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Inverse Semigroups and Sheu's Groupoid for Odd Dimensional Quantum Spheres

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Abstract. In this paper, we give a different proof of the fact that the odd dimensional quantum spheres are groupoid C^*-algebras. We show that the C^*-algebra $C(S_q^{2\ell+1})$ is generated by an inverse semigroup T of partial isometries. We show that the groupoid $\mathcal{G}_{tight}$ associated with the inverse semigroup T by Exel is exactly the same as the groupoid considered by Sheu.

1 Introduction

Quantization of mathematical theories is a major theme of research today. The theory of Quantum groups and Noncommutative geometry are two prime examples. The theory of compact quantum groups was initiated by Woronowicz in the late eighties in [8–10]. A main example in his theory is the quantum group $SU_q(n)$ and its homogeneous spaces. One of the problems in noncommutative geometry is understanding how these groups fit under Connes' formulation of NCG. Thus it becomes necessary to understand the C^*-algebra of these quantum groups.

Vaksman and Soibelman studied the irreducible representations of C^*-algebra of the quantum group $SU_q(n)$ in [7]. Exploiting their work, Sheu in [5] used the theory of groupoids and obtained certain composition sequences that are useful in understanding the structure of the C^*-algebra of $SU_q(n)$. Then in [6] the question of whether C^*-algebras of these quantum homogeneous spaces are in fact groupoid C^*-algebras was raised. In [6], an affirmative answer is given for the odd dimensional quantum spheres $S_q^{2n-1} := SU_q(n)/SU_q(n-1)$. The purpose of this paper is to give an alternative proof of the same result. We use the theory of inverse semigroups developed in [1] to reconstruct the groupoid given in [6]. We believe that the proof is constructive, as the groupoid in [6] is reconstructed from a combinatorial data naturally associated with S_q^{2n-1}.

The paper is organized as follows. In the next section, we recall the basics of inverse semigroups and the groupoid associated with it without proofs. We refer the reader to [1] for proofs. In Section 3, we recall the definition of the C^*-algebra of the odd dimensional quantum sphere $S_q^{2\ell+1}$ and associate a natural inverse semigroup with it. In Sections 4–6, we work out the groupoid associated with the inverse semigroup and show that the groupoid is isomorphic to Sheu's groupoid constructed in [6].

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2 Inverse Semigroups and their Groupoids

In this section, we briefly recall the construction of the groupoid associated with an inverse semigroup. We refer the reader to [1] for proofs and details.

Definition 2.1 An *inverse semigroup* T is an associative semigroup such that for every $s \in T$, there exists a unique element denoted s^* for which $s^*ss^* = s^*$ and $ss^*s = s$. Then $*$ is an involution that is antimultiplicative. We say that an inverse semigroup has 0 if there exists an element 0 such that $0.s = s.0 = 0$ for every $s \in T$.

2.1 The Unit Space of the Groupoid

Let T be an inverse semigroup with 0. We denote the set of projections in T by E, *i.e.*, $E := \{e \in T : e = e^* = e^2\}$. Then E is a commutative semigroup containing 0. Consider the set $\{0, 1\}$ as a multiplicative semigroup.

Definition 2.2 Let T be an inverse semigroup with 0 and let E be its set of projections. A character of E is a nonzero map $x\colon E \to \{0, 1\}$ such that:

(i) the map x is a semigroup homomorphism;
(ii) $x(0) = 0$.

We denote the set of characters of E by $\widehat{E}_0$. The set of characters $\widehat{E}_0$ is a locally compact Hausdorff topological space where the topology on $\widehat{E}_0$ is the subspace topology inherited from the product topology on $\{0, 1\}^E$.

The set of characters can also be described in terms of filters by considering its support. For a character x, let $A_x := \{e \in E : x(e) = 1\}$. Then A_x is nonempty and has the following properties:

(i) $0 \notin A_x$;
(ii) if $e \in A_x$ and $f \geq e$, then $f \in A_x$;
(iii) if $e, f \in A_x$, then $ef \in A_x$.

A nonempty subset A of E having properties (i), (ii), and (iii) is called a filter. Moreover, if A is a filter then the indicator function 1_A is a character. Thus, there is a bijection between filters and characters. A filter is called an ultra filter if it is maximal. By Zorn's lemma, ultra filters exist. Define

$$\widehat{E}_\infty := \{x \in \widehat{E}_0 : A_x \text{ is an ultrafilter }\}$$

and denote its closure by $\widehat{E}_{tight}$.

2.2 The Partial Action of T on $\widehat{E}_0$

The inverse semigroup T acts naturally on $\widehat{E}_0$ as partial homeomorphisms, which we now explain. We let T act on $\widehat{E}_0$ on the right as follows. For $x \in \widehat{E}_0$ and $s \in T$, define $(x.s)(e) = x(ses^*)$. Then

(i) the map $x.s$ is a semigroup homomorphism;
(ii) $(x.s)(0) = 0$.

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But $x.s$ is nonzero if and only if $x(ss^*) = 1$. For $s \in T$, define the domain and range of s as

$$D_s := \{x \in \widehat{E}_0 : x(ss^*) = 1\}, \quad R_s := \{x \in \widehat{E}_0 : x(s^*s) = 1\}.$$

Note that both D_s and R_s are compact and open. Moreover, s defines a homeomorphism from D_s to R_s with s^* as its inverse. Also observe that $\widehat{E}_{tight}$ is invariant under the action of T.

2.3 The Groupoid $\mathcal{G}_{tight}$

Consider the transformation groupoid $\Sigma := \{(x, s) : x \in D_s\}$ with the composition and the inversion given by:

$$(x, s)(y, t) := (x, st) \text{ if } y = x.s, \quad (x, s)^{-1} := (x.s, s^*)$$

Define an equivalence relation $\sim$ on Σ as $(x, s)\sim(y, t)$ if $x = y$ and if there exists an $e \in E$ such that $x \in D_e$ for which $es = et$. Let $\mathcal{G} = \Sigma/\sim$. Then $\mathcal{G}$ is a groupoid, as the product and the inversion respect the equivalence relation $\sim$. Now we describe a topology on $\mathcal{G}$ that makes $\mathcal{G}$ into a topological groupoid.

For $s \in T$ and U an open subset of D_s, let $\theta(s, U) := \{[x, s] \in \mathcal{G} : x \in U\}$. We refer to [1] for the proof of the following proposition. We denote $\theta(s, D_s)$ by θ_s.

Proposition 2.3 *The collection $\{\theta(s, U) : s \in T, U \text{ open in } D_s\}$ forms a basis for a topology on $\mathcal{G}$. The groupoid $\mathcal{G}$ with this topology is a topological groupoid whose unit space can be identified with $\widehat{E}_0$. Also, one has the following:*

(i) *for $s, t \in T$, $\theta_s\theta_t = \theta_{st}$;*
(ii) *for $s \in T$, $\theta_s^{-1} = \theta_{s^*}$;*
(iii) *the set $\{1_{\theta_s} : s \in T\}$ generates the C^*-algebra $C^*(\mathcal{G})$.*

We define the groupoid $\mathcal{G}_{tight}$ to be the reduction of the groupoid $\mathcal{G}$ to $\widehat{E}_{tight}$.

3 The Odd Dimensional Quantum Spheres

Before we proceed let us fix some notation. Throughout we assume that $q \in (0, 1)$ and ℓ is a positive integer. We denote the set of non-negative integers by $\mathbb{N}$. Let the Hilbert space $\ell^2(\mathbb{N})^{\otimes \ell} \otimes \ell^2(\mathbb{Z})$ be denoted by $\mathcal{H}_\ell$. The left shift operator on $\ell^2(\mathbb{N})$ is denoted by S, and the right shift on $\ell^2(\mathbb{Z})$ is denoted by t. The letter N stands for the number operator on $\ell^2(\mathbb{N})$, *i.e.*, on the standard orthornormal basis $\{e_n : n \in \mathbb{N}\}$, N is defined by $N(e_n) := ne_n$.

In this section, we recall a few well-known facts about the C^*-algebra of the odd dimensional quantum spheres. The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \dots, z_{\ell+1}$ satisfying the fol-

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lowing relations (see [2]):

$$
\begin{aligned}
z_i z_j &= q z_j z_i, \quad 1 \le j < i \le \ell+1,\\
z_i^* z_j &= q z_j z_i^*, \quad 1 \le i \ne j \le \ell+1,\\
z_i z_i^* - z_i^* z_i + (1-q^2)\sum_{k>i} z_k z_k^* &= 0, \quad 1 \le i \le \ell+1,\\
\sum_{i=1}^{\ell+1} z_i z_i^* &= 1.
\end{aligned}
$$

Note that for $\ell = 0$, the C^*-algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the circle and for $\ell = 1$, it is $C(SU_q(2))$.

Let $Y_{k,q}$ be the following operators on $\mathcal{H}_\ell$:

$$
Y_{k,q} = \begin{cases} \underbrace{q^N \otimes \dots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1-q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-k \text{ copies}}, & \text{if } 1 \le k \le \ell,\\ \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes t, & \text{if } k = \ell+1. \end{cases}
$$

Then $\pi_\ell \colon z_k \mapsto Y_{k,q}$ gives a faithful representation of $C(S_q^{2\ell+1})$ on $\mathcal{H}_\ell$ for $q \in (0,1)$ (see [2, Lemma 4.1 and Remark 4.5]). We let $Y_{k,0}$ to denote the limit of the operators $Y_{k,q}$ as q tends to zero. The formula for $Y_{k,0}$ is again the same as that of $Y_{k,q}$ where q^N stands for the rank one projection $p = |e_0\rangle\langle e_0|$.

Consider the unitary operator U on $\mathcal{H}_\ell$ defined by

$$
U(e_{m,z}) = e_{m,z+\sum_{i=1}^{\ell} m_i}.
$$

Define $Z_{k,q} := UY_{k,q}U^*$ for $q \in [0,1)$. The representation $z_k \to Z_{k,q}$ of $C(S_q^{2\ell+1})$ is the one considered in [5,6]. Let $A_\ell(q)$ be the image of $C(S_q^{2\ell+1})$ under this representation, *i.e.*, $A_\ell(q)$ is the C^*-algebra generated by $Z_{k,q}$. We refer to [4] for the proof of the following proposition.

Proposition 3.1 *For any $q \in (0,1)$, one has $A_\ell(0) = A_\ell(q)$.*

From now on, we simply denote $Z_{k,0}$ by Z_k. Note that Z_k's are in fact partial isometries. Let us introduce more notations. For $m, n \in \mathbb{N}^\ell$ and $r \in \mathbb{Z}$, let $B_k(r,m,n)$ be defined as

$$
B_k(r,m,n) = \begin{cases} \underbrace{S^{*m_1} p S^{n_1} \otimes S^{*m_2} p S^{n_2} \otimes \cdots \otimes S^{*m_{k-1}} p S^{n_{k-1}}}_{k-1 \text{ copies}} \otimes S^{*m_k} S^{n_k} \otimes 1 \otimes t^{(\sum_{i=1}^{k}(m_i-n_i))} & \text{if } 1 \le k \le \ell,\\ S^{*m_1} p S^{n_1} \otimes S^{*m_2} p S^{n_2} \otimes \cdots \otimes S^{*m_\ell} p S^{n_\ell} \otimes t^{r+\sum_{i=1}^{\ell}(m_i-n_i)}, & \text{if } k = \ell+1. \end{cases}
$$

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Note the following commutation relations.

If $i < j$, then

$$B_i(r,m,n)B_j(r',m',n') = \delta_{(n_1,n_2,\ldots,n_{i-1}),(m_1',m_2',\ldots m_{i-1}')} 1_{[n_i,\infty)}(m_i')B_j(r',m'',n'),$$

where $m'' = (m_1, m_2, \ldots, m_{i-1}, m_i + m_i' - n_i, m_{i+1}', \ldots, m_\ell')$.

If $i \leq \ell$ and $n_i \leq m_i'$, then

$$B_i(r,m,n)B_i(r',m',n') = \delta_{(n_1,n_2,\ldots,n_{i-1}),(m_1',m_2',\ldots,m_{i-1}')} B_i(r',m'',n'),$$

where $m'' := (m_1, m_2, \ldots, m_{i-1}, m_i + m_i' - n_i, m_{i+1}', \ldots m_l')$.

If $i \leq \ell$ and $n_i > m_i'$, then

$$B_i(r,m,n)B_i(r',m',n') = \delta_{(n_1,n_2,\ldots,n_{i-1}),(m_1',m_2',\ldots,m_{i-1}')} B_i(r',m,n''),$$

where $n'' := (n_1', n_2', \ldots, n_{i-1}', n_i' + n_i - m_i', n_{i+1}', \ldots, n_\ell')$.

Finally, $B_{\ell+1}(r,m,n)B_{\ell+1}(r',m',n') = \delta_{n,m'} B_{\ell+1}(r+r',m,n')$. Also

$$B_i(r,m,n)^* := B_i(-r,n,m).$$

It is clear from the above commutation relations that the set

$$T := \{0\} \cup \{B_i(r,m,n) : 1 \leq i \leq \ell+1, r \in \mathbb{Z}, m,n \in \mathbb{N}^\ell\}$$

is an inverse semigroup of partial isometries.

Proposition 3.2 *The set $T := \{0\} \cup \{B_i(r,m,n) : 1 \leq i \leq \ell+1, r \in \mathbb{Z}, m,n \in \mathbb{N}^\ell\}$ is an inverse semigroup of partial isometries. Moreover $\{Z_i : 1 \leq i \leq \ell+1\}$ generates T.*

Proof As already observed T is an inverse semigroup of partial isometries. Let e_i be the ℓ-tuple which is 1 on the i-th coordinate and zero elsewhere. Then $Z_k = B(0, e_k, 0)$ for $k \leq \ell$ and $Z_{\ell+1} = B_{\ell+1}(1,0,0)$. Thus Z_k's are in T. Moreover,

$$\begin{aligned} 0 &= Z_i^* Z_j \quad \text{if } i < j, \\ B_i(r,m,n) &= Z_1^{m_1} Z_2^{m_2} \cdots Z_i^{m_i} Z_i^{*n_i} Z_{i-1}^{*n_{i-1}} \cdots Z_1^{*n_1} \quad \text{if } i \leq \ell, \\ B_{\ell+1}(r,m,n) &= Z_1^{m_1} Z_2^{m_2} \cdots Z_\ell^{m_\ell} (Z_{\ell+1}^{r_+} Z_{\ell+1}^{*r_-}) Z_\ell^{*n_\ell} Z_{\ell-1}^{*n_{\ell-1}} \cdots Z_1^{*n_1}. \end{aligned}$$

where r_+ and r_- denote the positive and negative parts of r. Thus every element in T is a word in Z_i's and in Z_i^*'s. This completes the proof. ■

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4 The Tight Characters of the Inverse Semigroup T

In this section, we describe the tight characters of the inverse semigroup T defined in Proposition 3.2. The set of projections in T is denoted by E and the set of characters of E by $\widehat{E}_0$. Consider the one-point compactification $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$. We denote the projection from $\overline{\mathbb{N}}^\ell$ onto the first r components by π_r.

Let $p_i(m) := B_i(0, m, m)$. Then $E := \{0\} \cup \{p_i(m) : 1 \leq i \leq \ell + 1, m \in \mathbb{N}^\ell\}$. First observe that if Λ is a subsemigroup of E not containing 0, then the set

$$A_\Lambda := \{f \in E : f \geq e \text{ for some } e \in \Lambda\}$$

is a filter.

Let $k \in \overline{\mathbb{N}}^\ell$ be given. Let r be the least positive integer for which $k_{r+1} = \infty$. (For an ℓ-tuple k, we set $k_{\ell+1} = \infty$.) Define $\Lambda_k = \{p_{r+1}(\pi_r(k), n) : n \in \mathbb{N}^{\ell-r}\}$. It is easy to see that Λ_k is a subsemigroup of E not containing 0. Then A_{Λ_k} is a filter and thus gives rise to a character. We denote the character associated with A_{Λ_k} by $\phi(k)$. The following lemma gives a closed formula for $\phi(k)$.

Lemma 4.1 *Let $k \in \overline{\mathbb{N}}^\ell$ be given. The character $\phi(k)$ is given by*

$$\phi(k)\big(p_i(m)\big) := \begin{cases} \delta_{\pi_{i-1}(m),\pi_{i-1}(k)} 1_{[0,k_i]}(m_i) & \text{if } i \leq \ell, \\ \delta_{m,k} & \text{if } i = \ell + 1. \end{cases}$$

Proof Let r be the least positive integer for which $k_{r+1} = \infty$. Observe $p_i(m) \geq p_{r+1}(\pi_r(k), n)$ for some n if and only if $i \leq r + 1$, $\pi_{i-1}(m) = \pi_{i-1}(k)$ and $m_i \leq k_i$. The proof follows from this observation. ∎

An immediate consequence of the above lemma is that the map $\phi\colon \overline{\mathbb{N}}^\ell \to \widehat{E}_0$ is continuous. In the next proposition, we show that the image of ϕ is exactly $\widehat{E}_\infty$.

Proposition 4.2 *The image of the map ϕ is in fact $\widehat{E}_\infty$.*

Proof First we show that the image of ϕ is contained in $\widehat{E}_\infty$. Let $k \in \overline{\mathbb{N}}^\ell$ be given and let r be the least non-negative integer for which $k_{r+1} = \infty$. Recall that

$$\Lambda_k := \{p_{r+1}(\pi_r(k), n) : n \in \mathbb{N}^{\ell-r}\}.$$

We denote $\phi(k)$ by x. We claim that A_x is an ultrafilter. Suppose that there exists a character, say y, such that $A_x \subset A_y$. Then we need to show that $x = y$ or $A_x = A_y$. Since $x = 1$ on Λ_k and $A_x \subset A_y$, it follows that $y = 1$ on Λ_k. If $\pi_r(m) \neq \pi_r(k)$, then $p_{r+1}(\pi_r(m), v)$ is orthogonal to Λ_k. Hence x and y vanish on $p_{r+1}(\pi_r(m), v)$. Thus, $x(p_{r+1}(m)) = y(p_{r+1}(m))$ for every $m \in \mathbb{N}^\ell$.

Now let $i > r + 1$ be given. Let $m \in \mathbb{N}^\ell$. Choose an ℓ-tuple n such that $\pi_r(n) = \pi_r(k)$ and $n_{r+1} > m_{r+1}$. Then $p_i(m)$ and $p_{r+1}(n)$ are orthogonal, but $x = y = 1$ on $p_{r+1}(n)$. Thus, x and y vanish on $p_i(m)$.

Now let $i \leq r$ and $m \in \mathbb{N}^\ell$. If $m_i > k_i$, then $p_i(m)$ is orthogonal to Λ_k, but $x = y = 1$ on Λ_k. Thus, x and y vanish on $p_i(m)$ if $m_i > k_i$. Now suppose $m_i \leq k_i$.

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If $\pi_{i-1}(m) \neq \pi_{i-1}(k)$,then $p_i(m)$ is again orthogonal to Λ_k and thus x and y vanish on $p_i(m)$. Consider now the case where $m_i \leq k_i$ and $\pi_{i-1}(m) = \pi_{i-1}(k)$. Then $x(p_i(m)) = 1$ by definition, and, since $A_x \subset A_y$, it follows that $y(p_i(m)) = 1$. Thus we have shown that $x(p_i(m)) = y(p_i(m))$ for every i and m. Hence $x = y$ or, in other words, A_x is an ultrafilter. This proves that $\phi(\overline{\mathbb{N}}^{\ell})$ is contained in $\widehat{E}_\infty$.

Now let us prove that $\widehat{E}_\infty$ is contained in the range of ϕ. Let $x \in \widehat{E}_\infty$ be given. Let r be the largest non-negative integer for which there exists a $k^{'}$ such that $x = 1$ on $p_{r+1}(k^{'})$. Choose k such that $\pi_r(k) = \pi_r(k^{'})$ and $k_{r+1} = \infty$. We claim that $A_x \subset A_{\phi(k)}$. Then the maximality of A_x forces the equality $x = \phi(k)$.

Let $i \leq r+1$ be given. Consider an ℓ-tuple m such that either $m_i > k_i$ or $\pi_{i-1}(m) \neq \pi_{i-1}(k)$. Then $p_i(m)$ is orthogonal to $p_{r+1}(k^{'})$. Hence $x(p_i(m)) = 0$ if either $m_i > k_i$ or $\pi_{i-1}(m) \neq \pi_{i-1}(k)$. Also, x vanishes on $p_i(m)$ if $i > r+1$ by the choice of r. Thus we have shown that $A^c_{\phi(k)} \subset A^c_x$. Hence A_x is contained in $A_{\phi(k)}$. Since A_x is maximal, it follows that $x = \phi(k)$. This completes the proof. ∎

Corollary 4.3 *The set $\widehat{E}_\infty$ is compact, and $\widehat{E}_{tight} = \widehat{E}_\infty$.*

Proof The proof follows from the fact that ϕ is continuous, $\overline{\mathbb{N}}^{\ell}$ is compact, and from Proposition 4.2. ∎

Define an equivalence relation on $\overline{\mathbb{N}}^{\ell}$ as follows:

$$k \sim k^{'} \text{ if there exists } r \geq 0 \text{ such that } \pi_r(k) = \pi_r(k^{'}) \text{ and } k_{r+1} = k^{'}_{r+1} = \infty.$$

In the next proposition, we show that $\widehat{E}_{tight}$ is homeomorphic to the quotient space $\overline{\mathbb{N}}^{\ell}/\sim$.

Proposition 4.4 *The map $\phi\colon \overline{\mathbb{N}}^{\ell} \to \widehat{E}_{tight}$ factors through the quotient $\overline{\mathbb{N}}^{\ell}/\sim$ to give a map $\widetilde{\phi}\colon \overline{\mathbb{N}}^{\ell}/\sim \to \widehat{E}_{tight}$. The map $\widetilde{\phi}$ is a homeomorphism.*

Proof It is clear from the definition and from Lemma 4.1 that ϕ factors through the quotient to give a map $\widetilde{\phi}$. Since ϕ is continuous, it follows that $\widetilde{\phi}$ is continuous. We now show that $\widetilde{\phi}$ is one-to-one.

Let $k, k^{'}$ be such that $\phi(k) = \phi(k^{'})$. Let r_k (resp. $r_{k^{'}}$) be the least non-negative integer for which $k_{r_k+1} = \infty$ (resp. $k^{'}_{r_{k^{'}}+1} = \infty$). Then r_k is the largest integer for which there exists an m such that $\phi(k)$ is 1 on $p_{r_k+1}(m)$. Thus $r_k = r_{k^{'}}$. Moreover, $\phi(k)$ is 1 on $p_{r_k+1}(\pi_{r_k}(k), u)$. Thus $\phi(k^{'})$ is 1 on $p_{r_{k+1}}(\pi_{r_k}(k), u)$. Now Lemma 4.1 implies that $\pi_{r_k}(k) = \pi_{r_k}(k^{'})$. Hence $k \sim k^{'}$. This proves that $\widetilde{\phi}$ is one-to-one.

Now Proposition 4.2 implies that $\widetilde{\phi}$ is onto. As $\overline{\mathbb{N}}^{\ell}/\sim$ is compact and $\widehat{E}_{tight}$ is Hausdorff, it follows that $\widetilde{\phi}$ is a homeomorphism. This completes the proof. ∎

5 Sheu's Groupoid

In this section, we recall the groupoid for the odd dimensional quantum spheres $S_q^{2\ell+1}$ described in [6]. Consider the transformation groupoid $\mathbb{Z} \times (\mathbb{Z}^{\ell} \times \overline{\mathbb{Z}}^{\ell})$ where $\mathbb{Z}^{\ell}$ acts

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on $\overline{\mathbb{Z}}^{\ell}$ by translation. Let $\mathcal{F}$ be the restriction of the transformation groupoid to $\overline{\mathbb{N}}^{\ell}$. Define

$$\Sigma := \left\{ (z, x, w) \in \mathcal{F} : w_i = \infty \Rightarrow x_{i+1} = \cdots = x_\ell = 0 \text{ and } z = -\sum_{j=1}^{i} x_j \right\}$$

Then Σ is an open subgroupoid of $\mathcal{F}$. Define an equivalence relation $\sim$ on Σ as follows:

$$(z, x, w_1, w_2, \ldots, w_{i-1}, \infty, *, *, \ldots, *) \sim (z, x, w_1, w_2, \ldots, w_{i-1}, \infty, \infty, \ldots, \infty).$$

Let $\mathcal{G} := \Sigma/\sim$. The multiplication and the inversion on Σ factors through the equivalence relation making $\mathcal{G}$ into a groupoid. When $\mathcal{G}$ is given the quotient topology, it becomes the topological groupoid that was described in [6].

6 The Groupoid $\mathcal{G}_{tight}$ of the Inverse Semigroup T

In this section, we show that the groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T is isomorphic to Sheu's groupoid described in the previous section. For an ℓ-tuple m, we set $m_{\ell+1} = \infty$. We define a map $\psi \colon \Sigma \to \mathcal{G}_{tight}$ as follows. Let $(z, x, w) \in \Sigma$ be given. Let r be the least non-negative integer for which $w_{r+1} = \infty$. Then ψ on (z, x, w) is given by

$$\psi(z, x, w) := \left[(\phi(w), B_{r+1}(t, m, n) \right],$$

where t, m, n are given by $t := z + \sum_{j=1}^{r} x_j$, $m := (w_1, w_2, \ldots, w_r, |x_{r+1}|, 0, \ldots, 0)$ and $n := (x_1 + w_1, x_2 + w_2, \ldots, x_r + w_r, x_{r+1} + |x_{r+1}|, 0, \ldots, 0)$. Observe that ψ is well defined as $\pi_r(w) = \pi_r(m)$ and $w_{r+1} = \infty$.

Let us introduce the following notation. For $m, n \in \mathbb{N}^\ell$ and $0 \le r \le \ell$, let

$$A_{r+1}(m, n) := S^{*m_1} p S^{n_1} \otimes \cdots \otimes S^{*m_r} p S^{n_r} \otimes S^{*m_{r+1}} S^{n_{r+1}} \otimes 1.$$

We consider $A_{r+1}(m, n)$ as an operator on $\ell^2(\mathbb{N}^\ell)$.

Proposition 6.1 *The map ψ is continuous, and ψ factors through the equivalence relation $\sim$. Let $\widetilde{\psi}$ be the induced map from $\mathcal{G} \to \mathcal{G}_{tight}$. Then $\widetilde{\psi}$ is a topological groupoid isomorphism.*

Proof First we show that ψ factors through the equivalence relation. Let $(z, x, w) \sim (z, x, w')$ and let r (resp. r') be the least non-negative integer for which $w_{r+1} = \infty$ (resp $w'_{r'+1} = \infty$). By definition, $r = r'$ and $\pi_r(w) = \pi_r(w')$. Then by Proposition 4.4, it follows that $\phi(w) = \phi(w')$. Since the definition of ψ involves only the first r components of w, $\psi(z, x, w) = \psi(z, x, w')$. This proves that $\widetilde{\psi}$ is well defined.

The map $\widetilde{\psi}$ is one-to-one:

Suppose that $\psi(z, x, w) = \psi(z', x', w')$. Again let r and r' be the least non-negative integer for which w_{r+1} and $w_{r'+1}$ are both ∞. Then r is the largest integer for which there exists an m such that $\phi(w)$ is 1 on $p_{r+1}(m)$. Since $\phi(w) = \phi(w')$, we

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have $r = r'$ and $\pi_r(w) = \pi_r(w')$. As $\psi(z, x, w) = \psi(z', x', w')$, it follows that there exists a projection e such that

$$\phi(w)(e) = 1 \quad \text{and} \quad e(t^z \otimes A_{r+1}(m, n)) = e(t^{z'} \otimes A_{r+1}(m', n')).$$

But $\phi(w)(e) = 1$ implies that $e \geq p_{r+1}(\pi_r(w), 0)$. Hence we can choose e to be $p_{r+1}(\pi_r(w), 0)$. Thus it follows that $z = z'$ and $A_{r+1}(m, n) = A_{r+1}(m', n')$. Thus $m = m'$ and $n = n'$, which in turn implies $x_i = x_i'$ for $i \leq r+1$. Since $(z, x, w) \in \Sigma$, it follows that $x_i = x_i' = 0$ for $i \geq r+2$. Thus, we have shown that $(z, x, w) \sim (z', x', w')$. Hence $\widetilde{\psi}$ is one-to-one.

The map $\widetilde{\psi}$ is onto:

First note that if $a - b = c - d$, then there exists a projection $e = S^{*(b+c)}S^{b+c}$ such that $eS^{*b}S^a = eS^{*d}S^c$. Hence in the definition of ψ we can change the $r+1$st components of m and n such that $n_{r+1} - m_{r+1} = x_{r+1}$. Let $[(\phi(w), B_i(s, m, n))]$ be an element in $\mathcal{G}_{tight}$. Let r be the first non-negative integer for which $w_{r+1} = \infty$. Then $i \leq r+1$. By premultiplying $B_i(s, m, n)$ by $p_{r+1}(\pi_r(w), 0)$ we can assume that $i = r+1$ and m is such that $\pi_r(w) = \pi_r(m)$. Now if $r \leq \ell - 1$, then, for $z = \sum_j (m_j - n_j)$, for an x such that $\pi_{r+1}(x) = \pi_{r+1}(n) - \pi_{r+1}(m)$ and $x_i = 0$ for $i \geq r+2$, $\psi(z, x, w) = [(\phi(w), B_{r+1}(s, m, n))]$. If $r = \ell$, then with $z = s + \sum_{j=1}^{\ell}(m_j - n_j)$ and $x_j = n_j - m_j$, one has $\psi(z, x, w) = [(\phi(w), B_{\ell+1}(s, m, n))]$. This proves that $\widetilde{\psi}$ is onto.

The map $\widetilde{\psi}$ is continuous:

Let (z^n, x^n, w^n) be a sequence in Σ converging to $(z, x, w) \in \Sigma$. Let r be the least non-negative integer for which $w_{r+1} = \infty$. Then, eventually, $(z^n, x^n, \pi_r(w^n))$ coincides with $(z, x, \pi_r(w))$. Suppose that $\theta(s, U)$ is an open set containing $\psi(z, x, w)$. Without loss of generality we can assume that $s := t^z \otimes A_{r+1}(m, n)$, where m, n are defined as in the definition of ψ. Since U is an open set containing $\phi(w)$ and as ϕ is continuous, it follows that $\phi(w^n) \in U$ eventually. Let r_n be the least non-negative integer for which $w^n_{r_n+1} = \infty$. Then $r_n \geq r$. Let m^n and n^n be as in the definition of ψ for (z, x, w^n). If $e_n := p_{r_n+1}(m^n)$, then $\phi(w^n) \in D_{e_n}$ and $e_n(t^z \otimes A_{r_n+1}(m^n, n^n)) = e_n(t^z \otimes A_{r+1}(m, n))$ eventually. Thus, eventually, $\psi(z^n, x^n, w^n) = [(\phi(w^n), s)] \in \theta(s, U)$. This proves that $\widetilde{\psi}$ is continuous.

We leave it to the reader to check that $\widetilde{\psi}$ is a homeomorphism, and it is in fact a groupoid homomorphism. This completes the proof. ∎

Remark 6.2 In [6], it was shown that $C^*_{red}(\mathcal{G})$ is isomorphic to $C(S_q^{2\ell+1})$. One can construct a faithful representation of $C^*_{red}(\mathcal{G})$ onto $C(S_q^{2\ell+1})$. We refer to [3, 5, 6] for constructing such representations.

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