}$ be the corresponding curvature tensors. The Gauss and

Weingarten formulas are, respectively:

$$\bar\nabla_X Y = \nabla_X Y + B(X, Y), \tag{2.1}$$

$$\bar\nabla_X U = -C_U X + \nabla^{\perp}_X U, \tag{2.2}$$

where $X, Y \in \mathscr{X}(M)$, $U \in \mathscr{X}(M)^{\perp}$ and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$. Here B and C denote the corresponding second fundamental tensors and are related by $\bar g(B(X, Y), U) = g(C_U X, Y)$.

Next, we recall the fundamental Gauss, Codazzi and Ricci equations:

$$\begin{aligned}\bar R(X, Y, Z, W) = R(X, Y, Z, W) &+ \bar g(B(X, W), B(Y, Z)) \\ &- \bar g(B(X, Z), B(Y, W)),\end{aligned} \tag{2.3}$$

$$(\bar R_{XY} Z)^{\perp} = -(\tilde\nabla_X B)(Y, Z) + (\tilde\nabla_Y B)(X, Z), \tag{2.4}$$

$$\bar R(X, Y, U, V) = R^{\perp}(X, Y, U, V) - g(C_U X, C_V Y) + g(C_V X, C_U Y) \tag{2.5}$$

for $X, Y, Z, W \in \mathscr{X}(M)$, $U, V \in \mathscr{X}(M)^{\perp}$ and where $R^{\perp}$ denotes the curvature tensor of $\nabla^{\perp}$. $\tilde\nabla B$ is defined by

$$(\tilde\nabla_X B)(Y, Z) = \nabla^{\perp}_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \tag{2.6}$$

for all $X, Y, Z \in \mathscr{X}(M)$. The immersion f is said to be *totally geodesic* if $B = 0$, *parallel* if $\tilde\nabla B = 0$ and *minimal* if $\operatorname{tr} B = 0$ on M.

Now, let $\mathscr{F}_{\bar\xi}$ be an isometric flow on $(\bar M, \bar g)$ and let $\bar H$ be the tensor field defined in (1.1). Then f is said to be *tangent* if $\bar\xi$ is tangent to $f(M)$. In this case we shall denote by ξ the vector field on M given by $f_*\xi = \bar\xi \circ f$. In what follows we will consider a special kind of tangent isometric immersions which we define now.

DEFINITION 2.1. An isometric immersion f of (M, g) into $(\bar M, \bar g, \mathscr{F}_{\bar\xi})$ is said to be *invariant* (*with respect to the flow* $\mathscr{F}_{\bar\xi}$) if f is tangent and invariant with respect to $\bar H$; that is, $\bar H f_* T_m M \subset f_* T_m M$ for all $m \in M$.

For invariant immersions we have $\bar H (f_* T_m M)^{\perp} \subset (f_* T_m M)^{\perp}$ and the tensor field H on M related to $\bar H$ by $f_* \circ H = \bar H \circ f_*$ coincides with the one defined via (1.1) for the flow $\mathscr{F}_{\xi}$ on (M, g). Moreover, since f is tangent, we have $\bar H X = H X - B(X, \xi)$ for any X tangent to M and so we get

$$B(X, \xi) = 0, \qquad C_U \xi = 0 \tag{2.7}$$

for all $X \in \mathscr{X}(M)$ and all $U \in \mathscr{X}(M)^{\perp}$.

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We note that if $\bar{\mathscr{F}}_{\bar{\xi}}$ is a contact flow on $(\bar{M}, \bar{g})$, then $\mathscr{F}_{\xi}$ is a contact flow too, and hence M is also odd-dimensional.

The notion of an invariant immersion corresponds to a similar one in Sasakian geometry and many properties about these immersions may be proved easily in our case. We give some examples.

PROPOSITION 2.1. *Let $\mathscr{F}_{\bar{\xi}}$ be a contact flow on $(\bar{M}, \bar{g})$ and $f : (M, g) \to (\bar{M}, \bar{g})$ an isometric immersion. If $\bar{H}X$ is tangent to $f(M)$ for all X tangent to M, then f is tangent and hence invariant.*

PROOF. First, suppose $\bar{\xi} \circ f$ is normal. Then $\bar{h}(X, Y) = \frac{1}{2}\bar{\eta}([X, Y]) = 0$ for all $X, Y \in \mathscr{X}(M)$ and so $\bar{H}$ cannot be of maximal rank.

So, put $\bar{\xi} \circ f = \bar{\xi}^T + \bar{\xi}^N$ where $\bar{\xi}^T$ and $\bar{\xi}^N$ denote the tangential and normal part of $\bar{\xi}$ along M, respectively. Since $\bar{H}\bar{\xi}^T = \bar{H}\bar{\xi}^N = 0$ and since $\mathscr{F}_{\bar{\xi}}$ is contact, we obtain that $\bar{\xi}$ is tangent to $f(M)$ along $f(M)$. This proves the required result.

Next, we obtain easily

LEMMA 2.1. *Let $f: (M, g) \to (\bar{M}, \bar{g})$ be an invariant immersion and let $\mathscr{F}_{\bar{\xi}}$ be a normal flow on $(\bar{M}, \bar{g})$. Then $\mathscr{F}_{\xi}$ is also normal and moreover, we have*

$$(\bar{\nabla}_X \bar{H})Y = (\nabla_X H)Y, \tag{2.8}$$

$$B(X, HY) = \bar{H}B(X, Y) = B(HX, Y), \tag{2.9}$$

$$HC_U X = -C_U HX = C_{\bar{H}U} X. \tag{2.10}$$

From this we then get

PROPOSITION 2.2. *Let $\mathscr{F}_{\bar{\xi}}$ be a contact normal flow on $(\bar{M}, \bar{g})$ and f an invariant immersion of (M, g) into $(\bar{M}, \bar{g})$. Then f is minimal.*

PROOF. For each $m \in M$ we can choose an orthonormal basis $\{u_{2i-1}, u_{2i}; \xi; v_{2j-1}, v_{2j}; i = 1, \dots, (n-1)/2,\ j = 1, \dots, (\bar{n}-n-1)/2\}$ of $T_m\bar{M}$, where $(u_1, \dots, u_n, \xi)$ span $T_m M$ and $(v_1, \dots, v_{\bar{n}-n-1})$ span $T_m^{\perp}M$, and positive numbers λ_i, μ_j such that

$$Hu_{2i-1} = \lambda_i u_{2i}, \qquad Hu_{2i} = -\lambda_i u_{2i-1},$$
$$\bar{H}v_{2j-1} = \mu_j v_{2j}, \qquad \bar{H}v_{2j} = -\mu_j v_{2j-1}.$$

Then $\operatorname{tr} B_m = 0$ follows at once by using (2.7) and (2.9).

Next, we focus on the sectional curvatures and prove

PROPOSITION 2.3. *Let $f\colon (M, g) \to (\bar{M}, \bar{g}, \mathscr{F}_\xi))$ be an invariant immersion. Then for each $X \in \mathscr{X}(M)$ and orthogonal to ξ we have*

(i) $\bar{K}(X, \bar{\xi}) = K(X, \xi)$;

(ii) *if $\mathscr{F}_{\bar{\xi}}$ is normal, then $K(X, HX) \leq \bar{K}(X, HX)$. Moreover, if $\mathscr{F}_{\bar{\xi}}$ is also contact, then equality holds if and only if f is totally geodesic.*

PROOF. (i) follows immediately from (2.3) and (2.7). To prove (ii) one just uses the following consequence of (2.3) and (2.9):

$$\bar{R}(X, HX, X, HX) = R(X, HX, X, HX) + 2\bar{g}(\bar{H}B(X, X), \bar{H}B(X, X)).$$

Furthermore, we have

PROPOSITION 2.4. *Let $\mathscr{F}_{\bar{\xi}}$ be a normal contact flow on $(\bar{M}, \bar{g})$ and f an invariant immersion. If f is parallel, then it is totally geodesic.*

PROOF. From (2.6) and (2.9) we obtain

$$0 = (\tilde{\nabla}_X B)(Y, \xi) = B(Y, HX) = \bar{H}B(X, Y).$$

Now the result follows at once since $\bar{H}$ has maximal rank.

This result shows that the notion of a parallel invariant immersion is very restrictive. Therefore, we introduce the notion of an η-parallel immersion. Let $\mathscr{F}_{\bar{\xi}}$ be an isometric flow on $(\bar{M}, \bar{g})$ and let f be a tangent isometric immersion of (M, g) into $(\bar{M}, \bar{g})$. Then f is said to be *η-parallel* if $(\tilde{\nabla}_X B)(Y, Z) = 0$ for all $X, Y, Z \in \mathscr{X}(M)$ and orthogonal to ξ. Then we have

PROPOSITION 2.5. *Let f be an η-parallel invariant immersion of (M, g) into $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$. If $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ is a lo-cally* KTS*-space, then $(M, g, \mathscr{F}_\xi)$ is also a locally* KTS*-space.*

PROOF. From (2.3), (2.4) and (2.6) we get

$$\begin{aligned}(\bar{\nabla}_X \bar{R})(X, Y, X, Y) =& (\nabla_X R)(X, Y, X, Y) - \bar{g}((\tilde{\nabla}_X B)(X, X), B(Y, Y)) \\ &+ \bar{g}(4(\tilde{\nabla}_X B)(X, Y) - 2(\tilde{\nabla}_Y B)(X, X), B(X, Y)) \\ &+ \bar{g}(2(\tilde{\nabla}_Y B)(X, Y) - 3(\tilde{\nabla}_X B)(Y, Y), B(X, X)).\end{aligned}$$

Now the result follows from this relation by using Proposition 1.2 and Lemma 2.1.

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3. Fibrations and congruent immersions

Let f be a tangent isometric immersion of (M, g) into $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ and suppose that $\mathscr{F}_{\bar{\xi}}$ and the induced flow $\mathscr{F}_{\xi}$ on M are fibrable. We denote by $(\bar{M}, \bar{M}', \bar{\pi})$ and (M, M', π) the corresponding fibre bundles. Then there exists an isometric immersion $f': (M', g') \to (\bar{M}', g')$ satisfying $f' \circ \pi = \bar{\pi} \circ f$. Specifically, $f'(m') = \bar{\pi} f(m)$ with $m' = \pi(m)$, $m' \in M'$. This f' is called the *tranverse mapping* of f. Note that when f is one-to-one, then also f' is one-to-one. Moreover, since $\bar{\pi}$ is an open map [22, Corallary 3], it follows that if f is an embedding , then also f' is an embedding.

Now, let $\bar{\nabla}'$ and ∇' be the Levi-Civita connections of $\bar{M}'$ and M', respectively, and denote by B', C' and $\nabla'^{\perp}$ the second fundamental tensors and the normal connection of the transversal mapping f' of f. Then, from (1.3), (2.1) and (2.2), we get

$$(B'(X', Y'))^* = B(X'^*, Y'^*), \tag{3.1}$$

$$(C'_{U'}X')^* = C_{U'^*}X'^* - \bar{g}(B(X'^*, \xi), U'^*)\xi, \tag{3.2}$$

$$(\nabla'^{\perp}_{X'}U')^* = \nabla'^{\perp}_{X'^*}U'^* \tag{3.3}$$

for $X', Y' \in \mathscr{X}(M')$ and $U' \in \mathscr{X}(M')^{\perp}$. Moreover, from (2.6), (1.3), (3.1) and (3.3) we obtain

$$\begin{aligned}(\tilde{\nabla}_{X'^*}B, (Y'^*, Z'^*) =& ((\nabla'_{X'}B')(Y', Z'))^* - g(HX'^*, Y'^*)B(\xi, Z'^*) \\ &- g(HX'^*, Z'^*)B(\xi, Y'^*).\end{aligned} \tag{3.4}$$

From this immediately follows

PROPOSITION 3.1. *Let f be an invariant immersion. Then we have*

(i) *f is η-parallel if and only if f' is parallel;*
(ii) *f is totally geodesic if and only if f' is totally geodesic.*

In the rest of this section we focus on congruent isometric immersions.

DEFINITION 3.1. Let f_i, $i = 1, 2$, be tangent isometric immersions of (M_i, g_i) into $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$. Then f_1 and f_2 are said to be *$A(\bar{M})$-congruent* or briefly, *congruent*, if there exists an element $\bar{\varphi}$ of $A(\bar{M})$ and an isometry φ of M_1 into M_2 satisfying

$$\bar{\varphi} \circ f_1 = f_2 \circ \varphi. \tag{3.5}$$

Note that in this case $\varphi_* \xi_1 = \xi_2$ where ξ_i, $i = 1, 2$, denote the induced Killing vector fields on M_i. Moreover, it is easy to see that if one of these immersions is invariant, then also the other one is invariant. In that case the isometry φ satifies $\varphi_* \circ H_1 = H_2 \circ \varphi_*$ where H_i, $i = 1, 2$, are defined as before. Furthermore we have: If each f_i is an embedding, then they are congruent if and only if there exists a $\bar{\varphi} \in A(\bar{M})$ verifying $\bar{\varphi}(f_1 M_1) = f_2 M_2$.

Next, let $A(\bar{M}')$ denote the group of all $\bar{H}'$-preserving isometries of $\bar{M}'$. Since it is a closed subgroup of the full isometry group of $(\bar{M}', \bar{g})$, it is a Lie transformation group of $\bar{M}'$.

PROPOSITION 3.2. *Let f_1 and f_2 be congruent immersions into $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ and suppose that $\mathscr{F}_{\bar{\xi}}$, $\mathscr{F}_{\xi_1}$ and $\mathscr{F}_{\xi_2}$ are fibrable. Then the transverse isometric immersions f_i' are $A(\bar{M}')$-congruent.*

PROOF. The definition of the congruence implies that $\bar{\varphi}$ and φ preserve the corresponding flows. Therefore, we can define the mappings $\bar{\varphi}'$ and φ' as follows:

$$\varphi' \circ \pi_1 = \pi_2 \circ \varphi, \qquad \bar{\varphi}' \circ \bar{\pi} = \bar{\pi} \circ \bar{\varphi}$$

where π_i, $i = 1, 2$ and $\bar{\pi}$ denote the projections of M_i onto M_i' and $\bar{M}$ onto $\bar{M}'$, respectively. These mappings are isometries and $\bar{\varphi}'$ preserves $\bar{H}'$. Moreover, we have $\bar{\varphi}' \circ f_1' = f_2' \circ \varphi'$.

THEOREM 3.1. *Let $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ be a simply connected* KTS*-space and f_i, $i = 1, 2$, tangent isometric embeddings of M_i into $\bar{M}$ such that the induced flows $\mathscr{F}_{\xi_i}$ are fibrable. Then f_1 and f_2 are congruent if and only if the transverse embeddings f_i' are $A(\bar{M}')$-congruent.*

PROOF. The necessity follows from Proposition 3.2. So, suppose that f_1' and f_2' are $A(\bar{M}')$-congruent. Then there exists a $\bar{\varphi}' \in A(\bar{M}')$ verifying $\bar{\varphi}'(f_1' M_1') = f_2' M_2'$. Using the proof of Proposition 2.7 of [6] with slight modifications, we can guarantee the existence of an element $\bar{\varphi}$ of $A(\bar{M})$ such that $\bar{\varphi}' \circ \bar{\pi} = \bar{\pi} \circ \bar{\varphi}$. Then, since $f_i' \circ \pi_i = \bar{\pi} \circ f_i$, $i = 1, 2$, we have $\bar{\pi}(\bar{\varphi}(f_1 M_1)) = \bar{\pi}(f_2 M_2)$. Taking now into account the completeness of ξ_1 and ξ_2, it follows that $\bar{\varphi}(f_1 M_1) = f_2 M_2$ and so f_1 and f_2 are congruent.

PROPOSITION 3.3. *Let $\mathscr{F}_{\bar{\xi}}$ be a fibrable flow on $(\bar{M}, \bar{g})$ and let $f' : (M', g') \to (\bar{M}' = \bar{M}/\bar{\xi}, \bar{g}')$ be an isometric immersion. Then there exists a Riemannian manifold (M, g) equipped with an isometric flow $\mathscr{F}_{\xi}$ fibering over M' and an isometric immersion f of M into $\bar{M}$ such that $f_* \xi = \bar{\xi} \circ f$ with f' as associated transverse mapping.*

PROOF. Let M be the regular submanifold of $M' \times \bar{M}$ given by

$$M = \{(p, q) \in M' \times \bar{M} / f'(p) = \bar{\pi}(q)\}.$$

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Then M is a principal G^1-bundle over M' where G^1 denotes the one-parameter subgroup generated by $\bar{\xi}$. The projection π and the action of G^1 are defined by

$$\pi(p,q) = p, \qquad (p,q)\cdot s = (p, q\cdot s).$$

Let $f\colon M \to \bar{M}$ be the mapping $(p,q) \to q$. Then f is an immersion, $f' \circ \pi = \bar{\pi} \circ f$ and $f_* \varsigma = \bar{\varsigma}$, where ς and $\bar{\varsigma}$ denote the corresponding fundamental vector fields generated by d/dt of the G^1-bundles (M, M', π) and $(\bar{M}, \bar{M}', \bar{\pi})$, respectively. Moreover, if $\bar{\alpha}$ is a connection form on $\bar{M}$, then $\alpha = f^*\bar{\alpha}$ defines a connection form on M. Put $\eta = f^*\bar{\eta}$. If G^1 is isomorphic to $\mathbb{R}$ we put $\xi = \varsigma$ and η is a connection form on M. If G^1 is isomorphic to S^1 and l is the length of the integral curves of $\bar{\xi}$ on $\bar{M}$, we put $\xi = l^{-1}\varsigma$ and then $l^{-1}\eta$ is a connection form on M. In both cases $\eta(\xi) = 1$ and $f_*\xi = \bar{\xi} \circ f$.

Now, on M we consider the unique Riemannian metric g such that $g(\xi,\xi) = 1$, ξ is orthogonal to $\ker \eta$ and $\pi\colon M \to M'$ becomes a Riemannian submersion. Then ξ is a unit Killing vector field on M and the length of its integral curves is precisely l if they are closed. Since $\bar{\eta}(f'_*X'^*) = 0$ for all $X' \in \mathscr{X}(M')$, we have $f_*X'^* = (f_*X')^*$. Hence, taking into account that the G^1-bundles are Riemannian submersions and f' is an isometric immersion, it follows that f is also an isometric immersion.

REMARK 3.1. If $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ in Proposition 3.3 is a simply connected KTS-space, then it follows from Theorem 3.1 that (M, f) is unique up to congruence. Therefore we shall denote such a manifold by $((M', f'), \bar{\pi})$.

Then we have

PROPOSITION 3.4. *Let $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ be a simply connected* KTS-*space with non-vanishing constant ξ-sectional curvature and let (M', f') be a Kähler submanifold embedded into $(\bar{M}' = \bar{M}/\bar{\xi}, \bar{g}')$. Then $M = ((M', f'), \bar{\pi})$ is an invariant submanifold embedded into $\bar{M}$.*

PROOF. Theorem 3.2 of [6] implies that the orbit space $(\bar{M}', \bar{g}')$ is a Hermitian symmetric space with $J = c^{-1}\bar{H}'$ as a Hermitian structure, where c^2 denotes the ξ-sectional curvature of $(\bar{M}, \bar{g})$. Now, let $m \in M$, $X \in T_mM$ and $U \in (f_*T_mM)^\perp$. Since f' is a Kähler immersion, we have

$$\begin{aligned}\bar{g}(\bar{H} f_*X, U) &= \bar{g}'(\bar{H}'\bar{\pi}_* f_*X, \bar{\pi}_*U) = \bar{g}'(\bar{H}' \bar{f}'_*\pi_*X, \bar{\pi}_*U)\\ &= c\bar{g}'(J f'_*\pi_*X, \bar{\pi}_*U) = 0.\end{aligned}$$

Hence, M is invariant.

From this and from Theorem 3.1 in [6] we obtain

COROLLARY 3.1. *Let* $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ *be a simply connected contact* KTS-*space fibering over an irreducible Hermitian symmetric space* $(\bar{M}' = \bar{M}/\bar{\xi}, \bar{g}')$ *and let* (M', f') *be an embedded Kähler submanifold of* $(\bar{M}', \bar{g}')$. *Then* $M = ((M', f'), \bar{\pi})$ *is an invariant submanifold embedded into* $\bar{M}$.

Finally we prove

PROPOSITION 3.5. *Let* $(\bar{M}, \bar{g}, \mathscr{F}_{\bar{\xi}})$ *be a simply connected contact* KTS-*space fibering over a Hermitian symmetric space* $(\bar{M}' = \bar{M}'_1 \times \cdots \times \bar{M}'_r, \bar{g}', J)$ *where each* $\bar{M}'_i$, $i = 1, \ldots, r$ *is an irreducible Hermitian symmetric space. Let* f'_i, $i = 1, \ldots, r$ *be Kähler embeddings of Kähler manifolds* M'_i *into* $\bar{M}'_i$ *and let* $f' = f'_1 \times \cdots \times f'_r$ *be the product embedding of* $M' = M'_1 \times \cdots \times M'_r$ *into* $\bar{M}'$. *Then* $M = ((M', f'), \bar{\pi})$ *is an invariant submanifold embedded into* $\bar{M}$.

PROOF. Let $X \in T_m M$, $m \in M$. Then $f'_* \pi_* X$ can be decomposed as $f'_* \pi_* X = \sum_{i=1}^r f'_{i*} X_i$, where $\pi_* X = \sum_{i=1}^r X_i$, $X_i \in T_{\pi(m)} M'_i$. From [6, Theorem 3.2] we may conclude that there exist real numbers $c_1, \ldots, c_r$ such that $\bar{H}' f'_* \pi_* X = \sum_{i=1}^r c_i J f'_{i*} X_i$. Since f' is a Kähler embedding, this relation implies that f is invariant.

4. Invariant submanifolds in normal flow space forms

A Riemannian manifold (M, g) equipped with a contact flow $\mathscr{F}_\xi$ is said to be a *flow space form* if the H-sectional curvature is pointwise constant, that is, the sectional curvature of a two-plane $\{X, HX\}$ for horizontal $X \in T_m M$ is independent of X for each $m \in M$. Normal flow space forms have been studied in [7], where two cases are considered, according to whether the ξ-sectional curvature is constant or not. Now we shall treat invariant submanifolds in normal flow space forms for both cases.

I. Normal flow space forms with constant ξ-sectional curvature c^2 In this case it has been shown in [7] that for $\dim M \geq 5$ the H- sectional curvature is a *globally* constant k. In what follows we also suppose k to be globally constant for $\dim M = 3$. Then the normal flow space form is a (locally) KTS-space.

In what follows we shall denote such a space by $M^{2n+1}(c^2, k)$. Note that $c^2 = 1$ corresponds to the Sasakian space forms $M^{2n+1}(k)$. For each (c^2, k) the normal flow space form is locally isomorphic to one of the following model spaces: $(S^{2n+1} = SU(n+1)/SU(n))(c^2, k)$ for $k + 3c^2 > 0$, $H(n, 1)(k)$ for $k + 3c^2 = 0$, where $H(n, 1)$ is the $(2n+1)$-dimensional Heisenberg group, and $(U(1, n)/U(n))^\sim = (SU(1, n)^\sim / SU(n))(c^2, k)$ for $k + 3c^2 < 0$, where $\sim$ denotes the universal covering. See [7] for more details.

We firstly consider the case $k + 3c^2 \leq 0$.

PROPOSITION 4.1. *Let (M, g) be a $(2n+1)$-dimensional locally* KTS*-space invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$. If $\bar{k} + 3c^2 \leq 0$, then f is totally geodesic.*

PROOF. For each $m \in M$ we choose sufficiently small open neigborhoods $\mathscr{U}$ and $\bar{\mathscr{U}}$ of m and $f(m)$, respectively (where f denotes the immersion), such that ξ is regular on $\mathscr{U}$ and $\bar{\xi}$ on $\bar{\mathscr{U}}$ and let f' denote the transverse mapping of $f_{|\mathscr{U}}$. On $\bar{\mathscr{U}}' = \bar{\mathscr{U}}/\bar{\xi}$, $J = c^{-1}\bar{H}'$ defines a complex structure, and $(\bar{\mathscr{U}}', \bar{g}', J)$ is a Kähler manifold of constant holomorphic sectional curvature $\bar{k} + 3c^2$ [7]. It follows from Proposition 2.3 that the ξ-sectional curvature of M is constant and equals c^2. Then $\bar{\mathscr{U}}'$, equipped with the complex structure $J = c^{-1}H'$, is a locally Hermitian symmetric space (Lemma 2.1, Proposition 1.3 and [5, Theorem 3.2]) and f' is a Kähler immersion. Using [13, Theorem 3.2] it then follows that f' is totally geodesic. Finally, this and Proposition 3.1 yields that $f_{|\mathscr{U}}$ and hence f is totally geodesic.

Using Proposition 2.5 we then get

COROLLARY 4.1. *Let M be an $(2n + 1)$-dimensional invariantly immersed submanifold of $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ with η-parallel second fundamental form. It $\bar{k} + 3c^2$ is non-positive, then M is totally geodesic in $\bar{M}$.*

COROLLARY 4.2. *If $M^{2n+1}(c^2, k)$ is invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ and if $\bar{k} + 3c^2 \leq 0$, then M is totally geodesic in $\bar{M}$ and $k = \bar{k}$.*

The argument used in Proposition 4.1 indicates that there is a (local) correspondence between the theory of invariant submanifolds in (M, g) equipped with a normal flow of non-vanishing constant ξ-sectional curvature and the theory of Kähler submanifolds. This yields that several results about Kähler submanifolds may be translated directly to similar results about invariant submanifolds. We illustrate this by giving a series of examples. Their proofs, which we omit, are based on results for Kähler submanifolds given in [18–21, 24] and a similar reasoning as for Proposition 4.1.

PROPOSITION 4.2. *Let $M^{2n+1}(c^2, k)$ be invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ for $r < n(n+1)/2$. Then the immersion is totally geodesic.*

PROPOSITION 4.3. *Let $M^{2n+1}(c^2, k)$ be invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$. If $\bar{k} + 3c^2 > 0$, then either $k = \bar{k}$ and the immersion is totally geodesic, or $\bar{k} \geq 2k + 3c^2$.*

PROPOSITION 4.4. *Let $M^{2n+1}(c^2, k)$ be invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$. If the second fundamental form is η- parallel, then either $k = \bar{k}$ and the immersion is totally geodesic, or $\bar{k} = 2k + 3c^2$. This latter case arises only when $\bar{k} + 3c^2 > 0$.*

PROPOSITION 4.5. *Let $M^{2n+1}(c^2, k)$ be invariantly immersed in $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$. If $r = n(n+1)/2$, then either $k = \bar{k}$ and the immersion is totally geodesic, or $\bar{k} = 2k + 3c^2$. The latter case arises only when $\bar{k} + 3c^2 > 0$.*

PROPOSITION 4.6. *Let $M^3(c^2, k)$ be invariantly immersed in $\bar{M}^{2n+1}(c^2, \bar{k})$. If M cannot be immersed in any proper totally geodesic submanifold of $\bar{M}$, then $\bar{k} = 3c^2(n-1) + kn$.*

Further, by using (1.11), (1.12) and (1.13) and the papers cited above, we also get

PROPOSITION 4.7. *Let M^{2n+1} be invariantly immersed in $\bar{M}^{2n+3}(c^2, \bar{k})$. If $n \geq 2$ and if M is η-Einstein, then either the immersion is totally geodesic or $\rho(X, Y) = 2^{-1}\{n(\bar{k} + 3c^2) - 4c^2\}g(X, Y)$ for all horizontal X, Y on M. The latter case arises only when $\bar{k} + 3c^2 > 0$.*

PROPOSITION 4.8. *Let M^{2n+1} be a complete invariantly immersed submanifold of $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ where $\bar{k} + 3c^2 > 0$. If the H-sectional curvature K of M satisfies $K > (\bar{k} - 3c^2)/2$ and if the scalar curvature of M is constant, then the immersion is totally geodesic.*

Note that some of these results have been proved in a different way for the Sasakian case (that is, $c^2 = 1$). See, for example, [11, 12]. Moreover, the above list of results is not exhaustive. More results may be obtained using other papers as, for example, [16, 17].

Now we return to the case $\bar{k} + 3c^2 > 0$ and give some results based on [13] and [4].

Examples of Einstein-Kähler submanifolds embedded in complex projective space $\mathbb{C}P^N$ are described in [13]. These manifolds are compact irreducible C-spaces with $\dim H^2(M, \mathbb{R}) = 1$. C-spaces are closed, simply connected complex homogeneous spaces and are studied in [27]. All compact irreducible Hermitian symmetric spaces are C-spaces. Following the same notation as in [13], for every arbitrary complex simple Lie algebra $\mathfrak{g}$ with rank l and with fundamental root system $\{\alpha_1, \dots \alpha_l\}$ there are constructed compact C-spaces of the form $M_i' = G_u/H_{u,i}$, $i = 1, \dots, l$, where the Lie algebra of the connected Lie group G_u is a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$ and the center of the Lie algebra $\mathfrak{h}_{u,i}$ of $H_{u,i}$ is one-dimensional. Moreover, $\dim H^2(M_i', \mathbb{R}) = 1$. Furthermore, each compact irreducible C-space M with $\dim H^2(M, \mathbb{R}) = 1$ can be given in this way; and for each positive integer p, M_i' admits a full holomorphic embedding ρ_i^p into a $\mathbb{C}P^{N(p)}$ for some $N(p)$ such that the induced Kähler metric g_i^p is Einsteinian. (Here, a full embedding is an embedding such that the embedded submanifold cannot be embedded in a proper totally geodesic submanifold of $\mathbb{C}P^{N(p)}$.) We refer to [13] for more details.

Now, using Corollary 3.1, we can obtain an invariant embedding τ_i^p of $M_i^p = ((M_i', \rho_i^p), \bar{\pi})$ into $S^{2N(p)+1}(c^2, \bar{k})$, $\bar{k} + 3c^2 > 0$, by using the Hopf fibration $\bar{\pi}$. We shall call τ_i^p the *p-canonical embedding* of M_i^p into $S^{2N(p)+1}$. For $p = 1$ we call it the *canonical embedding*. It follows from [13, Theorem 4.3] (see also the local rigidity theorem [4, Theorem 9]) and Theorem 3.1 that all invariant embeddings of every M_i^p into $S^{2N(p)+1}$ are congruent to the p-canonical embedding τ_i^p.

Furthemore, let $\tilde{H}_{u,i}$ be the connected subgroup of $\tilde{G}_u$, the universal covering of G_u, associated to the Lie algebra $\mathfrak{h}_{u,i}$. Then M_i' may also be written as $\tilde{G}_u/\tilde{H}_{u,i}$, where $\tilde{G}_u$ is again compact. Since $\tilde{H}_{u,i}$ is compact, $\tilde{H}_{u,i}$ is locally the direct product of S^1 and a connected semisimple closed subgroup $\tilde{K}_{u,i}$. Hence, $\tilde{M}_i = \tilde{G}_u/\tilde{K}_{u,i}$ is a principal fiber bundle over M_i' with structural group $S^1 = \tilde{H}_{u,i}/\tilde{K}_{u,i}$. From [10, Theorem 11] it follows that there exists an integer q such that $M_i^p = q\tilde{M}_i = \tilde{M}_i/G_q$ where G_q is the cyclic subgroup of S^1 of order q.

Now, suppose that $M_i' = G_u/H_{u,i}$ is a compact irreducible Hermitian symmetric space. From Proposition 2.3 it follows that the ξ-sectional curvature on M_i^p is constant, say c^2, and from [6, 9] we may conclude that M_i^p is a KTS-space. In particular, it is a homogeneous space. Denote by (g_i^p, ξ_i^p) the structure tensor fields on M_i^p obtained from Proposition 3.3 for each ρ_i^p. Since the tensor field H' on M_i' is given by $H' = cJ$ and since (M_i', g_i^p) is Einsteinian, it follows that (M_i^p, g_i^p, ξ_i^p) is an η-Einsteinian manifold. We thus have

PROPOSITION 4.9. *Let $M_i' = G_u/H_{u,i}$ be a compact irreducible Hermitian symmetric space. Then (M_i^p, τ_i^p) is an η-Einstein invariant submanifold.*

Furthermore, let $\mathfrak{g}_u = \mathfrak{h}_{u,i} \oplus \mathfrak{m}^-$ be the canonical decomposition of $\mathfrak{g}_u$. Since all G_u-invariant Riemannian metrics on M_i' coincide up to a constant factor, we have that the Riemannian metric g_i^p on M_i' under the identification of $\mathfrak{m}^-$ with T_oM_i', o being the coset $H_{u,i}$, is given by $\beta_i^p B_{\mathfrak{m}^-}$ where B is the Killing form of $\mathfrak{g}_u$ and $\beta_i^p < 0$. Moreover, there exists a $Z_o \in Z(\mathfrak{h}_{u,i})$ such that $J_o = \mathrm{ad}_{\mathfrak{m}^-}(Z_o)$ defines the corresponding complex structure J on M_i'. (Here Z denotes the center.)

Let $\beta_i^o < 0$ be the scalar such that the Kähler form Φ_i^o for $\beta_i^o B_{\mathfrak{m}^-}$ on M_i' satisfies $e_{M_i'}(\tilde{M}_i) = [-(2\pi)^{-1}\Phi_i^o]$ (see [6]). Then the length l_i^p of the integral curves of the induced vector field ξ_i^p under τ_i^p on (M_i^p, g_i^p) satisfies $l_i^p = |2\pi c\beta_i^p/q\beta_i^o|$. On the other hand, considering $S^{2N(p)+1} = SU(N(p)+1)/SU(N(p))$ and $\mathbb{C}P^{N(p)} = SU(N(p)+1)/S(U(N(p))\times U(1))$, the scalars $\bar{\beta}$ and $\bar{\beta}^o$ for $\pi\colon S^{2N(p)+1}(c^2, \bar{k}) \to \mathbb{C}P^{N(p)}(\bar{k}+3c^2)$ are

$$\bar{\beta} = \frac{-1}{(\bar{k}+3c^2)N(p)}, \qquad \bar{\beta}^o = \frac{-1}{4N(p)}. \tag{4.1}$$

Hence the length l of the integral curves of $\bar{\xi}$ in $S^{2N(p)+1}(c^2, \bar{k})$ satisfies

$$l = \left|\frac{2\pi c\bar{\beta}}{\bar{\beta}^o}\right| \left|\frac{8\pi c}{\bar{k}+3c^2}\right|.$$

Taking into account all this information and the fact that l and l_i^p have to be equal, we have proved

THEOREM 4.1. *All invariantly embedded submanifolds in spheres, fibering over a compact irreducible Hermitian symmetric space $M_i' = G_u/H_{u,i}$ are, up to a congruence, of the form (M_i^p, τ_i^p) for some positive integer p. Moreover, M_i^p is isomorphic to the S^1-bundle $\tilde{M}_i/G_q$ over M_i' where $\tilde{M}_i = \tilde{G}_u/\tilde{K}_{u,i}$, $\tilde{K}_{u,i}$ being the commutator subgroup of $\tilde{H}_{u,i}$. Furthermore, q is given by*

$$q = (\bar{k} + 3c^2)\beta_i^p/4\beta_i^o \tag{4.2}$$

where $g_i^p = \beta_i^p B_{\mathfrak{m}^-}$ is the G_u-invariant Kähler metric on M_i' induced from the Fubini-Study metric on $\mathbb{C}P^{N(p)}(\bar{k} + 3c^2)$ and β_i^o is the negative scalar such that the corresponding Kähler form Φ_i^o on M_i' verifies $e_{M_i'}(\tilde{M}_i) = [-\Phi_i^0/2\pi]$.

COROLLARY 4.3. *Let $M^{2n+1}(c^2, k)$ and $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ be complete, simply connected normal flow space forms. If M is invariantly immersed into $\bar{M}$, then $k = \bar{k}$ and so, the immersion is totally geodesic.*

PROOF. For $\bar{k} + 3c^2 \leq 0$ the result follows from Corollary 4.2. So we suppose $\bar{k} + 3c^2 > 0$. Then $\bar{M}^{2(n+r)+1}(c^2, \bar{k})$ is isomorphic to the sphere $S^{2(n+r)+1}(c^2, \bar{k})$ which fibers over $\mathbb{C}P^{n+r}(\bar{k} + 3c^2)$. $M^{2n+1}(c^2, k)$ fibers over a complete and simply connected Kähler manifold of constant holomorphic sectional curvature $k + 3c^2$, say $M'^{2n}(k+3c^2)$. Then $M'^{2n}(k+3c^2)$ is a Kähler submanifold immersed in $\mathbb{C}P^{n+r}(\bar{k}+3c^2)$ and moreover, using [4, Theorem 11], M'^{2n} is isometrically embedded. Now, it follows from [4, Theorem 13] that $\bar{k}+3c^2 = p(k+3c^2)$ for a positive integer p, and so, taking into account (4.1) and (4.2), we have $q = p = 1$. Then the total geodesic property follows from Proposition 2.3.

In [4] Calabi proved that for each $p \in \mathbb{N}$, $\mathbb{C}P^n(h)$ can be isometrically embedded in $\mathbb{C}P^{N(p)}(ph)$ where $N(p) = \binom{n+p}{p} - 1$. The totally geodesic case corresponds to $p = 1$. Hence, using also Theorem 4.1 and [13], we have

PROPOSITION 4.10. *Let M^{2n+1} be an invariantly but non-totally geodesically embedded submanifold of $S^{2N+1}(c^2, \bar{k})$ which fibers over $\mathbb{C}P^n$. Then M is a normal flow space form $M(c^2, k)$ where $\bar{k} = 3c^2(p-1) + pk$ and $N = N(p) = \binom{n+p}{p} - 1$ for some $p > 1$. Moreover, M is isomorphic to the quotient manifold S^{2n+1}/G_p and the embedding $\alpha^p \colon S^{2n+1}/G_p \to S^{2N+1}$ is induced by the mapping of $\mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$ given by*

$$(z_0, \dots, z_n) \mapsto \left(z_0^p, \sqrt{p}z_0^{p-1}z_1, \dots, \sqrt{\frac{p!}{\alpha_0! \cdots \alpha_n!}}z_0^{\alpha_0} \cdots z_n^{\alpha_n}, \dots, z_n^p\right), \tag{4.3}$$

where $\sum_{i=0}^{n} \alpha_i = p$.

REMARK 4.1. Using the local rigidity theorem of [4] and Theorem 3.1, it follows that α^p coincides with the p-canonical embedding τ_i^p for $M_i' = \mathbb{C}P^n$, $i = n$.

In [13] another example of an embedded Kähler submanifold in $\mathbb{C}P^N$ is given. Namely, let f' be the mapping of $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}$ into $\mathbb{C}P^N$, $N = (n_1+1)\cdots(n_r+1) - 1$, given by

(4.4)
$$(z_0^1, \dots, z_{n_1}^1, \dots, z_0^r, \dots, z_{n_r}^r) \mapsto (z_0^1 \cdots z_0^r, \dots, z_{i_1}^1 \cdots z_{i_r}^r, \dots, z_{n_1}^1 \cdots z_{n_r}^r),$$
$$i_\alpha = 0, 1, \dots, n_\alpha, \quad \alpha = 1, \dots, r,$$

where $(z_0^\alpha, \dots, z_{n_\alpha}^\alpha)$ are homogeneous coordinates of $\mathbb{C}P^{n_\alpha}$. Then f' is a Kähler embedding of $\mathbb{C}P^{n_1}(h_1)\times\cdots\times\mathbb{C}P^{n_r}(h_r)$ into $\mathbb{C}P^N(h)$ if and only if $h = h_1 = \cdots = h_r$.

Hence we can consider the Riemannian manifold $((\mathbb{C}P^{n_1}(h)\times\cdots\times\mathbb{C}P^{n_r}(h), f'), \bar{\pi})$ as an invariant submanifold embedded into $S^{2N+1}(c^2, \bar{k})$ for $h = \bar{k} + 3c^2$. Then $((\mathbb{C}P^{n_1}(h) \times \cdots \times \mathbb{C}P^{n_r}(h), f'), \bar{\pi})$ equipped with the induced flow $\mathcal{F}_\xi$ is a KTS-space with constant ξ-sectional curvature equal to c^2. From [10, Theorem 11] it follows that there exist non-vanishing integers $q_1, \dots, q_r$ such that the above space is isomorphic to the quotient manifold

$$M = \left(\left(S^{2n_1+1}/G_{q_1}\right) \times \cdots \times \left(S^{2n_r+1}/G_{q_r}\right)\right)/T^{r-1}$$

where T^{r-1} denotes the $(r-1)$-dimensional torus and the action of T^{r-1} on $(S^{2n_1+1}/G_{q_1}) \times \cdots \times \left(S^{2n_r+1}/G_{q_r}\right)$ is defined by

$$(m_1, \dots, m_r)(s_2 \dots, s_r) = \left(m_1 \textstyle\prod_{i=2}^r s_i, m_2 s_2^{-1}, \dots, m_r s_r^{-1}\right).$$

From [6, Theorem 4.2] and taking into account that the lengths of the flow lines on M and $S^{2N+1}(c^2, \bar{k})$ coincide, we have

$$\frac{q_1\beta_1^0}{\beta_1} = \cdots = \frac{q_r\beta_r^0}{\beta_r} = \frac{\bar{\beta}^0}{\bar{\beta}}$$

where β_i, β_i^0, $i = 1, \dots, r$, $\bar{\beta}$, $\bar{\beta}^0$ are the scalars given in (4.1) for the fiber bundles $S^{2n_i+1} \to \mathbb{C}P^{n_i}$ and $S^{2N+1} \to \mathbb{C}P^N$, respectively. Hence, this implies $q_1 = \cdots = q_r = 1$. So, the KTS-space $\left(S^{2n_1+1} \times \cdots \times S^{2n_r+1}\right)/T^{r-1}$ over $\mathbb{C}P^{n_1}(h) \times \cdots \times \mathbb{C}P^{n_r}(h)$ is an invariant submanifold in $S^{2N+1}(c^2, \bar{k})$ with $h = \bar{k} + 3c^2$. Furthermore, since $\mathbb{C}P^{n_1}(h) \times \cdots \times \mathbb{C}P^{n_r}(h)$ is Einsteinian for $n_1 = \cdots = n_r$ and using (1.11) we have

PROPOSITION 4.11. *The quotient manifold* $M = \left(S^{2n_1+1} \times \cdots \times S^{2n_r+1}\right)/T^{r-1}$ *is an invariant submanifold embedded into* $S^{2N+1}(c^2, \bar{k})$, *where* $N = (n_1 + 1)\cdots(n_r + 1)$

-1. *Moreover, M is a* KTS*-space with its induced structure $(g, \mathscr{F}_\xi)$ fibering over $\mathbb{C}P^{n_1}(h) \times \cdots \times \mathbb{C}P^{n_r}(h)$ for $h = \bar{k} + 3c^2$ and M is η-Einsteinian if and only if $n_1 = \cdots = n_r$.*

REMARK 4.2. The embedding in Proposition 4.11 of M into S^{2N+1} is induced by f' given in (4.4) considered as a mapping of $\mathbb{C}^n$ into $\mathbb{C}^{N+1}$, where $n = r + n_1 + \cdots + n_r$.

Finally, using Propositions 3.1, 4.10, 4.11, Theorem 4.1 and [13, Theorem 7.4] we have the following classification theorem.

THEOREM 4.2. *Let (M, f) be a non-totally geodesic complete invariant submanifold embedded into $S^{2N+1}(c^2, \bar{k})$ where f is full and η-parallel. Then (M, f) is, up to congruence, one of the following submanifolds:*

(i) *the normal flow space form $M = \left(S^{2n+1}/G_2\right)(c^2, k)$, where $\bar{k} = 3c^2 + 2k$ and $N = N(2) = \binom{n+2}{n} - 1$. The embedding of M into S^{2N+1} is induced by the mapping of $\mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$ given in* (4.3) *for $p = 2$.*

(ii) *the* KTS*-space $M = \left(S^{2n_1+1} \times S^{2n_2+1}\right)/S^1$ over $\mathbb{C}P^{n_1}(h) \times \mathbb{C}P^{n_2}(h)$, where $h = \bar{k} + 3c^2$ and $N = (n_1+1)(n_2+1) - 1$. The embedding f is induced by the mapping of $\mathbb{C}^{n_1+n_2+2}$ into $\mathbb{C}^{N+1}$ given in* (4.4).

(iii) *$M = M^1 = ((M', \rho^1), \bar{\pi})$ where M' is a compact irreducible Hermitian symmetric space of rank 2, that is, a complex quadric $Q_n(\mathbb{C})$, a complex Grassmann manifold $G_{2,r}(\mathbb{C}) = SU(2+r)/S(U(2) \times U(r))$ with $r \geq 3$, $SO(10)/U(5)$ or $E_6/\operatorname{Spin}(10) \times S^1$. In this case f is the first canonical embedding and $N = n+1$, $\binom{r+2}{r} - 1$, 15 or 26, respectively.*

II. Normal flow space forms with non-constant ξ-sectional curvature and globally constant H-sectional curvature This second class of normal flow space forms has been treated in detail in [7]. We recall here some useful facts. Such a space is a (locally) KTS-space. Furthermore, in the complete case, $(\bar{M}, \bar{g}, \mathscr{F}_\xi)$ admits smooth distributions $\bar{\mathscr{H}}_1$ and $\bar{\mathscr{H}}_2$ such that for each $\bar{m} \in \bar{M}$, $\bar{\mathscr{H}}(\bar{m}) = \bar{\mathscr{H}}_1(\bar{m}) \oplus \bar{\mathscr{H}}_2(\bar{m})$ is an $\bar{H}$-invariant decomposition of the horizontal subspace $\bar{\mathscr{H}}(\bar{m})$ and each sectional curvature $\bar{K}(\bar{\mathscr{H}}_i, \xi)$, $i = 1, 2$, is a positive constant $\bar{c}_i^2$ ($\bar{c}_1^2 > \bar{c}_2^2$). Furthermore, such spaces are precisely the Riemannian manifolds $(\bar{M}^{2N+1}, \bar{g})$ equipped with a normal contact flow $\mathscr{F}_\xi$ which is transversally modelled on the Riemannian product $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ where $|h_2| < h_1$, $N_1 + N_2 = N$ and the ξ-sectional curvatures $\bar{c}_i^2$, $i = 1, 2$, are given by

$$\bar{c}_i^2 = (-1)^{i+1} h_i \frac{h_1 - h_2}{3(h_1 + h_2)}. \tag{4.5}$$

In what follows we denote this flow space form by $\bar{M}(N_1, N_2; h_1, h_2)$. The H-sectional curvature $\bar{k}$ is a strictly negative constant given by

$$\bar{k} = 2\frac{h_1 h_2}{h_1 + h_2}. \tag{4.6}$$

Now we have

PROPOSITION 4.12. *Let f be an invariant immersion of a Riemannian manifold (M^{2n+1}, g) into a complete normal flow space form $\bar{M}(N_1, N_2; h_1, h_2)$. If the ξ-sectional curvature on M is a constant c^2, then c^2 is either $\bar{c}_1^2$ or $\bar{c}_2^2$ and then for all $m \in M$, $f_* T_m M \subset \bar{\mathscr{H}}_1(f(m))$ or $\subset \bar{\mathscr{H}}_2(f(m))$, respectively. Hence M is locally transversally immersed into $\mathbb{C}P^{N_1}(h_1)$ or into $\mathbb{C}H^{N_2}(h_2)$, and so $n \leq \max(N_1, N_2)$.*

PROOF. For a horizontal vector field $X \in T_m M$, $m \in M$, we put $f_* X = \bar{X}_1 + \bar{X}_2$ where $\bar{X}_i \in \bar{\mathscr{H}}_i(f(m))$ for $i = 1, 2$. Since $H^2 X = -c^2 X$ and $\bar{H}^2 f_* X = -\bar{c}_1^2 \bar{X}_1 - \bar{c}_2^2 \bar{X}_2$ it follows, taking into account the invariance of f, that $c^2 = \bar{c}_i^2$ for $i = 1$ or 2. So $f_* T_m M \subset \bar{\mathscr{H}}_i(f(m))$ for some $i \in \{1, 2\}$.

Now, we can choose small neighborhoods $\mathscr{U}$ and $\bar{\mathscr{U}}$ of m and $f(m)$, respectively, such that ξ on $\mathscr{U}$ and $\bar{\xi}$ on $\bar{\mathscr{U}}$ are regular and the transverse mapping $f' : \mathscr{U}' = \mathscr{U}/\xi \to \bar{\mathscr{U}}' = \bar{\mathscr{U}}/\bar{\xi}$ is well defined. Then $\bar{\mathscr{U}}'$ can be written as $\bar{\mathscr{U}}' = \bar{\mathscr{U}}'_1 \times \bar{\mathscr{U}}'_2$, where $\bar{\mathscr{U}}'_i$, $i = 1, 2$, are connected open subsets of $\mathbb{C}P^{N_1}$ and $\mathbb{C}H^{N_2}$, respectively. Finally, it follows easily that $f'(\bar{\mathscr{U}}') \subset \bar{\mathscr{U}}'_i$ for some i.

From this, by using Proposition 4.1, we have

COROLLARY 4.4. *Let M^{2n+1} be a locally* KTS-*space with constant ξ-sectional curvature c^2 invariantly immersed in a complete normal flow space form $\bar{M}(N_1, N_2; h_1, h_2)$. If $c^2 = \bar{c}_2^2$, then M is totally geodesic.*

COROLLARY 4.5. *If $M^{2n+1}(c^2, k)$ is invariantly immersed in $\bar{M}(N_1, N_2; h_1, h_2)$ and $c^2 = \bar{c}_2^2$, then the immersion is totally geodesic.*

Note that for an invariant submanifold M of $\bar{M}(N_1, N_2; h_1, h_2)$ with constant ξ-sectional curvature $\bar{c}_1^2$ we may formulate a list of results similar to the ones given in Propositions 4.2–4.8 making only slight modifications. We omit the details.

Now, let $\bar{M}$ be a complete normal flow space form fibering over $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$. Then $\bar{M}$ is isomorphic to

$$\frac{(S^{2N_1+1}/G_q) \times \left(\dfrac{SU(1, N_2)^{\sim}}{SU(N_2)}/\mathbb{Z}[l]\right)}{S^1} \tag{4.7}$$

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for some $q \in \mathbb{Z}$ and where $l = |8\pi \bar{c}_1/qh_1|$ denotes the length of the flow lines. For each point $\bar{m}' \in \mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ the inclusion mappings i'_1, i'_2 of the totally geodesic submanifolds $\mathbb{C}P^{N_1}(h_1)$ and $\mathbb{C}H^{N_2}(h_2)$ through $\bar{m}'$ into the product define, using Proposition 3.3, totally geodesic submanifolds M_1 and M_2 of $\bar{M}(N_1, N_2; h_1, h_2)$ fibering over $\mathbb{C}P^{N_1}(h_1)$ and $\mathbb{C}H^{N_2}(h_2)$, respectively. Comparing lengths of the flow lines, such submanifolds are precisely

$$M_1 = \frac{S^{2N_1+1}}{G_q}(\bar{c}_1^2, \bar{k}), \qquad M_2 = \left(\frac{SU(1, N_2)^\sim}{SU(N_2)}/\mathbb{Z}[l]\right)(\bar{c}_1^2, \bar{k}).$$

These considerations lead to the following global version of Proposition 4.12.

THEOREM 4.3. *Let $\bar{M}$ be a complete normal flow space form fibering over $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ and let f be an invariant embedding of an (M, g) into $\bar{M}$. If the induced flow $\mathcal{F}_\xi$ is fibrable and the ξ-sectional curvature on M is a constant c^2, then c^2 equals $\bar{c}_1^2$ or $\bar{c}_2^2$. Moreover f is an invariant embedding of M into M_1 or M_2, respectively.*

REMARK 4.3. From the given theorem it follows that the invariant submanifolds with constant ξ-sectional curvature which are non-totally geodesically embedded in a complete normal flow space form fibering over $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ are essentially the non-totally geodesic invariant submanifolds of $M_1 = (S^{2N_1+1}/G_q)(\bar{c}_1^2, \bar{k})$, for some $q \in \mathbb{Z}$, which have been treated in I.

It has been proved in [6] that each complete contact locally KTS-space $(M, g, \mathcal{F}_\xi)$ is locally transversally modelled on a simply connected Hermitian symmetric space M'. Let $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ be its de Rham decomposition with Euclidean part $M'_0 = E^{2p}(x_1, \dots, x_{2p})$. Then we can consider each local submersion $\pi: \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ as a mapping into an open subset $\mathcal{U}' = \mathcal{U}'_0 \times \mathcal{U}'_1 \times \cdots \times \mathcal{U}'_r$ where $\mathcal{U}'_j$, $j = 0, 1, \dots, r$, is a connected open subset of M'_j. There exist $r + p$ real numbers $c_1, \dots, c_r, \mu_1, \dots, \mu_p$ and smooth distributions $\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_r$ on $\mathcal{U}$,obtained by taking the horizontal lifts of the tangent vectors of M'_j, such that, for each $m \in \mathcal{U}$, $\mathcal{H}(m) = \mathcal{G}_0(m) \oplus \mathcal{G}_1(m) \oplus \cdots \oplus \mathcal{G}_r(m)$ is an H-invariant orthogonal decomposition of the horizontal subspace $\mathcal{H}(m)$ and each sectional curvature $K(\mathcal{G}_l, \xi)$, $l = 1, \dots, r$, is a positive constant c_l^2. Moreover, $\{(\partial/\partial x_1)^*, \dots, (\partial/\partial x_{2p})^*\}$ is an orthonormal frame field of $\mathcal{G}_0$ which satisfies $K((\partial/\partial x_k)^*, \xi) = K((\partial/\partial x_{p+k})^*, \xi) = \mu_k^2$, $k = 1, \dots, p$.

Now, let f be an invariant immersion of $(M, g, \mathcal{F}_\xi)$ into a complete normal flow space form $\bar{M}(N_1, N_2; h_1, h_2)$. Using a similar argument as in the proof of Proposition 4.12 we have that each c_l^2 and each μ_k^2, $l = 1, \dots, r$, $k = 1, \dots, p$, is either $\bar{c}_1^2$ or $\bar{c}_2^2$. We denote by $\mathcal{H}_i(m)$, $i = 1, 2$, the subspace of $\mathcal{H}(m)$, $m \in M$, such that the sectional curvature $K(\mathcal{H}_i, \xi)$ equals $\bar{c}_i^2$. Then it follows that $f_*\mathcal{H}_i(m) \subset \bar{\mathcal{H}}_i(f(m))$. Moreover, it can be proved that $\mathcal{H}_1$ and $\mathcal{H}_2$ determine global distributions (see the proof of [7, Theorem 4.1]). Hence, we have

PROPOSITION 4.13. *Let (M, f) be a complete locally* KTS*-space and invariant submanifold immersed in a complete normal flow space form $\bar{M}(N_1, N_2; h_1, h_2)$. Then there exist smooth distributions $\mathscr{H}_1$ and $\mathscr{H}_2$ on M such that for each $m \in M$, $\mathscr{H}(m) = \mathscr{H}_1(m) \oplus \mathscr{H}_2(m)$, (it may occur that $\mathscr{H} = \mathscr{H}_1$ or $\mathscr{H} = \mathscr{H}_2$) is an H-invariant decomposition of the horizontal subspace $\mathscr{H}(m)$, the sectional curvatures $K(\mathscr{H}_i, \xi) = \bar{c}_i^2$ and $f_*\mathscr{H}_i \subset \bar{\mathscr{H}}_i$ for each $i = 1, 2$.*

Hence, using [6, Theorem 3.2] and [13, Theorem 3.2] we obtain

THEOREM 4.4. *Let $\bar{M}$ be a complete normal flow space form fibering over $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ and let f be an invariant embedding of a* KTS*-space $(M^{2n+1}, g, \mathscr{F}_\xi)$ with non-constant ξ-sectional curvature, into $\bar{M}$. If the orbit space $M' = M/\xi$ is simply connected, then we have:*

(i) *M' is a Riemannian product $M'^{n_1}_1 \times \mathbb{C}H^{n_2}(h_2)$, $n_1 + n_2 = n$, where M'_1 is a Hermitian symmetric space and the $(1, 1)$-tensor field $J = \bar{c}_1^{-1}H'_1 + \bar{c}_2^{-1}H'_2$ is a Hermitian structure on (M', g') where $H'_i = H' \circ p_i$, $i = 1, 2$, and where $p_i \colon M' \to M'_i$ denotes the projection of M' onto M'_i (for $M'_2 = \mathbb{C}H^{n_2}(h_2)$).*

(ii) *the transverse embedding f' of f is a product embedding $f'_1 \times f'_2$ where f'_1 is a Kähler embedding of M'_1 into $\mathbb{C}P^{N_1}(h_1)$ and f'_2 is a totally geodesic embedding of $\mathbb{C}H^{n_2}(h_2)$ into $\mathbb{C}H^{N_2}(h_2)$.*

The corresponding local version also holds.

Using Proposition 3.5 we can provide many examples of invariant submanifolds in a complete simply connected normal flow space form $\bar{M}(N_1, N_2; h_1, h_2)$. In fact, we obtain a full invariant embedding f of $((M'_i \times \mathbb{C}H^{N_2}(h_2), \rho_i^p \times \mathrm{id}), \bar{\pi})$ into $\bar{M}(N_1, N_2; h_1, h_2)$, where $M'_i = G_u/H_{u,i}$ is a compact irreducible Hermitian symmetric space, ρ_i^p is the p-canonical embedding of M'_i into $\mathbb{C}P^{N_1}$, $\bar{\pi}$ is the fibration of $\bar{M}(N_1, N_2; h_1, h_2)$ onto $\mathbb{C}P^{N_1} \times \mathbb{C}H^{N_2}$ and id: $\mathbb{C}H^{N_2} \to \mathbb{C}H^{N_2}$ is the identity mapping. In particular, for $\bar{M}$ given as in (4.7), we obtain the following families of invariant submanifolds:

(i) For each positive integer p, the quotient manifold

$$\left((S^{2n_1+1}/G_{pq}) \times \left(\frac{SU(1, N_2)^\sim}{SU(N_2)}\right)/\mathbb{Z}[l]\right)/S^1,$$

where $N_1 = N_1(p) = \binom{n_1+p}{p} - 1$, is an invariant submanifold embedded into $\bar{M}$ and fibering over $\mathbb{C}P^{n_1}(h_1 p^{-1}) \times \mathbb{C}H^{N_2}(h_2)$. The corresponding transverse embedding is the product $\rho_i^p \times \mathrm{id}$ where the p-canonical embedding ρ_i^p of $\mathbb{C}P^{n_1}$ into $\mathbb{C}P^{N_1}$ coincides with that one given in (4.3) in terms of homogeneous coordinates of $\mathbb{C}P^{n_1}$. Note that in this case, f is η-parallel only when $p = 2$.

(ii) The quotient manifold

$$\left((S^{2n_1+1}/G_q) \times \cdots \times (S^{2n_r+1}/G_q) \times \left(\frac{SU(1, N_2)^{\sim}}{SU(N_2)}/\mathbb{Z}[l]\right)\right)/T^r,$$

is an invariant submanifold embedded into $\tilde{M}$ which fibers over $\mathbb{C}P^{n_1}(h_1) \times \cdots \times \mathbb{C}P^{n_r}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ where $N_1 = (n_1 + 1) \cdots (n_r + 1) - 1$. Here the full transverse mapping is the product embedding of the mapping given in (4.4) and the identity mapping id: $\mathbb{C}H^{N_2} \to \mathbb{C}H^{N_2}$. For $r = 2$, f is η-parallel.

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