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Maximal sum-free sets in abelian groups of order divisible by three

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A subset S of an additive group G is called a maximal sum-free set in G if $(S+S) \cap S = \emptyset$ and $|S| \geq |T|$ for every sum-free set T in G. In this note, we prove a conjecture of Yap concerning the structure of maximal sum-free sets in finite abelian groups of order divisible by 3 but not divisible by any prime congruent to 2 modulo 3.

Given a finite additive abelian group G and non-empty subsets S, T of G, let $S + T$ denote the set $\{s+t \mid s \in S, t \in T\}$, $\bar{S}$ the complement of S in G and $|S|$ the cardinality of S. Define the subgroup $H(S)$ by $H(S) = \{g \in G \mid S+g = S\}$ so that

(a) $S + H(S) = S$ and

(b) if $S + K = S$ for some subgroup K, then $K \leq H(S)$.

Note that the subgroup generated by $H(S)$ and $H(T)$ is contained in $H(S+T)$. We call S a *sum-free set* in G if $(S+S) \subseteq \bar{S}$. If, in addition, $|S| \geq |T|$ for every sum-free set T in G, then we call S a *maximal sum-free set* in G.

Suppose that $|G|$ is not divisible by any prime congruent to 2 modulo 3, but is divisible by 3. Diananda and Yap [1] showed that if S is a maximal sum-free set in G, then

(a) $|S| = |G|/3$ and

(b) S is a union of cosets of a subgroup H of index $3m$ in G

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(for some m) such that either

(i) $|S+S| = 2|S| - |H|$ or

(ii) $|S+S| = 2|S|$ and $S \cup (S+S) = G$.

Yap [3] conjectured that case (ii) cannot in fact occur. Here we prove his conjecture. Part of the proof in [1] is restated for convenience.

Let A be a subset of G such that $A = -A$. Since $|G|$ is odd, $|A|$ is odd if and only if $0 \in A$.

$0 \notin S$ and $|S|$ is odd, so $S \neq -S$. $0 \in S-S = -(S-S)$, so $|S-S|$ is odd. Since S is sum-free, $S \cap (S+S) = \emptyset = \big(S\cup(-S)\big) \cap (S-S)$.

We apply Kneser's Theorem: the statement given in [2] is the most convenient. If $H = H(S+S)$, then $S + S + H = S + S$ and either $|S+S| \geq 2|S|$ or $|S+S| = 2|S+H| - |H|$.

If $S + H$ is not sum-free, then for some $s \in S$, $h \in H$ we have $s + h \in (S+H) + (S+H) = S + S + H = S + S$. But then $s \in S + S - h = S + S$ and S is not sum-free, which is a contradiction. Hence $S + H$ is a sum-free set containing S . By the maximality of S , $S + H = S$ and hence $H(S) = H$.

Since S is sum-free, $|S+S| \leq 2|S|$ and the possibilities for $|S+S|$ become

(i) $|S+S| = 2|S| - |H|$ or

(ii) $|S+S| = 2|S|$.

We know that $S - S + H = S - S$, so $H \leq H(S-S)$. A proof similar to that for $S + H$ shows that $S + H(S-S) = S$ also. Hence $H(S-S) = H$. Then the possibilities for $S - S$, just as for $S + S$, are

(i) $|S-S| = 2|S| - |H|$ or

(ii) $|S-S| = 2|S|$.

But $|S-S|$ is odd, so (ii) is ruled out and $|S-S| = 2|S| - |H|$.

If $|S+S| = 2|S|$, then exactly one coset of H occurs in $S + S$ but not in $S - S$. We show this is impossible.

Since S is a union of cosets of H and $S \neq -S$, we have $|S\cup(-S)| \geq |S| + |H|$. Since $\big(S\cup(-S)\big) \cap (S-S) = \emptyset$, we have

$|S\cup(-S)| \le |S| + |H|$. Hence $|S\cup(-S)| = |S| + |H|$, $|S\cap(-S)| = |S| - |H|$ and S consists of a union of pairs of cosets, $g_i + H$ and $-g_i + H$, $i = 1, \ldots, n$ for some n , together with one coset $g + H$ whose negative is not contained in S .

We want a representative of the coset of H which is contained in $S + S$ but not in $S - S$. Now $g_i + g_j = g_i - (-g_j)$ and $g + g_j = g - (-g_j)$, so both belong to $S - S$. So the only possibility is $2g$. Since $|G|$ is odd, $2g \ne -2g$ and we consider the coset $-2g + H \subseteq S + S$. If $-2g = g + g_j$, then $2g = (-g_j) - g \in S - S$; if $-2g = g_i + g_j$, then $2g = (-g_i) - (g_j) \in S - S$.

In either case we have a contradiction, so $|S+S| \ne 2|S|$.

References

[1] Palahenedi Hewage Diananda and Hian Poh Yap, "Maximal sum-free sets of elements of finite groups", *Proc. Japan Acad.* 45 (1969), 1-5.

[2] J.H.B. Kemperman, "On small sumsets in an abelian group", *Acta Math.* 103 (1960), 63-88.

[3] Hian-Poh Yap, "Structure of maximal sum-free sets in groups of order $3p$ ", *Proc. Japan Acad.* 46 (1970), 758-762.

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