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CYCLOTOMIC FACTORS OF BORWEIN POLYNOMIALS

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Abstract

A cyclotomic polynomial $\Phi_k(x)$ is an essential cyclotomic factor of $f(x) \in \mathbb{Z}[x]$ if $\Phi_k(x) \mid f(x)$ and every prime divisor of k is less than or equal to the number of terms of f. We show that if a monic polynomial with coefficients from $\{-1, 0, 1\}$ has a cyclotomic factor, then it has an essential cyclotomic factor. We use this result to prove a conjecture posed by Mercer ['Newman polynomials, reducibility, and roots on the unit circle', *Integers* **12**(4) (2012), 503–519].

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1. Introduction

Questions about the reducibility and irreducibility of polynomials with coefficients $0, \pm 1$ continue to attract attention (see [5, 7, 9, 10, 12, 13]). Selmer [12] gave a complete solution to the reducibility of $x^n \pm x \pm 1$. Ljunggren [7] extended Selmer's work to the irreducibility of $x^n \pm x^m \pm x^r \pm 1$. A polynomial $f(x)$ of degree n is said to be *reciprocal* if $f(x) = \pm x^n f(x^{-1})$. Let ζ_n be a *primitive* nth root of unity. The nth *cyclotomic polynomial*

$$\Phi_n(x) = \prod_{1 \le i \le n, (i,n)=1} (x - \zeta_n^i)$$

is a reciprocal polynomial with integer coefficients. Both Selmer and Ljunggren focused on finding the number of nonreciprocal factors when the special trinomials and quadrinomials shown above are reducible and proved that polynomials of the form $x^n + \epsilon_1 x^m + \epsilon_2 x^r \pm 1$, where $\epsilon_i \in \{-1, 0, 1\}$, have a cyclotomic factor whenever they are reducible. It is natural to ask whether this extends to polynomials with five or more terms. The following examples from Filaseta and Solan [4] suggest that such polynomials may not always have reciprocal factors:

$$\begin{aligned}
x^{14}+x^{12}+x^{10}+x^2-1 &= (x^6+x^2-1)(x^8+x^6+1),\\
x^{14}+x^{12}-x^4-x^2+1 &= (x^6+x^4-1)(x^8+x^2-1),\\
x^{14}-x^{12}+x^4-x^2-1 &= (x^6-x^4+1)(x^8-x^2-1),\\
x^{14}-x^{12}-x^4-x^2-1 &= (x^6-x^4-1)(x^8+x^2+1),\\
x^{2n-2}+x^{n+2}+x^{n-2}-x^4+1 &= (x^n-x^2+1)(x^{n-2}+x^2+1),
\end{aligned}$$

where $n \geq 5$ and $n \not\equiv 0, 3, 4, 6, 9 \pmod{12}$. Note that, in all these examples, at least one of the coefficients is negative. Filaseta and Solan proved the following result.

THEOREM 1.1 (Filaseta and Solan [4]). *If $f(x) = x^n + x^m + x^r + x^s + 1$ is reducible and $n > m > r > s > 0$, then f has a reciprocal factor together with at most one irreducible nonreciprocal factor.*

Mercer [8] made the following conjecture.

CONJECTURE 1.2 [8, Conjecture 6, page 5]. *If $f(x) = x^n + x^m + x^r + x^s + 1$ is reducible and $n > m > r > s > 0$, then f has a cyclotomic factor.*

Let $S^* = \{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \{-1, 0, 1\}\}$. A polynomial $f \in S^*$ is called a *Borwein* polynomial if $f(0) \neq 0$ and a *Newman* polynomial if its coefficients are either 0 or 1. In other words, a Newman polynomial $f(x)$ has the form

$$f(x) = x^m(x^{n_0} + x^{n_1} + \cdots + x^{n_{r-1}} + x^{n_r} + \cdots + x^{n_v} + 1), \tag{1.1}$$

where $m \geq 0$ and $n_0 > n_1 > \cdots > n_v > 0$. A polynomial $f \in S$ is called a *Littlewood* polynomial if its coefficients are either -1 or $+1$. The set of Borwein polynomials dividing a Newman polynomial or a Littlewood polynomial has been studied in [3]. We denote the set of Borwein polynomials by S in the rest of the paper.

Let $f(x) = x^{n_0} + \epsilon_1 x^{n_1} + \cdots + \epsilon_v x^{n_v} + \epsilon_{v+1} \in S$, where $\epsilon_i \in \{-1, 1\}$. The number of terms in f is called the length of the polynomial and is denoted by $\ell(f)$. A polynomial $f(x) = x^{n_0} + a_1 x^{n_1} + \cdots + a_v x^{n_v} + a_{v+1} \in \mathbb{Z}[x]$, where $a_i \neq 0$, is called *primitive* if $\gcd(n_0, n_1, n_2, \ldots, n_v) = 1$.

Mercer refined Conjecture 1.2 and proposed the following conjecture.

CONJECTURE 1.3. *Suppose $f(x)$ is a Newman polynomial of length five with a cyclotomic factor. Then $f(x)$ is divisible by either $\Phi_{5^\gamma}(x)$ or $\Phi_{2^\alpha 3^\beta}(x)$ for some $\alpha, \beta, \gamma \geq 1$.*

We prove Conjecture 1.3 at the end of this paper (see Theorem 3.2).

If $f(x)$ is not primitive, then $f(x) = g(x^d)$ for some $d \geq 2$, where $g(x)$ is a primitive polynomial. Let $(n, d) = t < n$. Then $(n/t, d/t) = 1$ and $\zeta_n^d = \zeta_{n/t}^{d/t}$ so that ζ_n^d is a primitive (n/t)th root of unity. From $f(\zeta_n) = g(\zeta_n^d) = 0$, we deduce that $\Phi_{n/t}(x) \mid g(x)$. Hence $\Phi_{n/t}(x^d) \mid f(x)$. Now, $\Phi_{n/t}(x^d) = \prod_{u \in D} \Phi_u(x)$, where $D = \{u \mid [u, d] = [n, d]\}$, and $[a, b]$ denotes the *least common multiple* of a and b. Hence $\prod_{u \in D} \Phi_u(x)$ divides $f(x)$.

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In particular, for every cyclotomic factor $\Phi_k(x)$ of $g(x)$, there exists a cyclotomic factor $\Phi_m(x)$ of $f(x)$ such that m and k have the same set of prime divisors.

If a Newman polynomial $f(x)$ is divisible by $\Phi_n(x)$, then $f(\zeta_n) = 0$, giving a *vanishing sum of roots of unity*. The study of minimal vanishing sums of roots of unity is of independent interest (see [6, 11]). Lam and Leung [6] proved that if $\Phi_n(x)$ divides $f(x)$, then $\ell(f) \in P(n)$, where $P(n)$ is the set of all nonnegative integer linear combinations of prime divisors of n: that is, if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then $P(n) = \{\sum_{i=1}^{r} n_i p_i \mid n_i \in \{0, 1, 2, \ldots\}\}$. As a consequence, the smallest prime factor of n is less than or equal to $\ell(f)$. It is natural to ask whether every prime divisor of n is less than $\ell(f)$, but this is not the case. If p is an odd prime, then $\Phi_{2p} \mid x^p + 1$. Similarly, if p is a prime greater than three, then $\Phi_{3p} \mid x^{2p} + x^p + 1$. Note that $x^p + 1$ and $x^{2p} + x^p + 1$ are not primitive. By definition, every Newman polynomial of length two except $x + 1$ is not primitive. From [7], every cyclotomic factor of $x^n + x^m + 1$ is of the form $\Phi_3(x^{(n,m)})$. Hence, it is not possible to find a primitive Newman polynomial with $\ell(f) \le 3$ divisible by $\Phi_k(x)$ with one prime divisor of k greater than $\ell(f)$.

2. Main results

The following result shows that, for $\ell(f) \ge 4$, one can find such a family of primitive Newman polynomials.

THEOREM 2.1. *For every positive integer $n > 3$ there exists a positive integer k with a prime factor greater than n such that $\Phi_k(x) \mid f_n(x)$, where $f_n(x)$ is a primitive Newman polynomial of length n.*

PROOF. If $6 \mid a$, then it is easy to see that $P(a) = \mathbb{N} \setminus \{1\}$, that is, every natural number $m \ge 2$ can be expressed as $m = 2t_1 + 3t_2$, where $t_1, t_2 \in \mathbb{N} \cup \{0\}$. Let $n, k \in \mathbb{N}$, where $n \ge 4$ and k has a prime factor greater than n. Then $n = 2r + 3s$ for some $r, s \in \mathbb{N} \cup \{0\}$. Also, $f_n(x) = \sum_{i=1}^{r} x^{n_i}\Phi_2(x^{3k}) + \sum_{j=1}^{s} x^{n_r+3k+j}\Phi_3(x^{2k})$, where $0 = n_1 < n_2 < \cdots < n_r$ and $\gcd(n_1, n_2, \ldots, n_r) = 1$, is the desired primitive Newman polynomial of length n and it is divisible by $\Phi_{6k}(x)$. $\square$

Now we show that the above result can be extended to monic polynomials with coefficients in $\{-1, 0, 1\}$. Suppose that $n \ge 2$, $n = 2t_1 + 3t_2$, where t_1, t_2 are nonnegative integers and k has a prime factor greater than n. One of the desired polynomials is

$$f_n(x) = \epsilon_1\Phi_2(x^{3k}) + \sum_{i=2}^{t_1} \epsilon_i x^{n_i}\Phi_2(x^{3k}) + \sum_{j=1}^{t_2-1} \delta_j x^{n_{t_1}+3k+j}\Phi_3(x^{2k}) + x^{n_{t_1}+3k+t_2}\Phi_3(x^{2k}),$$

where $0 < n_2 < \cdots < n_{t_1}$, $\gcd(n_1, n_2, \ldots, n_{t_1}) = 1$ and $\epsilon_i, \delta_j \in \{-1, 1\}$. Then f_n is a monic Borwein polynomial of length n. It can be seen that $f_n(x)$ is divisible by $\Phi_{6k}(x)$.

Next we investigate the following question. If a Borwein polynomial $f(x)$ is divisible by a cyclotomic polynomial $\Phi_k(x)$ and the smallest prime factor of k is p, is $p \le \ell(f)$? The polynomial $x^p - 1$, where p is an odd prime, is a counterexample. However, we can obtain a partial result.

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Lemma 2.2. *Suppose $f \in S$ and $\Phi_p(x)$ divides $f(x)$, where $p > \ell(f)$ is a prime. Then $f(1) = 0$.*

Proof. If $f(x)$ is not primitive, then $f(x) = h(x^d)$ for some $d > 1$, where $h \in S$ and h is primitive. If $d \equiv 0 \pmod p$, then $f(\zeta_p) = h(\zeta_p^d) = 0$, which gives $h(1) = 0$ and hence $f(1) = 0$. If $d \not\equiv 0 \pmod p$, then ζ_p^d is also a primitive pth root of unity and hence $\Phi_p(x) \mid h(x)$.

Hence, without loss of generality, we can assume that $f(x)$ is primitive. Since $\zeta_p^p = 1$, by reducing the exponents modulo p in the identity $f(\zeta_p) = 0$ we get a new identity $\bar{f}(\zeta_p) = 0$.

The number of terms in $\bar{f}(\zeta_p)$ is less than p. This is possible only when $f(\zeta_p)$ is a linear combination of terms of the form $\zeta_p^\delta(\zeta_p^{pt} - 1)$ for some δ with $0 \le \delta \le p$ and $t \ge 0$. From the primitivity of f,

$$f(x) = \sum_{i=0}^{r} \epsilon(x^{pt_i+\delta_i} - x^{\delta_i}),$$

where $\epsilon \in \{-1, 1\}, t_i, \delta_i \ge 0$ and at least one $\delta_i \neq 0$. The result follows. □

We need the following results to continue. We give the proof of the first for the sake of completeness.

Theorem 2.3 (Dresden [2]). *If $f, g \in \mathbb{Z}[x]$ and f has all nonreal roots, then the resultant,* $\mathrm{Res}(f, g)$*, of f and g is a nonnegative integer.*

Proof. Since all the roots of $f(x)$ are nonreal, $f(x)$ has even degree and we can denote its roots by $\zeta_1, \zeta_2, \ldots, \zeta_r, \bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_r$. Then

$$\mathrm{Res}(f, g) = \prod_{i=1}^{r} g(\zeta_i) \prod_{j=1}^{r} g(\bar{\zeta}_i) = \prod_{i=1}^{r} g(\zeta_i) \prod_{j=1}^{r} \overline{g(\zeta_i)} = \prod_{i=1}^{r} |g(\zeta_i)|^2.$$

Since $\mathrm{Res}(f, g)$ is the determinant of the Sylvester matrix associated with f and g, $\mathrm{Res}(f, g) \in \mathbb{Z}$. Thus $\mathrm{Res}(f, g) \in \mathbb{N} \cup \{0\}$. □

Theorem 2.4 (de Bruijn [1]). *Suppose $f \in \mathbb{Z}[x]$ is a polynomial of degree m. Then $\Phi_n(x)$ divides $f(x)$ if and only if $f(x) = \sum_{p|n} \Phi_p(x^{n/p}) f_p(x)$, where $f_p(x) \in \mathbb{Z}[x]$.*

Theorem 2.5. *If $f \in S$, $f(1) \neq 0$ and $\Phi_k(x)$ divides $f(x)$, then there is a prime divisor p of k such that $p \le \ell(f)$.*

Proof. Suppose $\Phi_k(x)$ divides $f(x)$ and $k = \prod_{i=1}^{r} p_i^{a_i}$, where $p_i > \ell(f)$ for all i. From Theorem 2.4,

$$f(x) = \sum_{i=1}^{r} \Phi_{p_i}(x^{k/p_i}) f_{p_i}(x), \tag{2.1}$$

where $f_{p_i} \in \mathbb{Z}[x]$ and $f_{p_i} \not\equiv 0$ for at least one i. Suppose $f_{p_i} \not\equiv 0$ exactly when $i \in \{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, r\}$. By rearranging the primes, if necessary, we can write (2.1) as $f(x) = \sum_{i=1}^{s} \Phi_{p_i}(x^{k/p_i}) f_{p_i}(x)$, where $f_{p_i} \not\equiv 0$ for $1 \le i \le s$.

Let $w = \zeta_k^{p_1}$, that is, w is a primitive (k/p_1)th root of unity. Then it is easy to see that $f(w) = p_1 f_{p_1}(w)$. Thus $|f(w)| = p_1|f_{p_1}(w)|$ or, equivalently, $|f_{p_1}(w)| \le \ell(f)/p_1 < 1$. The same is true for the remaining primitive (k/p_1)th roots of unity. Hence

$$|\mathrm{Res}(\Phi_{k/p_1}(x), f_{p_1}(x))| = \prod_{(i,k/p_1)=1} |f_{p_1}(w^i)| < 1.$$

From Theorem 2.3, $\mathrm{Res}(\Phi_{k/p_1}(x), f_{p_1}(x)) = 0$ and $\Phi_{k/p_1}(x) \mid f(x)$.

Continuing in this way, we can show that $\Phi_{p_i}(x) \mid f(x)$ for some i. But then, from Lemma 2.2, $f(1) = 0$, which is a contradiction. Hence the result follows. □

Definition 2.6. A cyclotomic factor $\Phi_k(x)$ of a polynomial $f \in \mathbb{Z}[x]$ is called an *essential cyclotomic factor* of f if $p \mid k$ and p prime implies that $p \le \ell(f)$.

For $\ell(f) = 2$, there are only two Borwein polynomials: $x \pm 1$. The cyclotomic factor is the same as the essential cyclotomic factor in this case. From [7], the cyclotomic factors of $x^n \pm x^m \pm 1$ are either $\Phi_3(x^{(n,m)})$ or $\Phi_6(x^{(n,m)})$. In other words, if a Borwein polynomial of length three has a cyclotomic factor, then it has an essential cyclotomic factor. In the following theorem, we show that this holds for arbitrary Borwein polynomials f provided $f(1) \neq 0$.

Theorem 2.7. *Suppose $f \in S$, $f(1) \neq 0$ and $\Phi_k(x)$ divides $f(x)$. Then there is a cyclotomic polynomial, $\Phi_{k_1}(x)$, dividing $f(x)$ such that $k_1 \mid k$ and every prime factor of k_1 is less than or equal to $\ell(f)$.*

Proof. If k itself has all its prime divisors less than or equal to $\ell(f)$, then there is nothing to prove. Suppose k has at least one prime factor greater than $\ell(f)$.

First, we consider $k = 2q$, where $q > \ell(f)$ is an odd prime. Since $\Phi_{2q}(x) \mid f(x)$, $f(x) = \Phi_q(-x)g(x)$ for some $g \in \mathbb{Z}[x]$. Since $\Phi_q(x)$ divides $f(-x)$, it follows from Lemma 2.2 that $f(-1) = 0$. Thus $\Phi_2(x) \mid f(x)$.

Now let $k = \prod_{i=1}^r p_i^{a_i} \prod_{j=1}^s q_j^{b_j}$, where $p_i \le \ell(f)$ for $1 \le i \le r$, $q_j > \ell(f)$ for $1 \le j \le s$ and $k/q_j > 2$ for $1 \le j \le s$. Since $\Phi_k(x)$ divides $f(x)$, from Theorem 2.4,

$$f(x) = \sum_{i=1}^{r} \Phi_{p_i}(x^{k/p_i}) f_{p_i}(x) + \sum_{j=1}^{s} \Phi_{q_j}(x^{k/q_j}) g_{q_j}(x) \tag{2.2}$$

for some $f_{p_i}(x), g_{q_j}(x) \in \mathbb{Z}[x]$. We divide the proof into two cases.

Case I: for every j, $g_{q_j} \equiv 0$. From (2.2), we see that $\Phi_{k_1}(x)$ divides a common factor of $\Phi_{p_1}(x^{k/p_1}), \ldots, \Phi_{p_r}(x^{k/p_r})$, where $k_1 = \prod_{i=1}^r p_i^{a_i}$.

Case II: some $g_{q_j} \not\equiv 0$. By rearranging the terms, if necessary, suppose $g_{q_1} \not\equiv 0$. Let $z = \zeta_k^{q_1}$ be a (k/q_1)th primitive root of unity. For a prime divisor p of k, if $p \neq q_1$, then $z^{k/p} = \zeta_k^{kq_1/p}$ is a primitive pth root of unity. On the other hand, if $p = q_1$, then $z^{k/p} = 1$. Hence, from (2.2),

$$f(z) = \Phi_{q_1}(1) g_{q_1}(z) = q_1 g_{q_1}(z)$$

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and

$$q_1|g_{q_1}(z)| = |f(z)| \leq \sum_{i=0}^{r} |z^{n_i}| + 1 = \ell(f).$$

Consequently, $|g_{q_1}(z)| \leq \ell(f)/q_1 < 1$. The same holds for the other primitive (k/q_1)th roots of unity. Since $k/q_1 > 2$, it follows that $\varphi(k/q_1)$ is even and $\mathrm{Res}(\Phi_{k/q_1}(x), g_{q_1}(x)) = \mathrm{Res}(g_{q_1}(x), \Phi_{k/q_1}(x))$. Hence

$$\prod_{i,(i,k/q_1)=1} |g_{q_1}(z^i)| = |\mathrm{Res}(\Phi_{k/q_1}(x), g_{q_1}(x))| < 1.$$

Further, all the roots of $\Phi_{k/q_1}(x)$ are nonreal. From Theorem 2.3, we conclude that $|\mathrm{Res}(\Phi_{k/q_1}(x), g_{q_1}(x))| = \mathrm{Res}(\Phi_{k/q_1}(x), g_{q_1}(x)) = 0$, that is, $g_{q_1}(z) = 0$ and $f(z) = 0$.

Now we replace k by k/q_1 and repeat the argument. We reach the required result after a finite number of steps. □

3. Applications

Theorem 3.1. *Let $q \geq 5$ be a prime and let $f \in S$ be a primitive polynomial of length q.*

(1) *If $\Phi_{kq}(x)$ is an essential cyclotomic factor of f, then $\Phi_k(x)$ divides $f(x)$.*

(2) *Suppose $f(x)$ is a Newman polynomial. Then, for every prime $p < q$, $\Phi_{pq}(x)$ cannot be an essential cyclotomic factor of $f(x)$. In particular, $\Phi_{2q}(x)$ and $\Phi_{3q}(x)$ do not divide $f(x)$.*

Proof. (1) Since $\Phi_{kq}(x)$ divides $f(x)$,

$$f(x) = \sum_{p|k} \Phi_p(x^{kq/p}) f_p(x) + \Phi_q(x^k) g_q(x).$$

If $g_q(x) \equiv 0$, then, as before, $\Phi_k(x)$ divides $f(x)$.

If, on the other hand, $g(x) \not\equiv 0$, then $f(\zeta_k) = q g_q(\zeta_k)$ so that $|g_q(\zeta_k)| \leq 1$. If $|g_q(\zeta_k)| = 1$, then $|f(\zeta_k)| = q$, which is possible if and only if all the exponents of $f(x)$ are multiples of k and all the coefficients of $f(x)$ are of same sign, which contradicts $f(x)$ being primitive. Hence $|g_q(\zeta_k)| < 1$ and, in the same way as before, $\Phi_k(x) \mid f(x)$.

(2) The proof of (2) follows from that of (1) together with the fact the only minimal vanishing sums of ζ_p of length at most p, up to a rotation, are $\zeta_p^p - 1$ and $\Phi_p(\zeta_p)$. □

We now resolve Conjecture 1.3.

Theorem 3.2. *Let f be a primitive Newman polynomial of length five. If $f(x)$ has a cyclotomic factor, then either $\Phi_5(x) \mid f(x)$ or $\Phi_{2^\alpha 3^\beta}(x) \mod f(x)$ for some $\alpha, \beta \geq 1$.*

Proof. From Theorem 2.7, $f(x)$ has an essential cyclotomic factor $\Phi_n(x)$ for some n. Suppose n' is the largest square-free part of n. Since $\Phi_n(x) = \Phi_{n'}(x^{n/n'})$, it is sufficient to consider square-free values of n. If n is square-free and $\Phi_n(x)$ divides $f(x)$, then $n = 5, 6, 10, 15$ or 30. From Theorem 3.1, $n = 5$ or 6. Since $f(x)$ is primitive, $\Phi_{5^\gamma}(x) \nmid f(x)$ for $\gamma > 1$. Hence, for a primitive Newman polynomial of length five, the cyclotomic factors are either $\Phi_5(x)$ or $\Phi_{2^\alpha 3^\beta}(x)$ for some $\alpha, \beta \geq 1$. □

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References

[1] N. G. de Bruijn, 'On the factorization of cyclic groups', *Indag. Math.* **15** (1953), 370–377.
[2] G. Dresden, 'Resultants of cyclotomic polynomials', *Rocky Mountain J. Math.* **42** (2012), 1461–1469.
[3] P. Drungilas, J. Jankauskas and J. Šiurys, 'On Littlewood and Newman polynomial multiples of Borwein polynomials', *Math. Comput.* **87**(311) (2018), 1523–1541.
[4] M. Filaseta and J. Solan, 'An extension of a theorem of Ljunggren', *Math. Scand.* **84**(1) (1999), 5–10.
[5] C. Finch and L. Jones, 'On the irreducibility of $\{-1, 0, 1\}$-quadrinomials', *Integers* **6** (2006), A16, 4 pages.
[6] T. Y. Lam and K. H. Leung, 'On vanishing sums of roots of unity', *J. Algebra* **224** (2000), 91–109.
[7] W. Ljunggren, 'On the irreducibility of certain trinomials and quadrinomials', *Math. Scand.* **8** (1960), 65–70.
[8] I. Mercer, 'Newman polynomials, reducibility, and roots on the unit circle', *Integers* **12**(4) (2012), 503–519.
[9] W. H. Mills, 'The factorization of certain quadrinomials', *Math. Scand.* **57** (1985), 44–50.
[10] A. M. Odlyzko and B. Poonen, 'Zeros of polynomials with 0, 1 coefficients', *Enseign. Math.* **39** (1993), 317–348.
[11] A. S. Reddy, 'The lowest degree 0, 1-polynomial divisible by cyclotomic polynomial', Preprint, arXiv:1106.1271.
[12] E. S. Selmer, 'On the irreducibility of certain trinomials', *Math. Scand.* **4** (1956), 287–302.
[13] H. Tverberg, 'On the irreducibility of the trinomials $x^n \pm x^m \pm 1$', *Math. Scand.* **8** (1960), 121–126.

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