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A DE RHAM THEOREM FOR GENERALISED MANIFOLDS

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0. Introduction

In (2) Bruhat has developed a theory of differentiable functions and distributions on a locally compact group in order to apply it to the study of the irreducible representations of the p-adic groups. Later, Whyburn (8) defined differentiable forms on a locally compact group and proved an analog of the de Rham theorem concerning the relationship between the Čech cohomology and the De Rham cohomology. In (4) I have introduced the notions of "generalised manifold" (roughly speaking a projective limit of smooth manifolds) and of "differentiable forms" on it, extending some of the results due to Bruhat and Whyburn.

The aim of the present paper is to prove for generalised manifolds a de Rham type theorem and then to extend in this context a theorem due to Chevalley and Eilenberg concerning the computation of the de Rham cohomology with the aid of the G-invariant forms, G being a compact connected group acting on the given manifold. As an application I show that the Čech cohomology of a compact connected Lie group is isomorphic to the cohomology of its Lie algebra (the Lie algebra of a locally compact group has been defined in (6)). For sake of completeness I have included in the first part of the paper some of the results proved in (4).

1. Smooth maps on generalised manifolds

Let I be an ordered set directed to the right and $\{V_i, \pi_j^i\}_{i,j\in I}$ be a projective system of C^∞-manifolds and C^∞-maps (all manifolds are assumed to be paracompact and with a countable basis). We shall assume that all the maps $\pi_j^i: V_i \to V_j$, $i \geq j$, are proper submersions.

Definition 1.1. $V = \varprojlim V_i$ endowed with the usual topology will be called the *generalised manifold* associated to the projective system $\{V_i, \pi_j^i\}_{i,j\in I}$.

From now on we shall fix a projective system $\{V_i, \pi_j^i\}$ as above and let $V = \varprojlim V_i$ be the associated generalised manifold.

Remarks 1.2. (i) V is locally compact and the canonical projections $\pi_i: V \to V_i$ are proper and surjective (1, 4).

(ii) Let $i \in I$; then $V_i = \bigcup_{r=1}^\infty K_r$, K_r being compact subsets of V_i. According to the above remark $V = \bigcup_{r=1}^\infty \pi_i^{-1}(K_r)$ and $\pi_i^{-1}(K_r)$ are compact. Being also locally compact, it follows that V is paracompact, hence normal.

(iii) A basis for the topology of V is given by the family $\{\pi_i^{-1}(D_i);\ D_i$ open in V_i, $i \in I\}$.

Definition 1.3. Let $D \subset V$ be an open subset. A map $f: D \to R$ is called *smooth* if for every $x \in D$ there exists an open neighborhood $D_x \subset D$ of x and $g \in C^\infty(V_i)$ for some $i \in I$ such that $f \mid D_x = g \circ \pi_i \mid D_x$. The set of all such maps will be denoted $C^\infty(D)$.

Clearly $C^\infty(D)$ is an algebra over R; moreover, given a family $\{f_\alpha\}_{\alpha\in A}$ of smooth maps with $\{\operatorname{supp} f_\alpha\}_{\alpha\in A}$ locally finite, then $\Sigma_\alpha f_\alpha$ is smooth.

Lemma 1.4. *Let $K \subset D \subset V$, K compact and D open. Then there exists $f \in C^\infty(V)$ such that $f \mid K = 1$, $\operatorname{supp} f \subset D$ and $0 \leq f \leq 1$.*

Proof. For every $x \in K$ there exists $i(x) \in I$ and D_x open in $V_{i(x)}$ such that $x \in D'_x = \pi_{i(x)}^{-1}(D_x) \subset D$. Since V is locally compact there exists an open set D''_x containing x and such that $\overline{D''_x}$ is compact and contained in D'_x. Since $\pi_{i(x)}(\overline{D''_x}) \subset D_x$ is compact, there exists $g_x \in C^\infty(V_{i(x)})$ such that $g \circ \pi_{i(x)}(D''_x) = 1$, $\operatorname{supp} g_x \subset D_x$ and $0 \leq g_x \leq 1$. Let $f_x = g_x \circ \pi_{i(x)}$; then $f_x \in C^\infty(V)$, $f_x | D''_x = 1$, $\operatorname{supp} f_x \subset D'_x$ and $0 \leq f_x \leq 1$. K being compact there exist $x_1, \ldots, x_n \in K$ such that $K = \cup_{i=1}^n D'_{x_i}$. Let $f_i = f_{x_i}$; then $f = 1-(1-f_1)\ldots(1-f_n)$ has the required properties.

Proposition 1.5. *(Partition of unity). Let $\{D_\alpha\}_{\alpha\in A}$ be an open covering of V. Then there exist smooth maps f_α on V, $f_\alpha \geq 0$, such that*: (i) $\operatorname{supp} f_\alpha \subset D$; (ii) $\{\operatorname{supp} f_\alpha\}_{\alpha\in A}$ *is locally finite*; (iii) $\Sigma_\alpha f_\alpha = 1$.

The proof of this proposition follows from Lemma 1.4 by standard arguments and we shall omit it (see (4)).

If $D \subset D'$ are open subsets of V, by restriction we obtain an algebra homomorphism $C^\infty(D') \to C^\infty(D)$. It is easy to see that in this way the family $\{C^\infty(D)$, D open in $V\}$ together with the restriction maps becomes a sheaf of algebras over R. We shall denote it by $\mathcal{D}^0$. Let also $\mathcal{D}_i^0$ be the sheaf of C^∞-maps on V_i. For any $i, j \in I$, $i \leq j$, the composition with π_i (resp. π_i^j) induces a π_i (resp. π_i^j)-cohomomorphism $\varphi_i: \mathcal{D}_i^0 \to \mathcal{D}^0$ (resp. $\rho_j^i: \mathcal{D}_i^0 \to \mathcal{D}_j^0$). Clearly $\{\mathcal{D}_i^0, \rho_j^i\}_{i,j\in I}$ is an inductive system of sheaves and cohomomorphisms and the cohomomorphisms φ_i induce a sheaf homomorphism $\varphi: \varinjlim \mathcal{D}_i^0 \to \mathcal{D}^0$. It follows from our definitions that φ is in fact an isomorphism. In general $C^\infty(V)$ is not isomorphic to $\varinjlim C^\infty(V_i)$, but if we restrict ourselves to $C_c^\infty(V)$ and $C_c^\infty(V_i)$ (maps with compact support) we obtain an isomorphism $\varphi_c: \varinjlim C_c^\infty(V_i) \to C_c^\infty(V)$ (see (4)). To summarise we have

Proposition 1.6. (i) *$\varphi: \varinjlim \mathcal{D}_i^0 \to \mathcal{D}^0$ is an isomorphism of sheaves such that for any $i \leq j$ the diagram*

$$\begin{tikzcd} \mathcal{D}_i^0 \arrow[rr, "\rho_j^i"] \arrow[dr, "\varphi_i"'] & & \mathcal{D}_j^0 \arrow[dl, "\varphi_j"] \\ & \mathcal{D}^0 & \end{tikzcd}$$

commutes;

(ii) φ *induces an isomorphism* $\varphi_c : \varinjlim C_c^\infty(V_i) \to C_c^\infty(V)$.

We shall conclude this section with an approximation theorem which follows from Lemma 1.4 by standard arguments.

Theorem 1.7. *$C_c^\infty(V)$ is dense in $C(V)$, the space of all real continuous maps on V endowed with the compact open topology.*

2. Tangent vectors and forms, de Rham theorem

Let $x = (x_i) \in V$ and T_xV be the vector space of all linear maps $v : C^\infty(V) \to R$ such that $v(fg) = v(f)g(x) + f(x)v(g)$, for all $f, g \in C^\infty(V)$. T_xV is called the *tangent space* of V at x.

We denote by $\mathcal{F}_x$ the ideal of $C^\infty(V)$ consisting of all maps f such that $f(x) = 0$. As usual we have an isomorphism θ: $\mathrm{Hom}(\mathcal{F}_x/\mathcal{F}_x^2, R) \to T_xV$ defined by $(\theta(A))(f) = A((f - f(x))\hat{\ })$ for $A \in \mathrm{Hom}(\mathcal{F}_x/\mathcal{F}_x^2, R)$ and $f \in C^\infty(V)$, where $g\hat{\ }$ denotes the class of $g \in \mathcal{F}_x$ in $\mathcal{F}_x/\mathcal{F}_x^2$. Let also $\mathcal{F}_{x_i}$ be the ideal of $C^\infty(V_i)$ consisting of all maps f such that $f(x_i) = 0$. Clearly $\rho_j^i(\mathcal{F}_{x_i}) \subset \mathcal{F}_{x_j}$ for $i \leq j$ and thus we obtain an inductive system $\{\mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2, \alpha_j^i\}_{i,j \in I}$, α_j^i being induced by ρ_j^i.

Lemma 2.1. *$\mathcal{F}_x/\mathcal{F}_x^2$ and $\varinjlim \mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2$ are isomorphic vector spaces.*

Proof. It is easy to see that the assignment $f\hat{\ } \to (f \circ \pi_i)\hat{\ }$ gives rise to an injection $\alpha : \varinjlim \mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2 \to \mathcal{F}_x/\mathcal{F}_x^2$. It remains to check the surjectivity of the map so defined. Let $f \in \mathcal{F}_x$ and choose $\varphi \in C_c^\infty(V)$ with $\varphi(x) = 1$; then $f\varphi \in C_c^\infty(V)$ and $f - f\varphi = f(1 - \varphi) \in \mathcal{F}_x^2$, hence $f\hat{\ } = (f\varphi)\hat{\ }$. According to Proposition 1.6 there exists $i \in I$ and $f_i \in C_c^\infty(V_i)$ such that $f\varphi = f_i \circ \pi_i$ and $f_i(x_i) = 0$. It follows that $\alpha(\widehat{f}_i) = f\hat{\ }$ ("$\frown$" denotes the class in $\varinjlim \mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2$).

If we view $T_{x_i}(V_i)$ as the vector space of all R-valued derivations of $C^\infty(V_i)$ at x_i, we can define a linear map $T_x(\pi_i) : T_xV \to T_{x_i}V_i$ by $(T_x(\pi_i)(v))(f) = v(f \circ \pi_i)$, $f \in C^\infty(V_i)$, $v \in T_xV$. In this way we obtain a linear map $\psi : T_xV \to \varprojlim T_{x_i}(V_i)$.

Proposition 2.2. *$\psi : T_xV \to \varprojlim T_{x_i}V_i$ is an isomorphism.*

Proof. It suffices to observe that ψ is the composition of the following sequence of isomorphisms: $T_xV \simeq \mathrm{Hom}(\mathcal{F}_x/\mathcal{F}_x^2, R) \simeq \mathrm{Hom}(\varinjlim \mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2, R) \simeq \varprojlim \mathrm{Hom}(\mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2, R) \simeq \varprojlim T_{x_i}V_i$, where the second one is induced by α, the third one is canonical and the last one is induced by the isomorphisms $\mathcal{F}_{x_i}/\mathcal{F}_{x_i}^2 \simeq T_{x_i}V_i$.

Endowed with the usual topology $\varprojlim T_{x_i}V_i$ becomes a complete locally convex topological vector space. We consider on T_xV the unique topology such that ψ is an isomorphism of topological vector spaces. Then T_xV is also complete and locally convex.

In what follows we shall denote the space $\mathcal{F}_x/\mathcal{F}_x^2$ by $\mathcal{D}_x^1(V)$ and call it the space of 1-forms of V at x.

Let $i \in I$ and define $\Phi_i : \mathcal{D}_{x_i}^1(V_i) \to L(T_xV, R)$ as follows: $\Phi_i(\omega) = \omega \circ T_x(\pi_i)$, $\omega \in \mathcal{D}_{x_i}^1(V_i)$ being viewed as a linear map $T_{x_i}V_i \to R$. For $i \leq j$ the diagram

$$\begin{tikzcd} \mathcal{D}_{x_i}^1(V_i) \arrow[rr, "(\pi_i^j)_{x_j}^*"] \arrow[dr, "\Phi_i"'] & & \mathcal{D}_{x_j}^1(V_j) \arrow[dl, "\Phi_j"] \\ & L(T_xV, R) & \end{tikzcd}$$

commutes. Hence by Lemma 2.1 the maps Φ_i induce a linear map $\Phi : \mathcal{D}_x^1(V) \to L(T_xV, R)$.

Proposition 2.3. *$\Phi : \mathcal{D}_x^1(V) \to L(T_xV, R)$ is an isomorphism.*

Proof. Let $\omega \in \mathcal{D}_x^1(V)$ be such that $\Phi(\omega) = 0$; if ω is the image of $\omega_i \in \mathcal{D}_{x_i}^1(V_i)$ under the canonical injection $(\pi_i)_x^* : \mathcal{D}_{x_i}^1(V_i) \to \mathcal{D}_x^1(V)$, then $0 = \Phi(\omega) = \Phi_i(\omega_i) = \omega_i \circ T_x(\pi_i)$. Since $T_x(\pi_i)$ is surjective, $\omega_i = 0$, hence $\omega = 0$. This proves the injectivity of Φ. Let now $\theta \in L(T_xV, R)$; then $\theta^{-1}(-1, 1)$ is a neighborhood of $0 \in T_xV$, hence there exist $i \in I$ and an open neighborhood D of $0 \in T_{x_i}V_i$ such that $(T_x(\pi_i))^{-1}(D) \subset \theta^{-1}(-1, 1)$. Let $w \in T_xV$ be such that $T_x(\pi_i)(w) = 0$; then any multiple of w lies in $\theta^{-1}(-1, 1)$ and thus $\theta(w) = 0$. This remark enables us to define $\omega : T_{x_i}V_i \to R$ by $\omega(v) = \theta(w)$, where $w \in T_xV$ is chosen such that $T_x(\pi_i)(w) = v$. There are no problems in proving that ω is linear (hence $\omega \in \mathcal{D}_{x_i}^1(V_i)$) and that $\Phi((\pi_i)_x^*(\omega)) = \theta$. It follows that Φ is also surjective.

Let $D \subset V$ be an open set, $f \in C^\infty(D)$ and $x \in D$. We shall denote by $df(x)$ the 1-form of V at x given by $df(x) = (f - f(x))^\wedge \in \mathcal{F}_x/\mathcal{F}_x^2 = \mathcal{D}_x^1(V)$.

Definition 2.5. A map $\omega : D \to \bigcup_{x \in V} \mathcal{D}_x^1(V)$ is called a 1-form on D if for every $x \in D$ there exist a neighborhood U of x in D and $f_1, \ldots, f_n, g_1, \ldots, g_n \in C^\infty(D)$ such that $\omega(y) = \sum_{i=1}^n g_i(y) df_i(y)$ for any $y \in U$. The set of all 1-forms on D will be denoted $\mathcal{D}^1(D)$.

For any $i \in I$ and any $\omega \in \mathcal{D}^1(V_i)$ there exists a 1-form $(\pi_i)^*(\omega) \in \mathcal{D}^1(V)$ given by $((\pi_i)^*(\omega))(x) = (\pi_i)_x^*(\omega(x_i))$. Using Proposition 1.6 it is not difficult to prove the following lemma.

Lemma 2.6. *A map $\omega : D \to \bigcup_{x \in D} \mathcal{D}_x^1(D)$ is a 1-form on D if and only if it coincides locally with a 1-form of the form $(\pi_i)^*(\tilde{\omega})$ for some $i \in I$ and $\tilde{\omega} \in \mathcal{D}^1(V_i)$. Moreover if it has compact support it coincides everywhere with such a 1-form.*

It is clear that the family $\{\mathcal{D}^1(D);\ D \text{ open in } V\}$ together with the obvious restriction maps $\mathcal{D}^1(D) \to \mathcal{D}^1(D')$ if $D' \subset D$ is a sheaf; more precisely a $\mathcal{D}^0$-module. Now, for any $i \in I$ let $\mathcal{D}_i^1$ denote the sheaf of 1-forms on V_i and let $(\pi_i)^*$ be the obvious π_i-cohomomorphism of $\mathcal{D}_i^1$ into $\mathcal{D}^1$.

Definition 2.7. For any $p > 1$ let $\mathcal{D}^p = \wedge^p \mathcal{D}^1$ (as $\mathcal{D}^0$-modules). A section of $\mathcal{D}^p$ will be called a *p-form.*

Let $\mathcal{D}_i^p$ be the sheaf of p-forms on V_i, $i \in I$. Since $\mathcal{D}_i^p = \wedge^p \mathcal{D}_i^1$, the π_i-cohomomorphisms $(\pi_i)^*: \mathcal{D}_i^1 \to \mathcal{D}^1$ induce π_i-cohomomorphisms of $\mathcal{D}_i^p$ into $\mathcal{D}^p$, denoted again by $(\pi_i)^*$. For $i \geq j$ the following diagram commutes

$$\begin{tikzcd} \mathcal{D}_j^p \arrow[rr, "(\pi_j^i)^*"] \arrow[dr, "(\pi_j)^*"'] & & \mathcal{D}_i^p \arrow[dl, "(\pi_i)^*"] \\ & \mathcal{D}^p & \end{tikzcd}$$

hence we obtain a sheaf homomorphism $\psi^p : \varinjlim \mathcal{D}_i^p \to \mathcal{D}^p$.

Proposition 2.8. *For any $p > 0$, ψ^p is an isomorphism.*

Proof. For $p = 1$ this follows immediately from Lemma 2.6. For $p > 1$ the assertion follows by standard arguments.

Remark 2.9. As in the case of the smooth maps, there exists a canonical isomorphism $\psi_c^p : \varinjlim \mathcal{D}_c^p(V_i) \to \mathcal{D}_c^p(V)$ induced by ψ^p (here $\mathcal{D}_c^p(V_i)$ and $\mathcal{D}_c^p(V)$ stand for the p-forms on V_i and V respectively with compact support).

Let $p \geq 1$, $D \subset V$ open, $f_0, f_1, \ldots, f_p \in \mathcal{D}^0(D)$ and $\omega = f_0 df_1 \wedge \ldots \wedge df_p \in \mathcal{D}^p(D)$. Denote $d^p(\omega) = df_0 \wedge df_1 \wedge \ldots \wedge df_p \in \mathcal{D}^{p+1}(D)$. It is not hard to see that d^p so defined induces a sheaf homomorphism $d^p: \mathcal{D}^p \to \mathcal{D}^{p+1}$ with the following properties:

(ii) $d^0 = d$ as defined before;

(ii) $d^{p+1} \circ d^p = 0$;

(iii) for every $i \in I$ the following diagram commutes

$$\begin{tikzcd} \mathcal{D}_i^p \arrow[r, "d_i^p"] \arrow[d, "(\pi_i)^*"'] & \mathcal{D}_i^{p+1} \arrow[d, "(\pi_i)^*"] \\ \mathcal{D}^p \arrow[r, "d^p"] & \mathcal{D}^{p+1} \end{tikzcd}$$

where d_i^p is the usual exterior differential;

(iv) the above property together with Proposition 2.8 imply that in fact d^p corresponds via the isomorphisms ψ^p and ψ^{p+1} to $\varinjlim d_i^p$. The "inductive limit" being an exact functor, it follows immediately that $0 \to \mathcal{R} \to \mathcal{D}^0 \to \mathcal{D}^1 \to \mathcal{D}^2 \to \ldots$ is a resolution of $\mathcal{R}$, the trivial sheaf over V with fibre R.

Now let $Z^r(V) = \{\omega \in \mathcal{D}^r(V) \mid d^r(\omega) = 0\}$ and $B^r(V) = d^{r-1}(\mathcal{D}^{r-1}(V))$; then by (ii) $B^r(V) \subset Z^r(V)$ and thus we can consider $H^r_{DR}(V) = Z^r(V)/B^r(V)$. Also let $H^*(V) = \bigoplus_{r=0}^{\infty} H^r(V)$ denote the Čech cohomology of V with value in R.

Theorem 2.10. *$H^*_{DR}(V) = \bigoplus_{r=0}^{\infty} H^r_{DR}(V)$ is isomorphic to $H^*(V)$.*

Proof. By Proposition 1.5, $\mathcal{D}^0$ is a fine sheaf. Since for any $r \geq 0$ $\mathcal{D}^r$ is a $\mathcal{D}^0$-module, it follows that $\mathcal{D}^r$ is soft. Combining this remark with (iv) above, the assertion of the theorem follows from a well known result in the theory of sheaves (see (**5**)).

Remark 2.11. One can define in an obvious way a multiplication on the graded vector space $\mathcal{D}^* = \bigoplus_{r=0}^{\infty} \mathcal{D}^r(V)$ such that $\mathcal{D}^*$ becomes a graded differential algebra. Then $H^*_{DR}(V)$ inherits a structure of graded algebra and the isomorphism in Theorem 2.10 is an isomorphism of graded algebras.

Remark 2.12. We shall give here an equivalent definition for the sheaf of p-forms on V. First let $\mathcal{D}^p_x(V) = \wedge^p \mathcal{D}^1_x(V)$. As in the case $p = 1$ we have an inductive system $\{\mathcal{D}^p_{x_i}(V_i),\ (\pi^i_j)^*_{x_i}\}_{i,j \in I}$ and canonical linear maps $(\pi_i)^*_x : \mathcal{D}^p_{x_i}(V_i) = \wedge^p \mathcal{D}^1_{x_i}(V_i) \to \mathcal{D}^p_x(V)$. From Lemma 2.1 we obtain that the maps $(\pi_i)^*$, $i \in I$, induce an isomorphism of $\varinjlim \mathcal{D}^p_{x_i}(V_i)$ onto $\mathcal{D}^p_x(V)$. Now it is easy to check that for any open subset $D \subset V$, any $\omega \in \mathcal{D}^p(D)$ can be viewed as a map $\tilde{\omega} : D \to \bigcup_{x \in D} \mathcal{D}^p_x(V)$ such that for any $x \in D$ there exist an open neighborhood U of x, $i \in I$ and $\omega_i \in \mathcal{D}^p(V_i)$ verifying $\tilde{\omega}(y) = (\pi_i)^*_y(\omega_i(y_i))$ for any $y \in U \cap D$. Conversely any such a map $\tilde{\omega}$ corresponds to a unique p-form $\omega \in \mathcal{D}^p(D)$.

Remark 2.13. Let $\mathrm{Alt}^p(T_xV, R)$ denote the vector space of all continuous skew symmetric multilinear maps of $T_xV \times \ldots \times T_xV$ (p times) into R. Since $\mathcal{D}^p_{x_i}(V_i)$ is canonically isomorphic to $\mathrm{Alt}^p(T_{x_i}V_i, R)$. The projections $T_x(\pi_i) : T_xV \to T_{x_i}V_i$ induce maps $\Phi^p_i : \mathcal{D}^p_{x_i}(V_i) \to \mathrm{Alt}^p(T_xV, R)$ which give rise to a linear map $\Phi^p : \varinjlim \mathcal{D}^p_{x_i}(V_i) \to \mathrm{Alt}^p(T_xV, R)$. Exactly as in Proposition 2.3 we can prove that Φ^p is an isomorphism of vector spaces. Thus $\mathcal{D}^p_x(V)$ and $\mathrm{Alt}^p(T_xV, R)$ are isomorphic vector spaces.

3. Invariant de Rham cohomology

Let G be a compact connected topological group. We shall assume that there exist left continuous actions through diffeomorphisms $G \times V_i \to V_i$, $(g, x) \mapsto gx_i$, such that the diagrams

$$\begin{array}{ccc} G \times V_i & \longrightarrow & V_i \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi^i_j} & & \Big\downarrow{\scriptstyle \pi^i_j} \\ G \times V_j & \longrightarrow & V_j \end{array}$$

commute for any $i \geq j$. Then $G \times V \to V$, $(g, (x_i)) \mapsto (gx_i)$ determines a continuous left action of G on V. For any $g \in G$ the homeomorphism (resp. diffeomorphism) $x \mapsto gx$ of V onto V (resp. V_i onto V_i) will be denoted by g.

Given $f \in \mathcal{D}^0(V)$ it is easy to see that $f^g = f \circ g$ is also smooth. Thus $(g, f) \mapsto f^g$ is a linear action of G on $\mathcal{D}^0(V)$; it extends to unique linear actions $(g, \omega) \mapsto \omega^g$ of G on $\mathcal{D}^p(V)$, $p = 1, 2, \ldots$, with the properties $d^p(\omega^g) = (d^p(\omega))^g$ and $\omega_1^g \wedge \omega_2^g = (\omega_1 \wedge \omega_2)^g$. With obvious notation we have $((\pi_i)^*(\omega))^g = (\pi_i)^*(\omega^g)$, $\omega \in \mathcal{D}^p(V_i)$.

Let $\mathcal{D}^p_G(V) = \{\omega \in \mathcal{D}^p(V) \mid \omega^g = \omega, g \in G\}$. Clearly $d^p(\mathcal{D}^p_G(V)) \subset \mathcal{D}^{p+1}_G(V)$ and hence we can consider $Z^p_G(V) = Z^p(V) \cap D^p_G(V)$, $B^p_G(V) = d^{p-1}(\mathcal{D}^{p-1}_G(V))$ and $H^p_G(V) = Z^p_G(V)/B^p_G(V)$. The inclusions $Z^p_G(V) \subset Z^p(V)$ induce a linear map $I: H^*_G(V) = \bigoplus_{p=0}^{\infty} H^p_G(V) \to H^*_{DR}(V)$.

Theorem 3.1. *$I: H^*_G(V) \to H^*_{DR}(V)$ is an isomorphism of graded vector spaces (and even of graded algebras).*

Proof. It is known (3) that for any $i \in I$ there exist linear maps $m^p_i: \mathcal{D}^p(V_i) \to \mathcal{D}^p_G(V_i)$ with the following properties:

(1) $m^p_i(\omega) = \omega$, if $\omega \in \mathcal{D}^p_G(V_i)$;

(2) $m^{p+1}_i \circ d^p_i = d^p_i \circ m^p_i$;

(3) if $\operatorname{supp} \omega \subset U$, U open in V_i, then $\operatorname{supp} m^p_i(\omega) \subset GU = \{gx \mid g \in G, x \in U\}$;

(4) if $\operatorname{supp} \omega \subset U$, U open in V_i, there exists $\omega' \in \mathcal{D}^{p-1}(V_i)$ with $\operatorname{supp} \omega' \subset GU$, such that $\omega - m^p_i(\omega) = d^{p-1}(\omega')$ (in (3) it is not proved that $\operatorname{supp} \omega' \subset GU$, but one can see this is indeed so);

(5) for any $i \geq j$, $(\pi^i_j)^* \circ m^p_j = m^p_i \circ (\pi^i_j)^*$.

The last property implies the existence of unique maps $m^p: \mathcal{D}^p_c(V) \to \mathcal{D}^p_G(V)$ such that

$$m^p((\pi_i)^*(\omega)) = (\pi_i)^*(m^p_i(\omega)), \quad \omega \in \mathcal{D}^p_c(V_i).$$

According to Remark 1.2. (ii) we can construct a sequence $\emptyset = D_0 \subset D_1 \subset D_2 \subset \ldots$ of open and relatively compact subsets of V such that $D_n \subset D_{n+1}$ and $D_0 \cup D_1 \cup D_2 \cup \ldots = V$. Let us denote $K_n = G\bar{D}_n$ and $E_n = \operatorname{int}(K_n)$. Then $\emptyset = E_0 \subset E_1 \subset E_2 \subset \ldots$, every E_n is open and relatively compact, $\bar{E}_n \subset E_{n+1}$, $GE_n = E_n$ and $E_0 \cup E_1 \cup E_2 \cup \ldots = V$. Finally let $U_n = E_{n+1} \setminus E_{n-1}$, $n \in N$. Clearly $\{U_n\}_{n \in N}$ is a locally finite covering of V with open, relatively compact and G-invariant subsets. Let $\{\psi'_n\}_{n \in N}$ be a partition of the unity with smooth maps subordinate to $\{U_n\}_{n \in N}$ (see Proposition 1.5). Then denoting $\psi_n = m^0(\psi'_n)$ we obtain another partition of the unity with smooth maps subordinate to the same covering.

Given $\omega \in \mathcal{D}^p(V)$ let $\omega_n = \psi_n \omega$. Then $\operatorname{supp}(\omega_n) \subset U_n$ and thus $\Sigma_n \omega_n$ makes sense and is equal to ω. By (3), $\operatorname{supp} m^p(\omega_n) \subset U_n$ and again $\Sigma_n m^p(\omega_n)$ makes sense. We shall define $m^p(\omega) = \Sigma_n m^p(\omega_n)$. An easy computation shows that m^p verifies the properties (1) – (4) with V_i replaced by V.

Let now prove the injectivity of I. Assume $\omega \in Z^p_G(V)$ and $\omega = d^{p-1}(\omega')$ for some $\omega' \in \mathcal{D}^{p-1}(V)$. Then $\omega = m^p(\omega) = m^p(d^{p-1}(\omega')) = d^{p-1}(m^{p-1}(\omega'))$ and $m^{p-1}(\omega') \in \mathcal{D}^{p-1}_G(V)$. It follows that $\omega \in B^p_G(V)$, hence I is injective. Since the surjectivity of I is an immediate consequence of (4), the theorem is completely proved.

4. Cohomology of homogeneous spaces

Let G be a locally compact group such that G/G_0 is finite (G_0 denotes the identity component of G) and let H be a closed topological subgroup of G. It is known (7) that there exists a family $\{K_i\}_{i\in I}$ of compact subgroups of G verifying:

(i) if $i,j \in I$ there exists $k \in I$ such that $K_i \cap K_j \supset K_k$; thus if we put $i \leqslant j$ if $K_i \supset K_j$, I becomes an ordered set directed to the right;

(ii) $G_i = G/K_i$ is a Lie group for any $i \in I$;

(iii) the canonical map $G \to \varprojlim G_i$ is an isomorphism of topological groups.

Let $L_i = H \cap K_i$ and $H_i = H/L_i$; according to (1, chap. III, §7, Proposition 3) the canonical map $G/H \to \varprojlim G_i/H_i$ is a homeomorphism. Thus we can view G/H as a generalised manifold (it should be verified that the maps $\pi^i_j: G_i/H_i \to G_j/H_j$ induced by the canonical projections $G_i \to G_j$, $i \geqslant j$, are proper submersions, but this is obvious since the K_i's are compact). Our results in Sections 1 and 2 can be applied and thus we obtain, in the case $H = \{e\}$, the results from (8) and some of the results from (2).

Assume now that G is connected and consider the canonical action of G on G/H. Let $\omega \in \mathcal{D}^p_G(G/H)$; we can view ω as a map $\tilde{\omega}: G/H \to \cup_{x\in G/H}\mathcal{D}^p_x(G/H)$. Being G-invariant ω is determined by $\tilde{\omega}(\epsilon) \in \mathcal{D}^p_\epsilon(G/H)$, where $\epsilon \in G/H$ is the class of the identity element $e \in G$. Since ϵ is left fixed by H, we have also an action of H on $\mathcal{D}^p_\epsilon(G/H)$; let $\mathcal{D}^p_{\epsilon,H}(G/H)$ be the set of all p-forms at ϵ fixed by H. Clearly $\tilde{\omega}(\epsilon) \in \mathcal{D}^p_{\epsilon,H}(G/H)$. Conversely, every element $\omega_\epsilon \in \mathcal{D}^p_{\epsilon,H}(G/H)$ determines a unique G-invariant form $\omega \in \mathcal{D}^p_G(G/H)$ such that $\tilde{\omega}(\epsilon) = \omega_\epsilon$. Indeed, the isomorphism $\mathcal{D}^p_\epsilon(G/H) \simeq \varinjlim \mathcal{D}^p_{\epsilon_i}(G_i/H_i)$ gives rise to an isomorphism $\mathcal{D}^p_{\epsilon,H}(G/H) \simeq \varinjlim \mathcal{D}^p_{\epsilon_i,H_i}(G_i/H_i)$, hence there exist $i \in I$, $\omega_{\epsilon_i} \in \mathcal{D}^p_{\epsilon_i,H_i}(G_i/H_i)$ such that ω_ϵ corresponds (through the above isomorphism) to the class of ω_{ϵ_i} in $\varinjlim \mathcal{D}^p_{\epsilon_i,H_i}(G_i/H_i)$. Now extend ω_{ϵ_i} to a G_i-invariant smooth form ω_i on G_i/H_i and let ω be its pull-back to G/H. Clearly $\omega \in \mathcal{D}^p_G(G/H)$ and $\tilde{\omega}(\epsilon) = \omega_\epsilon$. Thus $\omega \mapsto \omega(\epsilon)$ determines an isomorphism of $\mathcal{D}^p_G(G/H)$ onto $\mathcal{D}^p_{\epsilon,H}(G/H)$.

Let now $\mathfrak{g} = T_eG$ be the tangent space of G at e and $\mathfrak{h} = T_eH$. Since $G = \varprojlim G_i$ and $H = \varprojlim H_i$, it follows that $\mathfrak{g} = \varprojlim \mathfrak{g}_i$ and $\mathfrak{h} = \varprojlim \mathfrak{h}_i$, where $\mathfrak{g}_i$ (resp. $\mathfrak{h}_i$) is the Lie algebra of G_i (resp. H_i). Using the fact that the canonical projections $\mathfrak{g}_i \to \mathfrak{g}_j$ are (continuous) Lie homomorphisms, it follows that $\mathfrak{g}$ is topological Lie algebra and $\mathfrak{h}$ is a closed Lie subalgebra of $\mathfrak{g}$. The complex of vector spaces $C^*(\mathfrak{g}, \mathfrak{h})$ of the continuous real cochains on $\mathfrak{g}$ relative to $\mathfrak{h}$ can be defined as usual (see for example (3); the only difference consists in that we consider continuous cochains). We shall denote its cohomology by $H^*(\mathfrak{g}, \mathfrak{h})$.

Identifying $\mathcal{D}^p(G/H)$ to $\mathrm{Alt}^p(T_\epsilon(G/H), R)$ (cf. Remark 2.13) and observing that $T_\epsilon(G/H)$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$, we obtain an isomorphism of $\mathcal{D}^p_\epsilon(G/H)$ onto $\mathrm{Alt}^p(\mathfrak{g}/\mathfrak{h}, R)$ which sends $\mathcal{D}^p_{\epsilon,H}(G/H)$ onto $C^p(\mathfrak{g}, \mathfrak{h}) \subset \mathrm{Alt}^p(\mathfrak{g}/\mathfrak{h}, R)$ (the easiest way to check this is to consider the canonical isomorphisms $\mathcal{D}^p_{\epsilon,H}(G/H) \simeq \varinjlim \mathcal{D}^p_{\epsilon_i,H_i}(G_i/H_i)$, $\mathrm{Alt}^p(\mathfrak{g}/\mathfrak{h}, R) \simeq \varinjlim \mathrm{Alt}^p(\mathfrak{g}_i/\mathfrak{h}_i, R)$, $C^p(\mathfrak{g}, \mathfrak{h}) \simeq \varinjlim C_p(\mathfrak{g}_i, \mathfrak{h}_i)$ and $\mathcal{D}^p_{\epsilon_i,H_i}(G_i/H_i) \simeq C^p(\mathfrak{g}_i, \mathfrak{h}_i)$).

Finally, combining all the above remarks we obtain an isomorphism $\alpha: \mathcal{D}^*_G(G/H) \to C^*(\mathfrak{g}, \mathfrak{h})$ and one can see without difficulty that it commutes with the differentials.

Theorem 4.1. *Let G be a compact and connected topological group and H be a closed subgroup of G. Then the real Čech cohomology of G/H is isomorphic to $H^*(\mathfrak{g}, \mathfrak{h})$.*

Proof. The assertion follows from Theorems 2.11 and 3.1 and the above remark.

Corollary 4.2. *If G is a compact and connected topological group, then its Čech cohomology is isomorphic to $H^*(\mathfrak{g})$, the cohomology of its Lie algebra $\mathfrak{g}$.*

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