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On Gâteaux Differentiability of Pointwise Lipschitz Mappings

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Abstract. We prove that for every function $f: X \to Y$, where X is a separable Banach space and Y is a Banach space with RNP, there exists a set $A \in \tilde{\mathcal{A}}$ such that f is Gâteaux differentiable at all $x \in S(f) \setminus A$, where $S(f)$ is the set of points where f is pointwise-Lipschitz. This improves a result of Bongiorno. As a corollary, we obtain that every K-monotone function on a separable Banach space is Hadamard differentiable outside of a set belonging to $\tilde{\mathcal{C}}$; this improves a result due to Borwein and Wang. Another corollary is that if X is Asplund, $f: X \to \mathbb{R}$ cone monotone, $g: X \to \mathbb{R}$ continuous convex, then there exists a point in X, where f is Hadamard differentiable and g is Fréchet differentiable.

1 Introduction

The classical Rademacher theorem [9] concerning a.e. differentiability of Lipschitz functions defined on $\mathbb{R}^n$ was extended by Stepanoff to pointwise Lipschitz functions [10,11]. D. Bongiorno [2, Theorem 1] proved a version for infinite-dimensional mappings; namely, that for every $f: X \to Y$, where X is a separable Banach space and Y is a Banach space with RNP, there exists an Aronszajn null set $A \subset X$ (see [1] for the definition of Aronszajn null sets) such that f is Gâteaux differentiable at all $x \in S(f) \setminus A$ (here, $S(f)$ is the set of points where f is pointwise-Lipschitz). This generalized results for Lipschitz functions obtained by Aronszajn, Christensen, Mankiewicz, and Phelps; see [1] for the definitions of various notions of null sets they used. We prove a stronger version of infinite dimensional Stepanoff-like theorem, which asserts that under the same assumptions as in [2, Theorem 1], the set A can be taken in the class $\tilde{\mathcal{A}}$ defined by Preiss and Zajíček [8]; see Theorem 4.1. By results of [8], $\tilde{\mathcal{A}}$ is a strict subclass of Aronszajn null sets. Recently, Zajíček [12] proved that the sets in $\tilde{\mathcal{A}}$ (and even $\tilde{\mathcal{C}}$) are Γ-null, which is a notion of null sets due to Lindenstrauss and Preiss [7] (here, a definition and basic properties of this notion can be found). Thus, Theorem 4.1 has the following corollary: if X is a Banach space with separable dual (*i.e.,* an Asplund space) and Y is a Banach space with RNP, $f: X \to Y$ is pointwise-Lipschitz at all $x \in X \setminus A$ where $A \in \tilde{C}$, $g: X \to \mathbb{R}$ is continuous convex, then there exists $x \in X$ such that f is Gâteaux differentiable at x and g is Fréchet differentiable at x. In some sense, our proof of Theorem 4.1 is simpler than the proof of [2, Theorem 1]; some of the (rather cumbersome) measurability considerations from [2] are replaced by Lemma 3.2 and the construction of a total set from [2] is

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replaced by the Lipschitz property of certain restrictions of the given mapping. In the proof, we use several ideas from [8].

Let X be a Banach space and $\varnothing \neq K \subset X$ be a cone. Following [3], we say that $f\colon X \to \mathbb{R}$ is *K-monotone* provided f or $-f$ is K-increasing (we say that $f\colon X \to \mathbb{R}$ is *K-increasing* provided $x \leq_K y$ implies $f(x) \leq f(y)$ whenever $x, y \in X$; here, $x \leq_K y$ means $y - x \in K$). Borwein, Burke and Lewis [3] proved that every K-monotone $f\colon X \to \mathbb{R}$ is Gâteaux differentiable outside of a Haar null set (see [1] for the definition) provided X is separable and K is closed convex with $\operatorname{int}(K) \neq \varnothing$. This was strengthened by Borwein and Wang [4] who showed that "Haar null" can be replaced by "Aronszajn null". In Section 5, as a corollary to Theorem 4.1, we obtain that an analogous result holds if we replace "Haar null" by the class $\tilde{\mathcal{C}}$ defined by Preiss and Zajíček [8]; see Theorem 5.4 for details. The class $\tilde{\mathcal{C}}$ is a strict subclass of Aronszajn null sets (see [8, p. 19]) and thus our result improves a result due to Borwein and Wang [4, Proposition 16(iv)] who showed that instead of "Gâteaux differentiable" we can write "Hadamard differentiable" (see Corollary 5.5). Our result has another interesting corollary; namely, if X has a separable dual (*i.e.,* X is an Asplund space), $f\colon X \to \mathbb{R}$ is K-monotone, $g\colon X \to \mathbb{R}$ is continuous convex, then there exists $x \in X$ such that f is Hadamard differentiable at x, and g is Fréchet differentiable at x (see Corollary 5.6). This does not follow from the results of Borwein and Wang since Aronszajn null sets and Γ-null sets are incomparable. It seems to be a difficult open problem whether $\tilde{\mathcal{C}} = \tilde{\mathcal{A}}$ (see [8]). If this were true, then our theorem would also hold with $\tilde{\mathcal{A}}$ in place of $\tilde{\mathcal{C}}$. Thus, it remains open, whether we can replace $\tilde{\mathcal{C}}$ by $\tilde{\mathcal{A}}$ in Theorem 5.4 and Corollary 5.5. Going in another direction, the author [6] proved some results about a.e. differentiability of vector-valued cone monotone mappings.

The current paper is organized as follows. Section 2 contains basic definitions and facts. Section 3 contains auxiliary results. Section 4 contains the proofs of the main Theorem 4.1, and Corollary 4.2. Section 5 contains the proofs of Theorem 5.4, and Corollaries 5.5 and 5.6.

2 Preliminaries

All Banach spaces are assumed to be real. By λ we will denote the Lebesgue measure on $\mathbb{R}$. Let X be a Banach space. By $B(x, r)$ we will denote the open ball with center $x \in X$ and radius $r > 0$, and by S_X we denote $\{x \in X : \|x\| = 1\}$. If $M \subset X$, then by $d_M(x) := \inf\{\|y - x\| : y \in M\}$.

Let X, Y be Banach spaces. We say that $f\colon X \to Y$ is *pointwise Lipschitz at $x \in X$*, provided $\limsup_{y \to x} \frac{\|f(x) - f(y)\|}{\|x - y\|} < \infty$. By $S(f)$, we will denote the set of points of X where f is pointwise Lipschitz. By $\operatorname{Lip}(f)$ we will denote the usual Lipschitz constant of f.

In the following, let X be a Banach space. If f is a mapping from X to a Banach space Y and $x, v \in X$, then we consider the directional derivative $f'(x, v)$ defined by

$$f'(x, v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}. \tag{2.1}$$

If $x \in X$, $f'(x, v)$ exists for all $v \in X$, and $T(v) := f'(x, v)$ is a bounded linear operator from X to Y, then we say that f is *Gâteaux differentiable* at x. If f is Gâteaux

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differentiable at x and the limit in (2.1) is uniform in $\|v\| = 1$, then we say that f is *Fréchet differentiable* at x. If f is Gâteaux differentiable at x, and the limit in (2.1) is uniform with respect to norm-compact sets, then we say that f is *Hadamard differentiable* at x.

We will need the following notion of "smallness" of sets in Banach spaces from [8].

Definition 2.1 Let X be a Banach space, $M \subset X$, $a \in X$. Then we say that

(i) M is *porous* at a if there exists $c > 0$ such that for each $\varepsilon > 0$ there exist $b \in X$ and $r > 0$ such that $\|a - b\| < \varepsilon$, $M \cap B(b, r) = \varnothing$, and $r > c\|a - b\|$.

(ii) M is *porous at a in direction v* if the $b \in X$ from (i) verifying the porosity of M at a can always be found in the form $b = a + tv$, where $t \geq 0$. We say that M is *directionally porous at a* if there exists $v \in X$ such that M is porous at a in direction v.

(iii) M is *directionally porous* if M is directionally porous at each of its points.

(iv) M is *σ-directionally porous* if it is a countable union of directionally porous sets.

For a recent survey of properties of negligible sets, see [13]. We will also need the following notion of "null" sets in a Banach space. It was defined in [8].

Definition 2.2 Let X be a separable Banach space and $0 \neq v \in X$. Then $\tilde{\mathcal{A}}(v, \varepsilon)$ is the system of all Borel sets $B \subset X$ such that $\{t : \varphi(t) \in B\}$ is Lebesgue null whenever $\varphi\colon \mathbb{R} \to X$ is such that the function $t \to \varphi(t) - tv$ has Lipschitz constant at most ε, and $\tilde{\mathcal{A}}(v)$ is the system of all sets B such that $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k \in \tilde{\mathcal{A}}(v, \varepsilon_k)$ for some $\varepsilon_k > 0$.

We define $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{C}}$) as the system of those $B \subset X$ that can be, for every given complete[1] sequence $(v_n)_n$ in X (resp. for some sequence $(v_n)_n$ in X), written as $B = \bigcup_{n=1}^{\infty} B_n$, where each B_n belongs to $\tilde{\mathcal{A}}(v_n)$.

The following simple lemma shows that every directionally porous set is contained in a set from $\tilde{\mathcal{A}}$. As a corollary, we have the same result for σ-directionally porous sets.

Lemma 2.3 *Let X be a separable Banach space, and $A \subset X$ be directionally porous. Then there exists a set $\hat{A} \in \tilde{\mathcal{A}}$ such that $A \subset \hat{A}$.*

Proof This follows from the proof of [8, Theorem 10]; see also [8, Remark 6]. ■

The following simple lemma is proved in [2].

Lemma 2.4 **([2, Lemma 1])** *Given $f\colon X \to Y$ and $L, \delta > 0$, let S be the set of all points $x \in X$ such that $\|f(x + h) - f(x)\| \leq L\|h\|$ whenever $\|h\| < \delta$. Then S is a closed set.*

3 Auxiliary Results

The following is an extension of [4, Lemma 3] to a vector-valued setting.

[1] We say that $(v_n)_n \subset X \setminus \{0\}$ is a *complete* sequence provided $\overline{\operatorname{span}}(v_n) = X$.

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Lemma 3.1 *Let X, Y be Banach spaces, $f\colon X \to Y$. Fix $v_1, v_2 \in X$, $k, l, m \in \mathbb{N}$, and $y, z \in Y$. Then the set $A(k, l, m, y, z)$ of all $x \in X$ verifying*

(i) $\|\frac{f(x+tu)-f(x)}{t} - y\| < \frac{1}{l}$ *for* $\|u - v_1\| < 1/m$ *and* $0 < t < 1/k$,

(ii) $\|\frac{f(x+tu)-f(x)}{t} - z\| < \frac{1}{l}$ *for* $\|u - v_2\| < 1/m$ *and* $0 < t < 1/k$,

(iii) $\|\frac{f(x+s(v_1+v_2))-f(x)}{s} - (y+z)\| > \frac{3}{l}$ *occurs for arbitrarily small* $s > 0$,

is directionally porous in X.

Proof Let $x \in A(k, l, m, y, z)$. Choose $0 < s < 1/k$ such that the inequality in (iii) holds. We claim that $B(x + sv_1, \frac{s}{m}) \cap A(k, l, m, y, z) = \varnothing$.

Indeed, for $\|h\| < \frac{1}{m}$, if $x + s(v_1 + h)$ satisfies (ii), we have

$$\Big\| \frac{f(x + s(v_1 + h) + su) - f(x + s(v_1 + h))}{s} - z \Big\| < \frac{1}{l}, \tag{3.1}$$

for $\|u - v_2\| < \frac{1}{m}$. By (i) we get

$$\Big\| \frac{f(x + s(v_1 + h)) - f(x)}{s} - y \Big\| < \frac{1}{l}. \tag{3.2}$$

By the triangle inequality, (3.1), and (3.2) we get

$$\Big\| \frac{f(x + s(v_1 + h) + su) - f(x)}{s} - (y + z) \Big\| < \frac{2}{l}, \text{ for } \|u - v_2\| < \frac{1}{m}.$$

Taking $u = v_2 - h$, we have

$$\Big\| \frac{f(x + sv_1 + sv_2) - f(x)}{s} - (y + z) \Big\| < \frac{2}{l}.$$

This choice contradicts the choice of s. ■

Suppose that X, Y are Banach spaces, $f\colon X \to Y$. For $x \in X$, $0 \neq v \in X$, and $\varepsilon > 0$ by $O(f, x, v, \varepsilon)$ we denote the expression

$$\sup\Big\{ \Big\| \frac{f(x + tv) - f(x)}{t} - \frac{f(x + sv) - f(x)}{s} \Big\| : 0 < |t|, |s| < \varepsilon \Big\}.$$

We also define $O(f, x, v) := \lim_{\varepsilon \to 0+} O(f, x, v, \varepsilon)$. We borrow this definition from [8]. The following is true in general (in [8, Lemma 11] it is assumed that f is Lipschitz, but it is clearly not necessary):

$$f'(x, v) \text{ exists if and only if } O(f, x, v) = 0. \tag{3.3}$$

For the rest of this section, X will be a separable Banach space and Y will be a Banach space with RNP. Also, $G \subset X$ will be a closed set and $f\colon X \to Y$ a mapping such that there exist $L, \delta > 0$ with

$$\|f(y) - f(x)\| \leq L\|y - x\| \quad \text{whenever } y \in G,\ x \in B(y, \delta). \tag{3.4}$$

We also assume that D is a Borel subset of G such that the distance function $d_G(x)$ is Gâteaux differentiable at each point $x \in D$.

Lemma 3.2 *Let X be separable, $0 \neq v \in X$, and we put $g(x) := O(f, x, v)$. Then $g|_D$ is Borel measurable.*

Proof Let $w \in D$. Then $h = f|_{B(w,\delta/4)\cap G}$ is L-Lipschitz by (3.4), and thus $Z = h(B(w,\delta/4) \cap G)$ is separable. Thus, Z can be isometrically embedded into ℓ_∞, and by [1, Lemma 1.1(ii)], h can be extended to an L-Lipschitz mapping $H \colon X \to \ell_\infty$ (we identify Z with its isometric representation in ℓ_∞ for the moment). By [8, Lemma 11(ii)], $G(x) := O(H, x, v)$ is a Borel measurable function on X. We will prove that $g(x) = G(x)$ for all $x \in B(w, \delta/4) \cap D$, and conclude that $g|_D$ is Borel measurable (by separability of X).

Let $x \in B(w, \delta/4) \cap D$. Fix $\gamma > 0$ such that $B(x, 2\gamma) \subset B(w, \delta/4)$. Let $\varepsilon > 0$ and find $0 < \tau < \varepsilon$ such that $d_G(x + tv) < \frac{\varepsilon}{L}|t|$ and $x + tv \in B(x, \gamma)$ whenever $0 < |t| < \tau$. Take $\eta := \frac{1}{2}\min(\varepsilon, \tau, \frac{L\gamma}{\varepsilon})$. For $0 < |s|, |t| < \eta$ find $y, z \in G \cap B(w, \delta/4)$ such that $\|x + tv - y\| < \frac{\varepsilon}{L}|t|$ and $\|x + sv - z\| < \frac{\varepsilon}{L}|s|$. Then we have

$$\left\| \frac{f(x+tv) - f(y)}{t} \right\| \leq \frac{L}{|t|}\|x + tv - y\| \leq \varepsilon,$$

and similarly $\|\frac{f(x+sv)-f(z)}{s}\| \leq \varepsilon$. Also,

$$\left\| \frac{H(y) - H(x+tv)}{t} \right\| \leq \frac{L}{|t|}\|x + tv - y\| \leq \varepsilon,$$

and $\|\frac{H(x+sv)-H(z)}{s}\| \leq \varepsilon$. Thus using $f(x) = H(x)$, $f(y) = H(y)$, and $f(z) = H(z)$, we obtain

$$\begin{aligned}
(3.5)\quad & \left\| \frac{H(x+tv) - H(x)}{t} - \frac{H(x+sv) - H(x)}{s} \right\| \\
&\leq \left\| \frac{f(x+tv) - f(x)}{t} - \frac{f(x+sv) - f(x)}{s} \right\| \\
&\quad + \left\| \frac{f(x+tv) - f(y)}{t} \right\| + \left\| \frac{f(x+sv) - f(z)}{s} \right\| \\
&\quad + \left\| \frac{H(y) - H(x+tv)}{t} \right\| + \left\| \frac{H(x+sv) - H(z)}{s} \right\| \\
&\leq O(f, x, v, \varepsilon) + 4\varepsilon.
\end{aligned}$$

By taking a supremum over $0 < |s|, |t| < \eta$ in (3.5), we obtain $O(H, x, v, \eta) \leq O(f, x, v, \varepsilon) + 4\varepsilon$. Send $\eta \to 0+$ to get $O(H, x, v) \leq O(f, x, v, \varepsilon) + 4\varepsilon$, and then $\varepsilon \to 0+$ to see that $O(H, x, v) \leq O(f, x, v)$.

By (3.4) and H being L-Lipschitz, we can reverse the rôles of f and H in the above argument to show that $O(f, x, v) \leq O(H, x, v)$. ■

Lemma 3.3 *If $x \in D$, $0 \neq v \in X$, $O(f, x, v) > 0$, $\varphi \colon \mathbb{R} \to X$, $r \in \mathbb{R}$, $\varphi(r) = x$, and the mapping $\psi \colon t \to \varphi(t) - tv$ has Lipschitz constant strictly less than $O(f, \varphi(r), v)/8L$, then the mapping $f \circ \varphi$ is not differentiable at r.*

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Proof Denote $K := O(f,x,v) > 0$. To prove the lemma, let $\delta' > 0$ be such that $x + tv \in B(x, \delta/2)$ and $d_G(x+tv) < \frac{K}{16L}|t|$ for each $0 < |t| < \delta'$. Fix $\varepsilon > 0$ and let $\tau = \min(\varepsilon, \delta', \frac{16L\delta}{2K})$. By the assumptions on f, let $0 < |t|, |s| < \tau$ such that

$$\Big\| \frac{f(x+tv) - f(x)}{t} - \frac{f(x+sv) - f(x)}{s} \Big\| > \frac{3}{4} O(f,x,v),$$

and estimate

$$\begin{aligned} D &:= \Big\| \frac{f \circ \varphi(r+t) - f \circ \varphi(r)}{t} - \frac{f \circ \varphi(r+s) - f \circ \varphi(r)}{s} \Big\| \\ &\geq \Big\| \frac{f(x+tv) - f(x)}{t} - \frac{f(x+sv) - f(x)}{s} \Big\| - \Big\| \frac{f(x+tv) - f(\varphi(r+t))}{t} \Big\| \\ &\quad - \Big\| \frac{f(x+sv) - f(\varphi(r+s))}{s} \Big\|. \end{aligned}$$

Find $y, z \in G \cap B(x, \delta)$ such that $\|x + tv - y\| < \frac{K}{16L}|t|$ and $\|x + sv - z\| < \frac{K}{16L}|s|$. Then we have $\|\frac{f(x+tv) - f(y)}{t}\| \leq \frac{L}{|t|}\|x + tv - y\| \leq \frac{K}{16}$, and similarly

$$\begin{aligned} \Big\| \frac{f(y) - f(\varphi(r+t))}{t} \Big\| &\leq \frac{L}{|t|} \|y - \varphi(r+t)\| \\ &\leq \frac{L}{|t|} \|y - (x+tv)\| + \frac{L}{|t|} \|\varphi(r) + tv - \varphi(r+t)\| \\ &\leq \frac{K}{16} + \frac{L}{|t|} \|\psi(r) - \psi(r+t)\| \\ &\leq \frac{K}{16} + L \operatorname{Lip}(\psi) < \frac{K}{16} + \frac{K}{8} = \frac{3K}{16}. \end{aligned}$$

Thus

$$\begin{aligned} \Big\| \frac{f(x+tv) - f(\varphi(r+t))}{t} \Big\| &\leq \Big\| \frac{f(x+tv) - f(y)}{t} \Big\| + \Big\| \frac{f(y) - f(\varphi(r+t))}{t} \Big\| \\ &< \frac{K}{16} + \frac{3K}{16} = \frac{K}{4}. \end{aligned}$$

Since an analogous estimate holds for $\big\| \frac{f(x+sv) - f(\varphi(r+s))}{s} \big\|$, we obtain $D > \frac{3}{4}K - 2\frac{K}{4} = \frac{O(f,x,v)}{4}$; so $O(f \circ \varphi, r, 1) \geq O(f, \varphi(r), v)/4$ is strictly positive as required. ■

Lemma 3.4 *For each $0 \neq u \in X$, the set $\Delta = \{x \in D : f'(x,u)$ does not exist$\}$ belongs to $\tilde{A}(u)$.*

Proof Since $\Delta = \{x \in D : O(f,x,u) > 0\}$ by (3.3), and by Lemma 3.2 we have that $g(x) = O(f,x,u)$ is Borel on D, we obtain that Δ is Borel. By the same reasoning, each $A_k = \{x \in \Delta : O(f,x,u) > \frac{1}{k}\}$ is Borel for $k \in \mathbb{N}$, and we have $\Delta = \bigcup_k A_k$. To finish the proof of the lemma, it is enough to show that $A_k \in \tilde{A}(u, 1/16kL)$ for each $k \in \mathbb{N}$.

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Let $k \in \mathbb{N}$ be fixed. If $\varphi\colon \mathbb{R} \to X$ is such that the function $t \to \varphi(t) - tu$ has Lipschitz constant at most $1/16kL$, then Lemma 3.3 implies that $f \circ \varphi$ is not differentiable at any t for which $\varphi(t) \in A_k$. Hence $B_k := \{t \in \mathbb{R} : \varphi(t) \in A_k\}$ is a subset of the set of points at which $f \circ \varphi$ is not differentiable. Since $f \circ \varphi$ is pointwise Lipschitz at all t such that $\varphi(t) \in \Delta$, and since Y has RNP, [2, Proposition 1] implies that $\lambda(B_k) = 0$ as required for showing that $A_k \in \tilde{A}(u, 1/16kL)$. ∎

Lemma 3.5 *Let X be separable. Then there exists a set $R \in \tilde{A}$ such that $(N_f \cap D) \setminus R \in \tilde{A}$, where N_f is the set of all points $x \in X$ at which f is not Gâteaux differentiable.*

Proof Let $w \in D$, and denote $D_w = D \cap B(w, \delta/4)$. If $g := f|_{B(w,\delta/4)\cap G}$, then g is L-Lipschitz on its domain (by (3.4)). Since $T := g(B(w, \delta/4) \cap G)$ is separable, we will show that

$$Z := \overline{\operatorname{span}}\{u \in Y : u = f'(x, v) \text{ for some } x \in D_w, v \in X \setminus \{0\}\}$$

is a subset of $W := \overline{\operatorname{span}}(T)$ (and thus is separable). Suppose that $x \in D_w$, $0 \neq v \in X$, and $f'(x, v)$ exists. Fix $\gamma > 0$ such that $B(x, 2\gamma) \subset B(w, \delta/4)$. Let $\varepsilon > 0$ and find $\tau > 0$ such that for $0 < |t| < \tau$ we have $d_G(x + tv) < \frac{\varepsilon}{L}|t|$, $x + tv \in B(x, \gamma)$, and $\|\frac{f(x+tv)-f(x)}{t} - f'(x, v)\| < \varepsilon$. Let $\eta = \min(\tau, \frac{L\gamma}{2\varepsilon})$ and $0 < |t| < \eta$. Find $y \in G \cap B(w, \delta/4)$ with $\|x + tv - y\| < \frac{\varepsilon}{L}|t|$. Then

$$\begin{aligned}\Big\| f'(x, v) - \frac{f(y) - f(x)}{t}\Big\| &\leq \varepsilon + \Big\| \frac{f(x + tv) - f(x)}{t} - \frac{f(y) - f(x)}{t}\Big\| \\ &\leq \varepsilon + \frac{L}{|t|}\|x + tv - y\| \leq 2\varepsilon.\end{aligned}$$

Since $\frac{f(y)-f(x)}{t} \in W$, send $\varepsilon \to 0+$ to obtain $d_W(f'(x, v)) = 0$, and thus $f'(x, v) \in W$.

Since X, Z are separable, by R_w denote the set obtained as a union of all $A(k, l, m, y, y') \cap D$ (see Lemma 3.1) where $k, l, m \in \mathbb{N}$, y, y' are chosen from a countable dense subset of Z and v_1, v_2 are chosen from a countable dense subset of X. By Lemmas 3.1 and 2.3, there exists $R'_w \in \tilde{A}$ such that $R_w \subset R'_w$. We have the following: if $x \in D_w \setminus R'_w$, then the following implication holds [2]:

(∗) If the directional derivative $f'(x, u)$ exists in all directions u from a set $U_x \subset X$ whose linear span is dense in X, then $f'(x, v)$ exists for all $v \in \operatorname{span}_{\mathbb{Q}} U_x$; furthermore, $f'(x, \cdot)$ is bounded and linear on $\operatorname{span}_{\mathbb{Q}} U_x$.

The proof of (∗) is similar to the proof of [8, Theorem 2] and so we omit it.

For the rest of the proof, let $(v_n)_n$ be a complete sequence in X. Let $\Delta_n = \Delta_n(w)$ be the set Δ from Lemma 3.4 applied to v_n; the lemma implies that Δ_n is Borel and $\Delta_n \in \tilde{A}(v_n)$ for each $n \in \mathbb{N}$. Denote $F_w = D_w \setminus (\bigcup_n \Delta_n)$. It follows that $H_w := F_w \setminus R'_w$ is Borel. We will show that f is Gâteaux differentiable at each $x \in H_w$.

[2] Here, $\operatorname{span}_{\mathbb{Q}} V = \{\sum_{i=1}^{n} q_i v_i : q_i \in \mathbb{Q},\ v_i \in V, i = 1, \dots, n,\ n \in \mathbb{N}\}$.

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Let $x \in H_w$. Fix $\gamma > 0$ such that $B(x, 2\gamma) \subset B(w, \delta/4)$. Let $Q := \operatorname{span}_{\mathbb{Q}}\{v_n : n \in \mathbb{N}\}$. By $(*)$ we have a bounded linear mapping $\hat{T}\colon Q \to Z$ such that $\hat{T}(q) = f'(x, q)$ for each $q \in Q$. By the density of Q, $\hat{T}$ extends to a bounded linear mapping $T\colon X \to Y$. We must show that $f'(x, v) = T(v)$ for each $0 \neq v \in X$. Given $0 \neq v \in X$ and $\varepsilon > 0$, by the density of Q and continuity of T, there exists $q \in Q$ such that

$$\|v - q\| < \frac{\varepsilon}{9L} \quad \text{and} \quad \|T(v - q)\| < \frac{\varepsilon}{3}. \tag{3.6}$$

By the existence of $f'(x, q)$ and by the differentiability of the distance function $d_G(x)$ at the point x, there exists $\tau_\varepsilon > 0$ such that

$$\Big\| \frac{f(x + tq) - f(x)}{t} - f'(x, q) \Big\| < \frac{\varepsilon}{3}, \tag{3.7}$$

$x + tv \in B(x, \gamma)$, and $d_G(x + tv) < \frac{\varepsilon}{9L}|t|$ for each $0 < |t| < \tau_\varepsilon$. Let $0 < |t| < \min(\tau_\varepsilon, 9\gamma L/2\varepsilon)$ and let $y \in G \cap B(w, \delta/4)$ be such that $\|x + tv - y\| < \frac{\varepsilon}{9L}|t|$. Then $\|x + tq - y\| \leq \frac{2\varepsilon}{9L}|t|$. Thus we have

$$\Big\| \frac{f(x + tv) - f(x + tq)}{t} \Big\| \leq \Big\| \frac{f(x + tv) - f(y)}{t} \Big\| + \Big\| \frac{f(x + tq) - f(y)}{t} \Big\| \leq \frac{\varepsilon}{3}. \tag{3.8}$$

Now since $f'(x, q) = T(q)$, by (3.6), (3.7), and (3.8) it follows that

$$\Big\| \frac{f(x + tv) - f(x)}{t} - T(v) \Big\| \leq \Big\| \frac{f(x + tq) - f(x)}{t} - f'(x, q) \Big\| + \Big\| \frac{f(x + tv) - f(x + tq)}{t} \Big\| + \|T(v - q)\| \leq \varepsilon,$$

for each $0 < |t| < \tau_\varepsilon$. This proves that $f'(x, v)$ exists and $f'(x, v) = T(v)$. Thus f is Gâteaux differentiable at x.

Since there exist $w_k \in D$ such that $D = \bigcup_k (D \cap B(w_k, \delta/4))$, let $R = \bigcup_k R'_{w_k}$ we have that R is Borel and since

$$(N_f \cap D) \setminus R = \Big(\bigcup_k ((N_f \cap D_{w_k}) \setminus R'_{w_k}) \Big) \setminus R = \Big(\bigcup_k \big(D_{w_k} \setminus H_{w_k} \big) \Big) \setminus R, \tag{3.9}$$

we also obtain that $(N_f \cap D) \setminus R$ is Borel (strictly speaking, the right-hand side of (3.9) depends on the complete sequence (v_n), but the left-hand side does not, so $(N_f \cap D) \setminus R$ is indeed Borel since a complete sequence in X clearly exists by the separability of X).

Since we have the following simple observation: if $A \in \tilde{\mathcal{A}}(v)$ and $B \subset X$ is Borel, then $A \setminus B \in \tilde{\mathcal{A}}(v)$; we can conclude that $(N_f \cap D) \setminus R$ is indeed in $\tilde{\mathcal{A}}$. ∎

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4 Main Theorem

Theorem 4.1 *Let X be a separable Banach space and let Y be a Banach space with the RNP. Given $f\colon X \to Y$, let $S(f)$ be the set of all points $x \in X$ at which f is pointwise Lipschitz. Then there exists a set $E \in \tilde{\mathcal{A}}$ such that f is Gâteaux differentiable at every point of $S(f) \setminus E$.*

Proof We follow the proof from [2]. For each $n \in \mathbb{N}$ let G_n be the set of all $x \in X$ such that $\|f(x+h) - f(x)\| \le n\|h\|$ whenever $\|h\| < \frac{1}{n}$. Lemma 2.4 implies that each G_n is closed, and $S(f) = \bigcup_n G_n$. Since the distance function $d_{G_n}(x)$ is Lipschitz on X, by [8, Theorem 12] there exists a Borel set M_n such that $X \setminus M_n \in \tilde{\mathcal{A}}$ and $d_{G_n}(x)$ is Gâteaux differentiable on M_n. Let $D_n := G_n \cap M_n$. Thus, in particular, $G_n \setminus D_n \in \tilde{\mathcal{A}}$. By Ω_n denote the set of all points $x \in D_n$ at which f is not Gâteaux differentiable. By Lemma 3.5 applied to D_n we obtain $R_n \in \tilde{\mathcal{A}}$ such that $\Omega_n \setminus R_n \in \tilde{\mathcal{A}}$.

Define $E := (\bigcup_n (\Omega_n \setminus R_n) \cup R_n) \cup (\bigcup_n \big(G_n \setminus D_n\big))$. Then $E \in \tilde{\mathcal{A}}$ by the previous paragraph. If $x \in S(f) \setminus E$, then there exists $n \in \mathbb{N}$ such that $x \in G_n \setminus E$. The condition $x \notin E$ implies that $x \notin G_n \setminus D_n$ and $x \notin \Omega_n$. Therefore $x \in D_n \setminus \Omega_n$, and hence f is Gâteaux differentiable at x. ∎

Corollary 4.2 *Let X be a Banach space with X^* separable, Y be a Banach space with RNP, $f\colon X \to Y$ be pointwise Lipschitz outside some set $C \in \tilde{\mathcal{C}}$ (or even some set D which is Γ-null), $g\colon X \to \mathbb{R}$ be continuous convex. Then there exists a point $x \in X$ such that f is Gâteaux differentiable at x and g is Fréchet differentiable at x.*

Proof Assume that f is pointwise Lipschitz outside some $C \in \tilde{\mathcal{C}}$. By Theorem 4.1, there exists $A \in \tilde{\mathcal{A}}$ such that f is Gâteaux differentiable at each $x \in X \setminus (A \cup C)$. By [7, Corollary 3.11] there exists a Γ-null $B \subset X$ such that g is Fréchet differentiable at each $x \in X \setminus B$. Since $A \cup C$ is Γ-null by [12, Theorem 2.4], we have that $A \cup B \cup C$ is Γ-null and thus there exists $x \in X \setminus (A \cup B \cup C)$.

If f is pointwise Lipschitz outside a Γ-null set D, then the proof proceeds similarly. ∎

5 Cone Monotone Functions

Lemma 5.1 *Let X be a Banach space and $K \subset X$ a closed convex cone with $0 \ne v \in \operatorname{int}(K)$, and let $f\colon X \to \mathbb{R}$ be K-monotone. If $\limsup_{t\to 0} |t|^{-1}|f(x+tv) - f(x)| < \infty$, then f is pointwise-Lipschitz at x.*

Proof Without any loss of generality, we can assume that $v + B(0,1) \subset K$; then the proof is identical to the proof of [6, Lemma 2.5] (note that there we assume that f is Gâteaux differentiable at x, but, in fact, we are only using that f satisfies $\limsup_{t\to 0} |t|^{-1}|f(x+tv) - f(x)| < \infty$). ∎

Let $(X, \|\cdot\|)$ be a normed linear space. We say that $\|\cdot\|$ is *LUR at* $x \in S_X$ provided $x_n \to x$ whenever $\|x_n\| = 1$, and $\|x_n + x\| \to 2$. For more information about rotundity and renormings, see [5].

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Lemma 5.2 *Let X be a separable Banach space, $K \subset X$ be a closed convex cone, $v \in \operatorname{int}(K) \cap S_X$. Then there exists a norm $\|\cdot\|_1$ on X which is LUR at v, $x^* \in (X, \|\cdot\|_1)^*$ with $x^*(v) = \|v\|_1 = \|x^*\| = 1$, and $\alpha \in (0,1)$ such that*

$$K_1 := \{x \in X : \|x\|_1 \leq \alpha x^*(x)\}$$

is contained in K.

Proof The conclusion follows from [5, Lemma II.8.1] (see the proof of [6, Proposition 15]). ■

Lemma 5.3 *Let X be a Banach space, $v \in S_X$, $x^* \in X^*$ such that $\|v\| = \|x^*\| = x^*(v) = 1$, $\alpha \in (0,1)$. Let $K_{\alpha,x^*} = \{x \in X : \alpha\|x\| \leq x^*(x)\}$. Then there exists $\varepsilon = \varepsilon(K, v) \in (0,1)$ such that if $\varphi\colon \mathbb{R} \to X$ is a mapping such that $\psi\colon t \to \varphi(t) - tv$ has Lipschitz constant less than ε, then $s < t$ implies $\varphi(s) \leq_{K_{\alpha,x^*}} \varphi(t)$.*

Proof Since $x^*(v) = 1$, for each $\alpha < \alpha' < 1$ we have $v \in \operatorname{int}(K_{\alpha',x^*})$. Fix $\alpha' \in (\alpha, 1)$. Let $\varepsilon := \min\big(1, \frac{(\alpha'-\alpha)}{2\alpha'(1+\alpha)}\big)$. Take $s < t$, $s, t \in \mathbb{R}$. Then

$$
\begin{aligned}
(5.1)\qquad \alpha'\|\varphi(t) - \varphi(s)\| &\leq \alpha'\|\varphi(t) - tv - (\varphi(s) - sv)\| + \alpha'|t - s|\|v\| \\
&\leq \alpha'\varepsilon|t - s| + |t - s|x^*(v) \\
&= \alpha'\varepsilon|t - s| + x^*(tv - \varphi(t) - (sv - \varphi(s)) \\
&\qquad + x^*(\varphi(t) - \varphi(s)) \\
&\leq \alpha'\varepsilon|t - s| + \|tv - \varphi(t) - (sv - \varphi(s))\| \\
&\qquad + x^*(\varphi(t) - \varphi(s)) \\
&\leq (1 + \alpha')\varepsilon|t - s| + x^*(\varphi(t) - \varphi(s)).
\end{aligned}
$$

As in (5.1), we show that $x^*(tv - \varphi(t) - (sv - \varphi(s))) \leq \varepsilon|t - s|$, and from this we obtain $|t - s|(x^*(v) - \varepsilon) \leq x^*(\varphi(t) - \varphi(s))$. Then (5.1) implies that

$$\alpha'\|\varphi(t) - \varphi(s)\| \leq \Big(1 + \frac{(1 + \alpha')\varepsilon}{1 - \varepsilon}\Big) x^*(\varphi(t) - \varphi(s)).$$

The choice of ε shows that $\alpha\|\varphi(t) - \varphi(s)\| \leq x^*(\varphi(t) - \varphi(s))$, and therefore $\varphi(t) \geq_{K_{\alpha,x^*}} \varphi(s)$. ■

We prove the following theorem, which improves [4, Theorem 9]:

Theorem 5.4 *Let X be a separable Banach space, $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Suppose that $f\colon X \to \mathbb{R}$ is K-monotone. Then f is Gâteaux differentiable on X except for a set belonging to $\tilde{\mathcal{C}}$.*

Remark It is not known whether $\tilde{C} \subset \tilde{A}$ (see [8, p. 19]). If it is true, then Theorem 5.4 holds also with $\tilde{A}$ instead of $\tilde{\mathcal{C}}$.

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Proof Without any loss of generality, we can assume that f is K-increasing and lower semicontinuous (we can work with $\underline{f}$ instead by [4, Proposition 17 and Proposition 16(iii)], where $\underline{f}(x) = \sup_{\delta>0} \inf_{z\in B(x,\delta)} f(z)$ is the l.s.c. envelope of f). By Lemma 5.2, we can also assume that the norm on X is LUR at $v \in S_X$ and $K = K_{\alpha,x^*} = \{x \in X : \|x\| \le \alpha x^*(x)\}$ for some $x^* \in X^*$ and $\alpha \in (0,1)$ with $\|x^*\| = x^*(v) = 1$.

Find $\eta > 0$ such that $B(v,\eta) \subset \operatorname{int}(v/2 + K_{\alpha,x^*})$ (such an η exists since obviously $v \in \operatorname{int}(v/2 + K_{\alpha,x^*})$). Let $x \in X$ be such that $\|x\| = 1$ and $\beta\|x\| \le x^*(x)$ for some $0 < \beta < 1$. Since $1+\beta = 1+\beta\|x\| \le x^*(v)+x^*(x) \le \|x+v\|$, and the norm on X is LUR at v, there exists $\beta' \in (\alpha,1)$ such that $K_{\beta',x^*} \cap S(0,1) \subset B(v,\eta) \subset v/2 + K_{\alpha,x^*}$ and thus

$$K_{\beta',x^*} \cap S(0,t) \subset B(tv,\eta t) \subset tv/2 + K_{\alpha,x^*} \tag{5.2}$$

for each $t > 0$. Put $B := \left\{x \in X : \limsup_{t\to 0} \frac{|f(x+tv)-f(x)|}{|t|} = \infty\right\}$. Then Lemma 5.1 shows that $S(f) = X \setminus B$, and Lemma 2.4 shows that B is Borel. We will show that $B \in \tilde{\mathcal{A}}(v)$. Let $\varphi\colon \mathbb{R} \to X$ be a mapping such that $\psi(t) = \varphi(t) - tv$ has Lipschitz constant strictly less than $\varepsilon > 0$, where ε is given by application of Lemma 5.3 to K_{β',x^*}. Suppose that $r \in \mathbb{R}$ satisfies $\varphi(r) = x \in B$. Without any loss of generality, we can assume that there exist $t_k \to 0+$ such that $\frac{f(x+t_kv/2)-f(x)}{t_k/2} \ge k$ (otherwise work with $-f(-\,\cdot))$. For each k, find $r_k \in \mathbb{R}$ such that $\varphi(r_k) \in (x+K_{\beta',x^*}) \cap S(x,t_k)$. Such r_k exist since $\varphi(r) = x$, $\|\varphi(s)\| \to \infty$ as $s \to \infty$, and $\varphi(u) \in (x+K_{\beta',x^*})$ by the choice of ε. Then (5.2) implies that $\varphi(r_k) \ge_{K_{\alpha,x^*}} x+t_kv/2$, and thus $f(\varphi(r_k)) \ge f(x+t_kv/2)$. Now, since ψ is ε-Lipschitz, we have $(1-\varepsilon)|r-r_k| \le \|\varphi(r_k)-\varphi(r)\| = t_k$, and thus

$$k \le \frac{f(x+t_kv/2)-f(x)}{t_k/2} \le \frac{2}{1-\varepsilon} \cdot \frac{f(\varphi(r_k))-f(\varphi(r))}{r-r_k}.$$

It follows that $f\circ\varphi$ is not pointwise Lipschitz at r. By the choice of ε and Lemma 5.3, we have that $f \circ \varphi$ is monotone; thus $\lambda(\{r \in \mathbb{R} : \varphi(r) \in B\}) = 0$ (since monotone functions from $\mathbb{R}$ to $\mathbb{R}$ are known to be a.e. differentiable), and $B \in \tilde{\mathcal{A}}(v,\varepsilon/2)$.

We proved that $B \in \tilde{\mathcal{A}}(v)$. By Lemma 5.1 we have that $S(f) = X \setminus B$. By Theorem 4.1, there exists a set $A \in \tilde{\mathcal{A}}$ such that f is Gâteaux differentiable at all $x \in X \setminus (A \cup B)$. In [4, Theorem 9] it is proved that the set N_f of points of Gâteaux non-differentiability of f is Borel, and thus we obtain that $N_f \in \tilde{C}$ (since $N_f \subset A \cup B$). ∎

Theorem 5.4 and [4, Proposition 16(iv)] show the following.

Corollary 5.5 *Let X be a separable Banach space and $K \subset X$ a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Suppose that f is K-monotone. Then f is Hadamard differentiable outside of a set belonging to $\tilde{\mathcal{C}}$.*

We also have the following corollary.

Corollary 5.6 *Let X be a Banach space with X^* separable and $K \subset X$ a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Let $f\colon X \to \mathbb{R}$ be K-monotone and $g\colon X \to \mathbb{R}$ continuous convex. Then there exists a point $x \in X$ such that f is Hadamard differentiable at x and g is Fréchet differentiable at x.*

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Proof By Corollary 5.5, there exists $A \in \tilde{\mathcal{C}}$ such that f is Hadamard differentiable at each $x \in X \setminus A$. By [7, Corollary 3.11] there exists a Γ-null $B \subset X$ such that g is Fréchet differentiable at each $x \in X \setminus B$. Since A is Γ-null by [12, Theorem 2.4], we have that $A \cup B$ is Γ-null and thus there exists $x \in X \setminus (A \cup B)$. ■

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