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JORDAN DERIVATIONS ON PRIME RINGS

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The purpose of this paper is to present a brief proof of the well known result of Herstein which states that any Jordan derivation on a prime ring with characteristic not two is a derivation.

Throughout this paper all rings will be associative. We shall denote by $Z(R)$ the centre of a ring R. An additive mapping $D\colon R \to R$ will be called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. We call an additive mapping $D\colon R \to R$ a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. In this paper we present an alternative proof of the following theorem.

THEOREM 1. (Herstein [**1**]) *Let R be a prime ring with characteristic not two and let $D\colon R \to R$ be a Jordan derivation. Then D is a derivation.*

For the proof of Theorem 1 we need several steps. First we have

PROPOSITION 2. *Let R be a ring with characteristic different from two and let $D\colon R \to R$ be a Jordan derivation. Then the following hold:*

(1) $D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a)$ *for all* $a, b \in R$;
(2) $D(aba) = D(a)ba + aD(b)a + abD(a)$ *for all* $a, b \in R$;
(3) $D(abc + cba) = D(a)bc + aD(b)c + abD(c) + D(c)ba + cD(b)a + cbD(a)$ *for all* $a, b, c \in R$.

(1) is immediate. The proof of (2) is not difficult and can be found in [**1**] and [**2**]. (3) follows immediately from (2). For any Jordan derivation D we shall write a^b for $D(ab) - D(a)b - aD(b)$. From (1) in Proposition 2 we see that

$$b^a = -a^b \tag{1}$$

holds for all $a, b \in R$. It is easy to see that for all $a, b, c \in R$ the relation

$$a^{b+c} = a^b + a^c \tag{2}$$

holds. All is prepared for the proof of the result below.

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THEOREM 3. *Let R be a ring of characteristic not two, and let $D: R \to R$ be a Jordan derivation. In this case for all $a, b, r \in R$ we have*

$$a^b r(ab - ba) + (ab - ba)ra^b = 0 \tag{3}$$

PROOF OF THEOREM 3: Let us write W for $abrba + barab$. Then by (2) of Proposition 2 we obtain

$$\begin{aligned} D(W) &= D(a(brb)a + b(ara)b) \\ &= D(a)brba + aD(brb)a + abrbD(a) + D(b)arab + bD(ara)b + baraD(b) \\ &= D(a)brba + aD(b)rba + abD(r)ba + abrD(b)a + abrbD(a) \\ &+ D(b)arab + bD(a)rab + baD(r)ab + barD(a)b + baraD(b). \end{aligned}$$

On the other hand we obtain using (3) of Proposition 2

$$\begin{aligned} D(W) &= D((ab)r(ba) + (ba)r(ab)) \\ &= D(ab)rba + abD(r)ba + abrD(ba) + D(ba)rab + baD(r)ab + barD(ab). \end{aligned}$$

By comparing and using (1) we obtain (3). The proof of the theorem is complete. ∎

The proof of Theorem 1 is an almost immediate consequence of Theorem 3 and Lemma 3.10 in [**2**].

PROOF OF THEOREM 1: Let a and b be fixed elements from R. If $ab \neq ba$ then from Theorem 3 and Lemma 3.10 in [**2**] one obtains immediately that $a^b = 0$. If a and b are both in $Z(R)$ then $a^b = 0$ follows from (1) in Proposition 2. It remains to prove that $a^b = 0$ also in the case when $a \notin Z(R)$ and $b \in Z(R)$. There exists $c \in R$ such that $ac \neq ca$. Since $ac \neq ca$ and $a(b+c) \neq (b+c)a$ we have $a^c = 0$ and $a^{b+c} = 0$. Then we obtain using (2) $0 = a^{b+c} = a^b + a^c = a^b$. The proof of the theorem is complete. ∎

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