ON THE GEOMETRY OF THE UNIT SPHERES OF THE LORENTZ SPACES $L_{w,1}$

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We identify the extreme points of the unit sphere of the Lorentz space $L_{w,1}$. This yields a characterization of the surjective isometries of $L_{w,1}(0,1)$. Our main result is that every element in the unit sphere of $L_{w,1}$ is the barycenter of a unique Borel probability measure supported on the extreme points of the unit sphere of $L_{w,1}$.

1. Notation and terminology. For a measurable function f defined on $(0,\infty)$ we define the distribution of f by $d_f(t)=|\{x:|f(x)|>t\}|$, $0<t<\infty$ ($|A|$ denotes the Lebesgue measure of the set A), and the decreasing rearrangement of f by $f^*(t)=\inf\{s>0:d_f(s)\le t\}$. Following **[5]** we define the Lorentz space $L_{w,1}(0,\infty)$ as the space of all (equivalence classes of) measurable functions f on $(0,\infty)$ for which $\|f\|=\int_0^\infty f^*(t)w(t)\,dt<\infty$, where $w:(0,\infty)\to(0,\infty)$ is a strictly decreasing function satisfying $\lim_{t\to 0} w(t)=\infty, \lim_{t\to\infty} w(t)=0$, $\int_0^1 w(t)\,dt=1$, and $\int_0^\infty w(t)\,dt=\infty$. $L_{w,1}$ is sometimes referred to as Λ_ϕ where $\phi(t)=\int_0^t w(s)\,ds, t\ge 0$. The fact that w is strictly decreasing implies that ϕ is strictly concave.

For $M>0, L_{w,1}(0,M)$ is the subspace of $L_{w,1}(0,\infty)$ consisting of those functions which are supported on $[0,M]$. We shall write $L_{w,1}$ when the domain does not affect the argument.

$I(A)$ denotes the characteristic function of a set $A\subset[0,\infty)$. If $0<|A|<\infty$, we write $e(A)=I(A)/\phi(|A|)$ (so that $e(A)$ is of norm one in $L_{w,1}$). A^c denotes the complement of A, and $\{f>t\}$ denotes the set $\{s:f(s)>t\}$.

Given a Banach space X, Ba(X) denotes its closed unit ball. For a subset B of X, conv(B) denotes the convex hull of B.

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2. Preliminary results. Before we can prove our main result (Theorem 3.5), we need a description of the extreme points of Ba$(L_{w,1})$. For the sequence space $l_{p,1}$ **[3]** and the function space $L_{p,1}$ **[2]**, a characterization of the extreme points is well-known, but we do not know a reference for the general result. Because this general result (Lemma 2.1) will be needed later, we supply a proof below. As a consequence, we get a characterization of the surjective isometries of $L_{w,1}$ (Theorem 2.3).

LEMMA 2.1 (cf. **[2**, Lemma 2.1]). *If $f,g\in L_{w,1}(0,\infty)$ satisfy $\|f+g\|=\|f\|+\|g\|$, then $(f+g)^*=f^*+g^*$.*

Proof. Let $h(t)=f^*(t)+g^*(t)-(f+g)^*(t)$, and let $H(t)=\int_0^t h(s)\,ds$. Then $\int_0^\infty h(t)w(t)\,dt=\|f\|+\|g\|-\|f+g\|=0$, and $H(t)\ge 0$ by definition of the decreasing rearrangement. Integration by parts yields

$$\int_0^\infty H(t)\,d(-w(t))=-H(t)w(t)\,|_0^\infty+\int_0^\infty h(t)w(t)\,dt=0+0=0.$$

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Since H is continuous and w is strictly decreasing, it follows that $H(t) \equiv 0$. This implies that $h(t) = 0$ a.e. ∎

We prove our result on extreme points for the space $L_{w,1}(0, \infty)$, but the same proof works for $L_{w,1}(0, M)$.

PROPOSITION 2.2. *Let $f \in L_{w,1}$. Then f is an extreme point of* $\mathrm{Ba}(L_{w,1})$ *if and only if $|f| = e(A)$ for some $A \subset (0, \infty)$ of finite, positive measure.*

Proof. Suppose first that $f \in L_{w,1}(0, \infty)$ and that there exists $A \subset (0, \infty)$ of finite, positive measure such that $|f| = e(A)$. Then $f^* = I(0, |A|)/\phi(|A|)$. If $f = g + h$ with $\|g\| + \|h\| = 1$, then $f^* = g^* + h^*$ by Lemma 2.1. Since g^* and h^* are non-increasing, this implies that g^* and h^* are multiples of f^*. But then $|g|$ and $|h|$ must be multiples of $|f|$, and so g and h are multiples of f. Thus, f is an extreme point of $\mathrm{Ba}(L_{w,1})$.

Now suppose that $|f|$ is not a multiple of $I(A)$ for any $A \subset (0, \infty)$ of finite, positive measure. Then there exists $\lambda > 0$ such that if $A = \{|f| > \lambda\}$, we have $|A| > 0$ and $\|fI(A^c)\| > 0$. Let $g = (f - \lambda\,\mathrm{sgn}(f))I(A)$, and let $h = f - g$. Clearly, $\|g\| > 0$, $\|h\| > 0$, and $f \neq g/\|g\|$. But $f^* = g^* + h^*$, and so $\|f\| = \|g\| + \|h\|$. Since $f = \|g\|(g/\|g\|) + \|h\|(h/\|h\|)$, it follows that f is not an extreme point of $\mathrm{Ba}(L_{w,1})$. ∎

We now characterize the surjective isometries of $L_{w,1}(0, 1)$. Our proof is based on the description of the isometries of $L_p(0, 1)$, $1 < p < \infty$ [**6**, pp. 415–418]. We present only a sketch of the proof, but the details are easy to check.

THEOREM 2.3. *Let T be a surjective isometry of the space $L_{w,1}(0, 1)$. Then there exists a ± 1-valued Borel measurable function ε and a Borel measurable map σ from* $[0, 1]$ *to* $[0, 1]$ *which is measure-preserving (i.e., $|\sigma^{-1}(A)| = |A|$) such that*

$$(Tf)(t) = \varepsilon(t) f(\sigma(t)), \qquad 0 \leq t \leq 1.$$

Proof. Since T is a surjective isometry, T maps the set of extreme points of $\mathrm{Ba}(L_{w,1})$ onto itself. By Proposition 2.2, we know that for every Borel set $A \subset [0, 1]$ there is a Borel set A' such that $|T(e(A))| = e(A')$. Define a mapping ψ from $\mathscr{B}$, the collection of Borel sets of $[0, 1]$, into $\mathscr{B}/\mathscr{N}$, where $\mathscr{N}$ is the collection of Borel sets of measure zero, by setting $\psi(A) = A'$. It is easy to check that ψ sends disjoint sets to disjoint sets, that $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \psi(A_n)$ for disjoint sets (A_n), and that $|\psi(A)| = |A|$. In particular, $\psi([0, 1]) = [0, 1]$. Arguing now as in [**6**, p. 417], there exists a Borel mapping σ on $[0, 1]$ such that $\psi(A) = \sigma^{-1}(A)$ for every $A \in \mathscr{B}$. Finally, define a ± 1-valued Borel measurable function ε by $\varepsilon = T(I(0, 1))$. The conclusion of the theorem now follows easily. ∎

3. A uniqueness theorem. It is well-known that $L_{w,1}$ is a separable dual space [**5**]. One consequence of this fact is that for every $f \in L_{w,1}$ with $\|f\| = 1$, there is a probability measure μ on $\mathrm{Ba}(L_{w,1})$ which is supported on the extreme points of $\mathrm{Ba}(L_{w,1})$ such that f is the barycenter of μ:

$$f = \int_{\mathrm{Ba}(L_{w,1})} x \, d\mu(x).$$

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Our goal in this section is to show that this representation of f is unique. We begin with two technical lemmas. The proof of the first is straightforward (see e.g. [**1**, Section 2.7]).

Lemma 3.1. *Let f be a nonnegative, locally integrable function on $(0,\infty)$. For each $t>0$ there exists $A\subset(0,\infty)$ such that $|A|=t$ and $\int_0^t f^*=\int_A f$. Moreover, any such A is necessarily of the form $\{f>\lambda\}\cup C$, where $C\subset\{f=\lambda\}$, for some $\lambda\geq 0$.*

Lemma 3.2. *Let f and g be nonnegative functions in $L_{w,1}$ such that $\|f+g\|=\|f\|+\|g\|$, and let $B=\{f>\lambda\}$, where $\lambda>0$. Then $\operatorname{ess\,inf}_B g\geq\operatorname{ess\,sup}_{B^c} g$.*

Proof. Let $t=|B|$. Applying Lemma 3.1 to $f+g$, there exists $A\subset(0,\infty)$ such that $|A|=t$ and $\int_0^t(f+g)^*=\int_A(f+g)$. By Lemma 2.1, $(f+g)^*=f^*+g^*$, so $\int_0^t(f^*+g^*)=\int_A(f+g)$. This implies that $\int_0^t f^*=\int_A f$ and $\int_0^t g^*=\int_A g$. By Lemma 3.1 we have $A=B=\{f>\lambda\}$, and there exists an $\alpha>0$ such that $A=\{g>\alpha\}\cup C$, where $C\subset\{g=\alpha\}$. The conclusion of the lemma follows immediately. ∎

Let G denote the collection of extreme points of $\mathrm{Ba}(L_{w,1})$ and let μ be a regular Borel probability measure on $\mathrm{Ba}(L_{w,1})$ such that $\mu(G)=1$. Let us say that an extreme point e belongs to the support of μ if $\mu(U)>0$ for every norm-open neighborhood U of e, and let H denote the collection of extreme points in the support of μ. Then H is a G_δ-set in $L_{w,1}$ and $G\backslash H$ is contained in a union of μ-null open sets. Since $L_{w,1}$ is separable it is a Lindelöf space, and so $\mu(G\backslash H)=0$, whence $\mu(H)=1$. We are now in a position to give the main technical ingredient in the proof of our theorem.

Lemma 3.3. *Suppose that f is a nonnegative function on $(0,\infty)$, that $\|f\|=1$, and that f is the barycenter of a Borel probability measure μ as described above. Then every extreme point e in the support of μ is of the form $e(E)$, where $E=\{f\geq\lambda\}$ or $E=\{f>\lambda\}$, for some $\lambda\geq 0$.*

Proof. It is clear that every extreme point in the support of μ is nonnegative, and so e is of the form $e(E)$ for some $E\subset(0,\infty)$. If E is not of the form described in the statement of the lemma, then there exist a $\lambda>0$ and disjoint sets A and B of positive Lebesgue measure such that $A\subset\{f\geq\lambda\}\cap E^c$ and $B\subset\{f\leq\lambda\}\cap E$. Thus $e|_A=0$ and $e|_B=\rho$ for some $\rho>0$. Let $\varepsilon>0$; since e lies in the support of μ there exists a neighborhood U of e of diameter less than ε such that $\mu(U)>0$. Let g and h be defined by $g=\int_U x\,d\mu(x)$ and $h=\int_{U^c}x\,d\mu(x)$. Now $g/\mu(U)$ and e are close in measure, since $\|g/\mu(U)-e\|<\varepsilon$. By choosing $\varepsilon>0$ sufficiently small we may assume that there exist $A'\subset A$ and $B'\subset B$ of positive Lebesgue measure such that $g/\mu(U)>\rho/4$ on A' and $g/\mu(U)>3\rho/4$ on B'. Recall that $A'\subset A\subset\{f\geq\lambda\}$ and $B'\subset B\subset\{f\leq\lambda\}$, and that $f=g+h$. Thus, for almost all $a'\in A'$ we have

$$h(a')=f(a')-g(a')\geq\lambda-\rho\mu(U)/4,$$

and for almost all $b'\in B'$ we have

$$h(b')=f(b')-g(b')\leq\lambda-3\rho\mu(U)/4,$$

and so

$$\operatorname*{ess\,inf}_{B'} h\leq\operatorname*{ess\,sup}_{A'} h-\rho\mu(U)/2. \qquad (*)$$

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To derive a contradiction, first observe that $B' \subset \{g > 3\rho\mu(U)/4\}$ and $A' \subset \{g < \rho\mu(U)/4\}$. Also, $\|g+h\| = \|g\| + \|h\|$ since f and the support of μ both lie in the unit sphere of $L_{w,1}$. Combined with Lemma 3.2 it now follows that $\operatorname{ess\,inf}_{B'} h \geq \operatorname{ess\,sup}_{A'} h$, which contradicts ($*$). ■

LEMMA 3.4. *Let $f \in L_{w,1}$ with $\|f\| = 1$. Then f^* is the barycenter of a unique Borel probability measure supported on the extreme points of* $\mathrm{Ba}(L_{w,1})$.

Proof. By Lemma 3.3, the support of every μ for which f^* is a barycenter is contained in $S = \{e(0,u): u > 0\}$. The homeomorphism $u \mapsto e(0,u)$ from $(0,\infty)$ onto S induces a bijection between the regular Borel probability measures supported on S and those on $(0,\infty)$. Suppose μ corresponds to $\hat{\mu}$ under this bijection. Then

$$f^*(u) = \int_{(u,\infty)} \frac{d\hat{\mu}}{\phi}$$

for all u by the right continuity of f^*. Now every regular Borel measure on $(0,\infty)$ is the Lebesgue–Stieltjes measure defined by its indefinite integral [4, p. 331]; thus $(1/\phi)\,d\hat{\mu} = d(-f^*)$, and $d\hat{\mu} = \phi d(-f^*)$. The uniqueness of μ follows at once. ■

THEOREM 3.5. *Let $f \in L_{w,1}$ with $\|f\| = 1$. Then f is the barycenter of a unique Borel probability measure supported on the extreme points of $Ba(L_{w,1})$.*

Proof. The mapping $g \mapsto g\,.\,\varepsilon$, where ε is a ± 1-valued measurable function, defines an isometry from $L_{w,1}$ onto $L_{w,1}$, and hence we may assume that f is non-negative. Given a nonnegative function $g \in L_{w,1}$, define the set $S(g)$ by

$$S(g) = \{e(\{g > \lambda\}): \lambda > 0\} \cup \{e(\{g \geq \lambda\}): \lambda > 0\}.$$

Define a map T from $S(f)$ onto $S(f^*)$ by $T(e(\{f > \lambda\})) = e(\{f^* > \lambda\})$, $T(e(\{f \geq \lambda\})) = e(\{f^* \geq \lambda\})$. By linearity, T extends from $\mathrm{conv}(S(f))$ onto $\mathrm{conv}(S(f^*))$. Since f and f^* have the same distribution, it is easily verified that T is an affine isometry. So, T extends to an affine isometry from the closed convex hull of $S(f)$ onto the closed convex hull of $S(f^*)$ with $T(f) = T(f^*)$. By Lemma 3.3, if f is the barycenter of a Borel probability measure μ of the required type, then the support of μ is contained in $S(f)$. Under the affine isometry T, the measure μ corresponds to a measure μ^* whose barycenter is f^*. By Lemma 3.4, μ^* is unique, whence μ is unique. ■

The proof of Theorem 3.5 implies the following corollary.

COROLLARY 3.6. *Suppose $f \in L_{w,1}$ with $\|f\| = 1$. Then f admits a representation of the form $f = \sum_{n=1}^{\infty} \lambda_n e_n$, where $\lambda_n \geq 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$, and each e_n is an extreme point of* $\mathrm{Ba}(L_{w,1})$, *if and only if $d(f^*)$ is purely atomic* (*that is, if and only if f^* is a saltus function*; *cf.*, *e.g.*, [4, *p.* 335]).

REMARK. Of course, if $\|f\| < 1$, then a representing measure supported on the extreme points will no longer be unique. In fact, it is easy to see that f can always be expressed in the form $f = \sum_{n=1}^{\infty} \lambda_n e_n$ as described in Corollary 3.6.

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