Canad. Math. Bull. Vol. **44** (3), 2001 pp. 292–297

An Analogue of Napoleon's Theorem in the Hyperbolic Plane

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Abstract. There is a theorem, usually attributed to Napoleon, which states that if one takes any triangle in the Euclidean Plane, constructs equilateral triangles on each of its sides, and connects the midpoints of the three equilateral triangles, one will obtain an equilateral triangle. We consider an analogue of this problem in the hyperbolic plane.

1 Introduction

In the Euclidean Plane, Napoleon's Theorem is easily proven (see below), and hence, an obvious question will be whether or not the theorem holds in the Hyperbolic Plane. In what follows, we shall consider a slight variation of Napoleon's Theorem; that is, we shall push out a fixed distance d along the perpendicular bisector of a side, whereas in Napolean's Theorem, the distance is proportional to the length of the side. We show that, for a given d, the space of triangles (modulo orientation preserving isometry) under this map has a fixed point (an equilateral triangle whose side length can be written down explicitly in terms of d), and furthermore, that this fixed point is attracting. First, however, we state and prove Napoleon's Theorem. The research for this article was done as a master's thesis at the University of Maryland, College Park under the direction of Dr. Richard Schwartz.

Theorem 1.1 **(Napoleon's Theorem)** *Given any triangle, T, construct an equilateral triangle on each side of T (see figure). Then the new triangle, T', formed by connecting the midpoints of the three equilateral triangles will be equilateral.*

Although Napoleon's Theorem can be easily verified using basic trigometry, the following well-known proof is somewhat more insightful:

Proof Write a given triangle as a triple of points in the complex plane. We will then define a map

$$\tau\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3,$$

where

$$\tau(z_1, z_2, z_3) = \left(z_1 + \frac{z_2 - z_1}{2}\left(1 - \frac{i}{\sqrt{3}}\right), z_2 + \frac{z_3 - z_2}{2}\left(1 - \frac{i}{\sqrt{3}}\right),\ z_3 + \frac{z_1 - z_3}{2}\left(1 + \frac{i}{\sqrt{3}}\right)\right).$$

Received by the editors June 17, 1999.
AMS subject classification: 37D40.

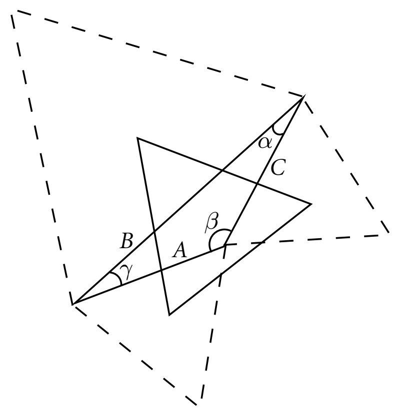

Figure 1: Napoleon's Theorem

It is easy to see that for $a \in \mathbb{C}$,

$$\tau(az_1, az_2, az_3) = a\tau(z_1, z_2, z_3),$$

and hence that τ is $\mathbb{C}$-linear. Note that if the vertices of the triangle are entered in the appropriate order, the map τ yields the triangle T' described in the theorem. Now, notice that any 3-tuple of vertices can be rewritten as $a\Sigma_1 + b\Sigma_2 + c\Sigma_3$, where $\Sigma_1 = (1, \omega, \omega^2)$, $\Sigma_2 = (1, \omega^2, \omega)$, $\Sigma_3 = (1, 1, 1)$ and $a, b, c \in \mathbb{C}$ (ω is a third root of unity). Since τ is $\mathbb{C}$-linear, $\tau(a\Sigma_1 + b\Sigma_2 + c\Sigma_3) = a\tau(\Sigma_1) + b\tau(\Sigma_2) + c\tau(\Sigma_3)$. The Σ_i are eigenvectors for τ with the appropriate eigenvalues. Now, $\tau(a\Sigma_1 + b\Sigma_2 + c\Sigma_3) = a\mu\Sigma_1 + c\Sigma_3$, which is obtained from the standard equilateral triangle Σ_1 by applying a linear similarity transformation (complex scalar multiplication by $a\mu$ and then translating by c). Thus, this is simply a translation of an equilateral triangle. ■

2 An Analogue of Napoleon's Theorem in the Hyperbolic Plane

Take S to be the space of triangles in the hyperbolic plane modulo orientation preserving isometries, and let T be a triangle in the hyperbolic plane. From T, construct the map $f\colon S \longrightarrow S$ as follows: Let d be any fixed real number. On each side of T, locate the point on the perpendicular bisector at distance d from the midpoint. Let the three points thus obtained be the vertices of $f(T)$. We wish to know what $\lim_{n\to\infty} f^n(T)$ looks like.

Notice that the question under consideration is not precisely the hyperbolic analogue of Napoleon's Theorem, because whereas we push out the same distance d from each side, in Napoleon's Theorem the distance depends on the length of the side in question.

We shall prove the following theorem:

https://doi.org/10.4153/CMB-2001-029-3 Published online by Cambridge University Press

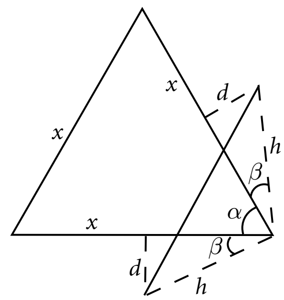

Figure 2: Proof of the Existence of a Fixed Point

Theorem 2.1 *For any $d > 0$ there is an equilateral T_d in S and a neighborhood $N_d(T_d)$ such that for any triangle T, $T \subseteq N_d$ implies $\lim_{n\to\infty} f(T) = T_d$.*

Calculations are greatly simplified if we use the substitution

$$\cosh d = \sqrt{\frac{8}{9 - A^2}},$$

where A is a variable which ranges between 1 and 3 in absolute value. Thus, the use of this substitution will be assumed.

Lemma 2.2 *The equilateral triangle T_d of side length $x = \cosh^{-1} \frac{2A}{3-A}$ is fixed under iteration by f.*

Proof Let α, β, h be as in the figure above. Then

$$\cos\alpha = \frac{2A}{A+3}$$

and

$$\cosh h = \frac{2}{3-A}.$$

So, using the law of cosines, $f(T_d)$ will have side length

$$\cosh^{-1}\left(\cosh^2 h - \sinh^2 h \cos(\alpha + 2\beta)\right),$$

or

$$\left(\frac{2}{3-A}\right)^2 - \left(\left(\frac{2}{3-A}\right)^2 - 1\right)\left(\frac{2A}{A+3}\cos 2\beta - \sin\alpha\sin 2\beta\right).$$

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Again via the law of cosines, we obtain

$$\cos 2\beta = \frac{-9 + 2A - A^2}{(-5 + A)(3 + A)}$$

and

$$\sin\alpha \sin 2\beta = \pm \frac{12(3 - A)(1 + A)}{(-5 + A)(3 + A)^2}.$$

Since finding $\sin\alpha \sin 2\beta$ involves taking a square root, we must stop at this point and establish which of the two possible values is appropriate. Notice, however, that

$$-\frac{12(3 - A)(1 + A)}{(-5 + A)(3 + A)^2}$$

is a strictly positive function, while

$$\frac{12(3 - A)(1 + A)}{(-5 + A)(3 + A)^2}$$

is a strictly negative function. Furthermore, the value of $\sin\alpha \sin 2\beta$ must be positive, since by construction, both α and 2β must be less than π. Thus, we conclude that

$$\sin\alpha \sin 2\beta = -\frac{12(3 - A)(1 + A)}{(-5 + A)(3 + A)^2} l$$

Substituting for $\cos 2\beta$ and $\sin\alpha \sin 2\beta$, we obtain an expression which simplifies to $\frac{2A}{3-A}$, the same side length we started with. Thus, f fixes T_d. ■

Lemma 2.3 *T_d is an attracting fixed point of f.*

Proof It suffices to show that the eigenvalues of the Jacobian are less than one in the norm. Given T_d, create the new triangle T_{d_ϵ}, with side lengths x, x, $x + \epsilon$, and let x_1, x_2, x_3 be the side lengths of $f(T_{d_\epsilon})$. Then if $a_1 = \lim_{\epsilon\to 0} \frac{x_1 - x}{\epsilon}$, $a_2 = \lim_{\epsilon\to 0} \frac{x_2 - x}{\epsilon}$, and $a_3 = \lim_{\epsilon\to 0} \frac{x_3 - x}{\epsilon}$, we need to show that the eigenvalues of

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

are all less than one in the norm. Notice that we actually only need to do two of the above calculations, since $x_2 = x_3$.

Since the calculation of the Jacobian involves a great deal of computation and very few ideas, we will not show the derivation here, but rather describe the method we used to obtain our results.

The calculation of a_1 and a_2 rest almost entirely on the observation that if x_l is sufficiently close to x, to find $\lim_{\epsilon\to 0} \frac{(x_l - x)}{\epsilon}$, we can simply find

$$\frac{1}{\sinh x} \lim_{\epsilon\to 0} \frac{(\cosh x_l - \cosh x)}{\epsilon}.$$

This is done using the laws for sine, cosine, cosh, sinh, and the appropriate trigonometric identities to reduce the original limit to summands of the form

$$k \lim_{\epsilon\to 0} \frac{\cosh(x+\epsilon) - \cosh x}{\epsilon},$$

or

$$k \lim_{\epsilon\to 0} \frac{\sinh(x+\epsilon) - \sinh x}{\epsilon},$$

where k is a constant. Using this procedure, we obtain

$$\lim_{\epsilon\to 0} \frac{(x_1 - x)}{\epsilon} = \frac{\sinh^2 y}{\sinh x}\left(\frac{\cos 2\beta}{\sinh x} + \frac{\sin 2\beta \cos\alpha}{\sinh x \sin\alpha}\right) = \frac{1}{3}$$

and

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{(x_2 - x)}{\epsilon}\\
&= \frac{\cosh^2 d}{4} - \cosh^2 d \cosh x \left(\frac{1}{2(\cosh x + 1)^2} + \frac{1}{4(\cosh x + 1)}\right)\\
&\quad + \frac{\sinh^2 d \cosh x}{(\cosh x + 1)^2(\cosh x - 1)} - \frac{\cosh^2 x \cosh d \sinh d}{(\cosh x + 1)^2\sqrt{2(2\cosh^2 x - \cosh x - 1)}}\\
&\quad + \frac{\sinh 2d}{2\cosh x}\left(\frac{1}{2}\sqrt{\frac{(\cosh x + 1)(2\cosh^3 x - 3\cosh^2 x + 1)}{2\sinh^4 x}}\right.\\
&\quad \left. + \sqrt{\frac{(\cosh x - 1)\sinh^4 x}{2(2\cosh^3 x - 3\cosh^2 x + 1)}}\,\frac{\cosh^2 x - \cosh x}{\sinh^2 x} \coth x \left(\frac{1}{\cosh^2 x} - \frac{\cosh x}{\sinh^2 x}\right)\right)\\
&= \frac{2 - 3A}{3(3 + A)}.
\end{aligned}$$

Thus,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix} = \begin{pmatrix} 1/3 & \frac{2-3A}{3(3+A)} & \frac{2-3A}{3(3+A)} \\ \frac{2-3A}{3(3+A)} & 1/3 & \frac{2-3A}{3(3+A)} \\ \frac{2-3A}{3(3+A)} & \frac{2-3A}{3(3+A)} & 1/3 \end{pmatrix}$$

In order to achieve our objective, namely to show that fixed points are attracting, we need to show that eigenvalues of the Jacobian are less than one in the norm. Since (given our substitution) $|A|$ can only take on values between 1 and 3, it follows that $\lim_{\epsilon\to 0} \frac{x_2 - x}{\epsilon}$ can only take on values between $\frac{1}{12}$ and $-\frac{1}{6}$, and hence that all eigenvalues are less than one in the norm. Thus, our fixed point is attracting. ■

Our theorem now follows directly from the lemmas above.

3 Concluding Remarks

In the above, we show that for every d, space of triangles under f has an attracting fixed point. Computational results suggest, however, that the function f is in fact a great deal stronger, *i.e.*, that f is in fact a contraction map which sends every initial triangle to the fixed point in the infinite limit. Computer results also suggest that a similar result holds for polygons.

References

[1] Svetlanna Katok, *Fuchsian Groups.* University of Chicago Press, Chicago, 1992
[2] Herbert Meschkowski, *Noneuclidean Geometry.* Academic Press, New York, 1964.

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