A GENERAL PERRON INTEGRAL

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1. Introduction and notation. In this paper integrals are considered from the point of view of inverting differential operators. In order to do this it is necessary to introduce integrals more general than the Lebesgue integral and these integrals turn out to have other interesting properties (**6**, **7**, **12**). The integral introduced here is defined in the setting of axiomatic potential theory (**2**, **4**). By defining it as generally as possible it not only includes the James P^2-integral but inverts many of the standard second-order differential operators.

In this section the concepts of potential theory are introduced very quickly; details can be found in the references (**2**, **4**). In particular the notation will follow that of Bauer (**2**). A generalized second-order derivative is introduced in the second section, and it is used to obtain necessary and sufficient conditions for a function to be hyperharmonic. In Sections 3 and 4 the general Perron integral that inverts this derivative is defined. Finally in Section 5 some examples are given; it is intended to develop these further in a later paper. At the beginning of each of Sections 2–4, the general conditions that are to hold in the section are stated and are to be understood as a part of the hypotheses of any of the results proved in that section. Extra restrictions introduced in the course of a section will be stated explicitly each time they are used.

Functions will be defined on a locally compact space X with values in the extended real line. On X there is a harmonic structure as defined by Bauer (**2**). $\mathfrak{U}$ will denote the family of non-empty open sets of X, and U, with or without indices, will always denote a member of $\mathfrak{U}$. For every set U, $\mathfrak{H}(U)$ will be the set of real (that is finite-valued) continuous functions on U, called the *harmonic functions on* U. $\mathfrak{H}(U)$ satisfies

AXIOM H. *The mapping* $U \to \mathfrak{H}(U)$, *defined on* $\mathfrak{U}$, *is a sheaf.*

$\mathfrak{U}$ contains a subfamily $\mathfrak{V}$ of *regular sets* with the properties that for all $V \in \mathfrak{V}$: (a) V is relatively compact; (b) the boundary of V, denoted by V^*, is not empty; (c) if f is a real continuous function on V^*, written $f \in \mathfrak{C}(V^*)$, then there exists a unique continuous extension of f to $\bar{V}$, whose restriction to V, H_f, is harmonic in V; further if $f \geqslant 0$, then $H_f \geqslant 0$. V, with or without indices, will always denote a member of $\mathfrak{V}$. If $\bar{V} \subset U$, then V will be said to be *regular in* U, written $V \in \mathfrak{V}(U)$. $\mathfrak{V}$ satisfies

AXIOM B. $\mathfrak{V}$ *is a base for the topology of* X.

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Bauer (**2**, §1.2) shows that for all V and all $x \in V$ there is a non-negative Radon measure, the *harmonic measure*, $\mu(V; x)$, supported by V^*, and such that

$$H_f(x) = \int f d\mu(V; x), \qquad \text{for all } f \in \mathfrak{C}(V^*).$$

If u is a numerical (that is, not necessarily finite-valued) function on U, then u is said to be *hyperharmonic on* U, $u \in \mathfrak{H}^*(U)$, if (a) $u > -\infty$; (b) u is lower semi-continuous (l.s.c.); (c) for all $V \in \mathfrak{V}(U)$ and $f \in \mathfrak{C}(V^*)$, $f(z) \leqslant u(z)$ for all $z \in V^*$ implies $H_f \leqslant u$. Bauer (**2**) has proved

THEOREM 1. *Let u be a l.s.c. numerical function on U with $u > -\infty$; then $u \in \mathfrak{H}^*(U)$ if and only if for every $V \in \mathfrak{V}(U)$ and every $x \in V$*

$$\int^* u \, d\mu(V; x) \leqslant u(x).$$

COROLLARY 2. *If $u \in \mathfrak{C}(U)$, then $u \in \mathfrak{H}(U)$ if and only if for every $V \in \mathfrak{V}(U)$ and every $x \in V$, $\int u \, d\mu(V; x) = u(x)$.*

This defines the harmonic structure and using two more axioms, T and K below, Bauer is able to extend to this situation many of the results of classical potential theory.

Let $t \in \mathfrak{C}(X)$ and $t > 0$; then a function u on U is said to be *t-harmonic on U* if $ut \in \mathfrak{H}(U)$. This class of functions defines another harmonic structure on X with the same regular sets. Further, the t-hyperharmonic functions are just those functions with $ut \in \mathfrak{H}^*(U)$ and the t-harmonic measure is $t\mu(V; x)/t(x)$ (**2**). If $t \in \mathfrak{H}(X)$, then the constant functions are t-harmonic on X.

AXIOM T. *There exists a positive function t, harmonic on X, such that the t-hyperharmonic functions on X separate points of X.*

Throughout the paper t will denote a positive harmonic function whose existence is assured by Axiom T. Let $\mathfrak{F}$ be a non-empty set of functions harmonic on U that is filtered to the right; then we state

AXIOM K. *If $\mathfrak{F}$ is bounded above, then* $\sup \mathfrak{F} \in \mathfrak{H}(U)$.

A generalization of H_f will now be defined. Suppose that U is relatively compact, then for a numerical function f on U^* define

$$\uparrow H_f = \inf\{u; u \text{ bounded below}, u \in \mathfrak{H}^*(U), \text{ and } \liminf_{x \to z,\, x \in U} u(x) \geqslant f(z), \text{ for all } z \in U^*\},$$

and $\downarrow H_f = -\uparrow H_{-f}$. f is said to be *resolutive on* U^* if $\uparrow H_f$ is real and $\uparrow H_f = \downarrow H_f$; this function is written H_f. If, further, H_f is harmonic, then f is said to be *harmonically resolutive on* U^*. If Axioms T and K are assumed, then all continuous functions are harmonically resolutive (**2**, Theorems 14 and 24). Then for all relatively compact U and all $x \in U$ there exists a non-negative harmonic measure, $\mu(U; x)$, supported by U^*, such that

$$H_f(x) = \int f\, d\mu(U; x) \text{ for all } f \in \mathfrak{C}(U^*).$$

From Bauer (**2**, Lemma 13, Theorem 19 and 7.6–7.9), we have

THEOREM 3.

(a) (i) *If $f \leqslant g$, then $\uparrow H_f \leqslant \uparrow H_g$.*
(ii) *For every positive real number a, $\uparrow H_{af} = a \uparrow H_f$.*
(iii) $\uparrow H_{f+g} \leqslant \uparrow H_f + \uparrow H_g$.
(iv) $\downarrow H_f \leqslant \uparrow H_f$.

(b) *If $V \in \mathfrak{B}(U)$, then for every numerical function f on V^* and all $x \in V$,*

$$\uparrow H_f(x) = \int^* f d\mu(V; x).$$

If U is relatively compact, then a set $A \subset U^*$ is said to be *negligible* if $\uparrow H_f = 0$, where f is the characteristic function of A. If a property holds on U^* except on a negligible set, it is said to hold *nearly everywhere*. In particular if two functions, f and g, are equal nearly everywhere on U^*, then $\uparrow H_f = \uparrow H_g$ (**2**, Lemma 5). An enumerable union of negligible sets is negligible.

If U is relatively compact and $z \in U^*$, then z is a *regular point* if

$$\lim_{x \to z,\ x \in U} H_f(x) = f(z) \text{ for all } f \in \mathfrak{C}(U^*).$$

A relatively compact set U is regular if and only if z is regular for all $z \in U^*$. If z is regular, then for any resolutive function f,

$$\lim_{x \to z,\ x \in U} H_f(x) = f(z); \qquad \text{cf. } (\mathbf{2}, \text{III}).$$

For each $x \in X$ let $\mathfrak{N}(x)$ be a fundamental system of regular neighbourhoods of x. Using Corollary 2, a class of $\mathfrak{N}$-*harmonic functions* can be defined (**2**, Definition 3). The theory of these functions, which parallels the above discussion, is an apparent generalization. However, it can be shown that the $\mathfrak{N}$-harmonic structure is just the original harmonic structure (**2**, Theorem 5). Throughout the paper, $\mathfrak{N}$ will denote such a map on X.

2. A generalized derivative. In this section, X is assumed to be a locally compact metric space, with the metric denoted by r and diameters of sets by R. Further, X is assumed to have a harmonic structure as defined in the previous section and satisfying all the axioms introduced there.

If f is a numerical function on X, $x \in X$ and $V \in \mathfrak{N}(x)$, then

$$\uparrow \Delta f(x; V) = \int^* f d\mu(V; x) - f(x) = \uparrow H_\phi(x) - f(x), \tag{1}$$

where ϕ is the restriction of f to V^*, is the formal definition of an *upper difference operator*. In order that this be meaningful, both $\uparrow H_\phi(x)$ and $f(x)$ cannot be ∞, or both $-\infty$. In particular, $\uparrow H_\phi(x)$ is certainly finite if f is resolutive on V^*.

A similar definition can be made for a *lower difference operator* $\downarrow \Delta f(x; V)$. When both are defined, then $\downarrow \Delta f(x; V) \leqslant \uparrow \Delta f(x; V)$. If they are equal, the common value will be written $\Delta f(x;\, V)$. In particular, this is the case when f is resolutive on V^*.

Using (1) the *generalized upper derivative of f at x (relative to $\mathfrak{N}$)* is

$$\uparrow Df(x;\mathfrak{N}) = \limsup_{\mathfrak{N}(x)} \frac{\uparrow\Delta f(x;\, V)}{R^2(V)}. \tag{2}$$

This upper derivative is properly defined if either (a) $f(x)$ is finite, or (b) $f(x) = \infty\ (-\infty)$ and for all $V \subset V_0$, V and V_0 in $\mathfrak{N}(x)$, $\uparrow H_\phi(x) \neq \infty\ (-\infty)$, where ϕ is the restriction of f to V^*. When no ambiguity can result, the letter $\mathfrak{N}$ will be omitted from the left-hand side of (2).

By replacing lim sup and $\uparrow\Delta$ by lim inf and $\downarrow\Delta$, respectively, we define the *generalized lower derivative* $\downarrow Df(x)$. If both derivatives are defined, then $\downarrow Df(x) \leqslant \uparrow Df(x)$, and if they are equal, the common value will be written $Df(x)$.

The above definitions can be introduced in the t-harmonic structure. In that case the notation will be $\uparrow\Delta f(x;\, V;t)$, $\uparrow Df(x;\,\mathfrak{N};t)$, etc. It can easily be shown by using the results of the first section that

$$\uparrow\Delta f(x;\, V;t) = \frac{1}{t(x)}\uparrow\Delta(tf)(x;\, V)$$

if either side is defined, with similar results for the derivatives.

Finally, if f is not defined on the whole of X but only on some U, all the V that occur in the above definitions must be taken to be regular in U.

THEOREM 4. *If $f \in \mathfrak{H}^*(X)$, then for any $\mathfrak{N}$ for which $\uparrow Df(x;\,\mathfrak{N})$ is defined, $\uparrow Df(x;\,\mathfrak{N}) \leqslant 0$.*

The proof follows immediately from the definitions and Theorem 1.

To obtain a converse of Theorem 4 we need

AXIOM 1. *There is a real function v, hyperharmonic on X, for which $\uparrow Dv(x;\,\mathfrak{N}) < 0$, for all $x \in X$ and all $\mathfrak{N}$.*

THEOREM 5. *If Axiom 1 is assumed and f is a numerical l.s.c. function on X such that for some $\mathfrak{N}$ and all $x \in X$, $\downarrow Df(x;\,\mathfrak{N}) \leqslant 0$, then $f \in \mathfrak{H}^*(X)$.*

Proof. For simplicity it will be assumed that for all x $\mathfrak{N}(x)$ consists of all regular sets containing x.

The proof will require the positive constants to be hyperharmonic. There will be no loss in generality in assuming this, or even that constants are harmonic. Let us suppose that Theorem 5 has been proved under this last assumption. The hypotheses of Theorem 5 imply that if $f' = f/t$, then for some $\mathfrak{N}$, $\downarrow Df'(x;\,\mathfrak{N};t) \leqslant 0$, for all $x \in X$. Hence, since constants are t-harmonic, $f' \in t - \mathfrak{H}^*(X)$; but this is equivalent to $f \in \mathfrak{H}^*(X)$.

It is also sufficient to prove the theorem with the inequality assumed to be strict. Let us suppose that this weaker form of Theorem 5 has been proved and let v be the real hyperharmonic function whose existence is assured by Axiom 1. If then $g_n = v/n$, $n = 1, 2, \ldots, g_1, g_2, \ldots$ is a sequence of hyperharmonic functions that converges everywhere to zero; further $\uparrow Dg_n = \uparrow Dv/n < 0$, for all n. Suppose f satisfies the conditions of Theorem 5. Then $\downarrow D(f + g_n) \leqslant \downarrow Df + \uparrow Dg_n < 0$. Hence, by the above assumption, $f + g_n \in \mathfrak{H}^*(X)$, for all n. So

$$\uparrow\Delta f(x, V) \leqslant \uparrow\Delta(f + g_n)(x; V) - \downarrow\Delta g_n(x; V) \leqslant -(1/n)\downarrow\Delta v(x; V),$$

for all n, all V, and all $x \in V$. Hence $\uparrow\Delta f(x; V) \leqslant 0$ for all V and all $x \in V$, which implies that $f \in \mathfrak{H}^*(X)$, using Theorem 1.

Now let us commence the main part of the proof, subject to these extra conditions. Since $\downarrow Df(x)$ exists and is not ∞ it follows that $f > -\infty$. Hence it is sufficient, from the definition of hyperharmonic functions, to show that for all V and all $v \in \mathfrak{C}(V^*)$, $v(z) \leqslant f(z)$ for all $z \in V^*$ implies $H_v \leqslant f$.

Suppose it does not. Then there exists an x_1, a $V_1 \in \mathfrak{N}(x_1)$, and a $h_1 \in \mathfrak{C}(\overline{V_1})$ such that (i) the restriction of h_1 to V_1 is harmonic in V_1, (ii) $h_1(z) < f(z)$ for all $z \in V_1^*$, and (iii) $h_1(x_1) > f(x_1)$. In fact we can take h_1 to be equal to v on V_1^* and H_v in V_1 for some suitably chosen $v \in \mathfrak{C}(V_1^*)$.

Write $g(x) = f(x) - h_1(x)$ for $x \in \overline{V_1}$. Then (a) g is l.s.c., (b) $g > -\infty$, (c) $g(z) > 0$ for all $z \in V_1^*$, (d) $g(x_1) < 0$, (e) $\downarrow Dg(x) = \downarrow Df(x) < 0$ for all $x \in V_1$.

By its first four properties g assumes a finite negative minimum at a point $x_2 \in V_1$. Choose then $V \in \mathfrak{B}(V_1) \cap \mathfrak{N}(x_2)$ and we have

$$\downarrow\Delta g(x_2; V) = \downarrow H_g(x_2) - g(x_2) \geqslant g(x_2)\Delta(1; V) = 0$$

by Theorem 3(a) and the assumption that $1 \in \mathfrak{H}(X)$. Hence at x_2, $\downarrow Dg(x_2) \geqslant 0$, which contradicts property (*e*).

This completes the proof of the theorem.

COROLLARY 6. (i) *If Axiom* 1 *is assumed and* $f \in \mathfrak{C}(X)$ *is such that for some* $\mathfrak{N}$, $\downarrow Df(x; \mathfrak{N}) \leqslant 0 \leqslant \uparrow Df(x; \mathfrak{N})$ *for all* $x \in X$, *then* $f \in \mathfrak{H}(X)$.

(ii) *If Axiom* 1 *is not assumed, the conclusion of Theorem* 5 *is still valid if the inequality there is taken to be strict.*

Proof. The proof of (i) is an immediate consequence of Theorem 5 and part (ii) follows from the proof of that theorem.

In applying these results to the definition of an integral it is of importance to allow an exceptional enumerable set on which the conditions are weaker than those of the above theorem. In order to get a result of this type, the space X must be further restricted. Assume now that X is a finite-dimensional vector space over the reals with a scalar product denoted by $\langle x, y\rangle$. Then

$$r(x, y) = |x - y| = \langle x - y, x - y\rangle^{\frac{1}{2}}.$$

If f is a numerical function on X define

$$f'(x, y) = \lim_{a \to 0} \frac{f(x + ay) - f(x)}{a}$$

whenever the limit exists. If $f'(x, y)$ exists for all (x, y), is continuous in x for each y, and is a linear functional in y for each x, then f will be said to be *continuously differentiable*. Then for each $x \in X$ there exists a unique point of X, denoted by $\nabla f(x)$, such that $f'(x, y) = \langle \nabla f(x), y \rangle$, for all $y \in X$.

Assume further that the sets of all the $\mathfrak{N}(x)$ are convex. Then we shall say that $\mathfrak{N}$ is *convex*. If x, y are distinct points of X and $V \in \mathfrak{N}(x)$, let y_V be that point of V^* of the form $ax + (1 - a)y$, for some $a > 0$. Then define

$$\downarrow df(x) = \inf_{y} \liminf_{\mathfrak{N}(x)} \left\{ \frac{f(y_V) - f(x)}{r(x, y_V)} \right\}.$$

If f is continuously differentiable, then simple calculations show that

$$\downarrow df(x) = \inf_{y} \left\{ \pm f'\left(x, \frac{x - y}{|x - y|} \right) \right\} = - |\nabla f(x)|.$$

Now let the following axioms be assumed.

Axiom 2. *If $h \in t - \mathfrak{H}(X)$, then h is continuously differentiable.*

Axiom 3. *There exists an $h \in t - \mathfrak{H}(X)$ such that for all $x \in X$, $|\nabla h(x)| \neq 0$.*

Axiom 4. (i) *For each $x \in X$ there exists a $V' \in \mathfrak{N}(x)$ and non-negative Radon measures $\mu_1(x)$, $\mu_2(x)$ such that if f is a non-negative numerical function on X and $V \subset V'$, $V \in \mathfrak{N}(x)$, then*

$$\begin{aligned} \int_{*V^*} f(z) d\mu(V; x)(z) &= \int_{*V^*} f(ax + (1 - a)z_{V'}) d\mu(V; x)(z) \\ &\geqslant \int_{*V'^*} f(ax + (1 - a)z_{V'}) d\mu_1(x)(z_{V'}) \\ &\qquad + s(x; V) \int_{*V'^*} f(ax + (1 - a)z_{V'}) d\mu_2(x)(z_{V'}) \end{aligned}$$

where

$$s(x; V) = \min_{z \in V^*} r(x, z).$$

(ii) *If $A \subset V^*$ and $\mu_1(A) = 0$, then A is non-dense in V^*.*

This axiom expresses the dependence of $\mu(V; x)$ on V. It is satisfied in particular if $\mu(V; x)$ does not depend on V; that is, abridging the above notation, if $\mu(V; x) = \mu_1(x)$ instead of $\mu(V; x) \geqslant \mu_1(x) + s(x; V)\mu_2(x)$, as above. This particular case occurs classically when, for all x and V, $\mu(V; x)$ is the uniform distribution of unit mass on the sphere V of centre x.

THEOREM 7. *If Axioms* 1, 2, 3, *and* 4 *are assumed and f is a numerical l.s.c. function on the finite-dimensional vector space X such that for some convex $\mathfrak{N}$*

(*i*) $\downarrow Df(x; \mathfrak{N}) \leqslant 0$ *for all* $x \in X - E$, E *an enumerable set*,

(*ii*) $\liminf_{\mathfrak{N}(x)} \left\{ \dfrac{\downarrow \Delta f(x; V)}{s(x; V)} \right\} \leqslant 0$ *for all* $x \in E$,

then $f \in \mathfrak{H}^*(X)$. *If Axiom* 1 *is not assumed, the same conclusion holds if the inequality in* (i) *is strict.*

Proof. The first part of the proof will follow closely that of Theorem 5. In particular $\mathfrak{N}(x)$ will be assumed to consist of all the regular sets containing x; constants will be assumed to be harmonic and so Axioms 2 and 3 will apply directly to the harmonic structure; and only the last part of the statement of the theorem will be proved.

If h is the function whose existence is assured by Axiom 3, redefine the function g of the proof of Theorem 5 as $f - h_1 + ah$ where $a > 0$ and is small enough not to change the properties (a)–(e) of g. Since the set E is enumerable, this number a can also be chosen so that

$$\downarrow df(x) \neq -|\nabla(h_1 - ah)(x)| \qquad \text{for all } x \in E.$$

The argument of Theorem 5 shows that g has a local minimum at x_2 and that $x_2 \in E$ and further that

$$\liminf_{\mathfrak{N}(x_2)} \frac{\downarrow \Delta g(x_2; V)}{s(x_2; V)} \geqslant 0.$$

But $\downarrow \Delta g(x_2; V) = \downarrow \Delta f(x_2; V)$, so by Hypothesis (ii)

$$0 = \liminf_{\mathfrak{N}(x_2)} \frac{\downarrow \Delta g(x_2; V)}{s(x_2; V)} = \liminf_{\mathfrak{N}(x_2)} \frac{1}{s(x_2; V)} \int_* \{g(z) - g(x_2)\} d\mu(V; x_2)(z).$$

Now apply the first part of Axiom 4 (the notation will be abbreviated somewhat, the full form being given in the statement of the axiom):

$$0 \geqslant \liminf_{\mathfrak{N}(x_2)} \int_* \frac{g(z) - g(x_2)}{r(z, x_2)} d\mu_1(x_2)(z) + \liminf_{\mathfrak{N}(x_2)} \int_* \{g(z) - g(x_2)\} d\mu_2(x_2)(z)$$
$$\geqslant \int_* \liminf_{\mathfrak{N}(x_2)} \frac{g(z) - g(x_2)}{r(z, x_2)} d\mu_1(x_2)(z) + 0 \geqslant 0,$$

by Fatou's Lemma, the fact that g is l.s.c. and x_2 a local minimum of g. Hence

(3) $$\liminf_{\mathfrak{N}(x_2)} \frac{g(z) - g(x_2)}{r(z, x_2)} = 0 \quad \text{for } \mu_1 - \text{almost all } z \in V^*,$$

and in any case the left-hand side of (3) is non-negative for all $z \in V^*$. That is,

(4) $$\liminf_{\mathfrak{N}(x_2)} \frac{f(z) - f(x_2)}{r(z, x_2)} = \left\langle \nabla(h_1 - ah)(x_2), \frac{x_2 - z}{|x_2 - z|} \right\rangle$$

for μ_1-almost all $z \in V^*$, and for all $z \in V^*$ the left-hand side of (4) is not less than the right-hand side. By Axioms 2 and 4(ii) this implies that $\downarrow df(x_2) = -|\nabla(h_1 - ah)(x_2)|$, which is a contradiction. This completes the proof of the theorem.

3. Major and minor functions. In this section X is assumed to be a finite-dimensional vector space with a harmonic structure as defined in Section 1 and satisfying the Axioms 1, 2, 3, and 4. A function $\mathfrak{N}$ is given and will be assumed to be convex. In particular, therefore, Theorem 7 can be applied. The discussion will occur in some fixed relatively compact open set U.

Let f be a numerical function defined on U. A function m defined on $\bar{U}$ is called a *minor function of f on U*, written $m \in \downarrow\mathfrak{M}(f)$, if and only if there exists an enumerable subset E of U such that

(i) $m \in \mathfrak{C}(\uparrow U)$,

(ii) $\uparrow Dm(x) < \infty$ if $x \in U - E$,

(iii) $\uparrow Dm(x) \leqslant f(x)$, if $x \in U - E$,

(iv) $\limsup\limits_{\mathfrak{N}(x_2)} \dfrac{\uparrow\Delta m(x; V)}{s(x; V)} \leqslant 0$, if $x \in E$,

(v) $m(z) \geqslant 0$ for nearly all $z \in U^*$.

If $-M \in \downarrow\mathfrak{M}(-f)$, then M is said to be a *major function of f on U*, $M \in \uparrow\mathfrak{M}(f)$.

LEMMA 8. *If $M \in \uparrow\mathfrak{M}(f)$ and $m \in \downarrow\mathfrak{M}(f)$, then*

(i) *$m - M$ is a real continuous hyperharmonic function on U,*

(ii) *$m \geqslant M$.*

Proof. (i) Note first that $\downarrow D(f - g) \leqslant \uparrow Df - \downarrow Dg$, and similarly that the same holds with D replaced by Δ, provided the terms are defined. If $x \in U - E$, then $\uparrow Dm - \downarrow DM$ is defined and hence $\downarrow D(m - M) \leqslant \uparrow Dm - \downarrow DM \leqslant 0$. If $x \in E$, then

$$\liminf_{\mathfrak{N}(x_2)} \frac{\downarrow\Delta(m - M)(x; V)}{s(x; V)} \leqslant 0.$$

Hence since $m - M \in \mathfrak{C}(\bar{U})$, Theorem 7 shows that $m - M \in \mathfrak{H}^*(U)$.

(ii) By part (i), $m - M \in \mathfrak{H}^*(U)$ and hence, by definition, $m - M \geqslant H_{m-M}$. But for nearly all $z \in U^*$, $m(z) - M(z) \geqslant 0$ and hence $H_{m-M} \geqslant 0$, by Theorem 3(a)(i). This completes the proof of the lemma.

A numerical function f on U is said to be *$\mathfrak{H}$-integrable on U*, written $f \in \mathfrak{J}(U)$, if given any $a > 0$ there exists an $m \in \downarrow\mathfrak{M}(f)$ and an $M \in \uparrow\mathfrak{M}(f)$ such that $0 \leqslant m(x) - M(x) \leqslant a$ for all $x \in \bar{U}$.

LEMMA 9. *If $f \in \mathfrak{J}(U)$, there exists a function F defined on U such that*

(i) $F = \sup\{M; M \in \uparrow\mathfrak{M}(f)\} = \inf\{m; m \in \downarrow\mathfrak{M}(f)\}$,

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(ii) $F \in \mathfrak{C}(\bar{U})$,

(iii) *for all* $M \in \uparrow\mathfrak{M}(f)$ *and all* $m \in \downarrow\mathfrak{M}(f)$ *the functions* $F - M$ *and* $m - F$ *are hyperharmonic in* U.

Proof. (i) Let $F = \inf\{m; m \in \downarrow\mathfrak{M}(f)\}$. Then, by Lemma 8, F is an upper bound of $\uparrow\mathfrak{M}(f)$. Given an $a > 0$, let $m \in \downarrow\mathfrak{M}(f)$ and $M \in \uparrow\mathfrak{M}(f)$ be chosen so that $0 \leqslant m(x) - M(x) \leqslant a$ for all $x \in \uparrow U$. Then $0 \leqslant F(x) - M(x) \leqslant a$ and so $F = \sup\{M; M \in \uparrow\mathfrak{M}(f)\}$.

(ii) and (iii) follow immediately from (i) and Lemma 8.

If $f, g, \ldots$ are $\mathfrak{H}$-integrable on U, then $F, G, \ldots$ will denote the function defined by Lemma 9 for $f, g, \ldots$ respectively.

It is important to allow an additional exceptional set in the definitions of $\downarrow\mathfrak{M}(f)$ and $\uparrow\mathfrak{M}(f)$. This set in the classical case is a set of measure zero **(7)**; it ensures that functions equal except on such sets are $\mathfrak{H}$-integrable together. Let $\mathfrak{Z}(U)$ denote the subsets Z of U for which there is associated a $v_Z \in \mathfrak{C}(\bar{U})$ whose restriction to U is hyperharmonic, such that $Dv_Z(x) = -\infty$ for all $x \in Z$. $\mathfrak{Z}(U)$ contains at least the empty set and a simple calculation shows that it is closed for finite unions. Further properties for $\mathfrak{Z}(U)$ will be assumed later, Axioms 5 and 6. Now modify Part (iii) of the definition of $\downarrow\mathfrak{M}(f)$ to read

(iii)′ $\uparrow Dm(x) \leqslant f(x)$ for all $x \in U - Z - E$, where $Z \in \mathfrak{Z}(U)$ and E is an enumerable subset of U.

This gives rise to new classes of major and minor functions, $\uparrow\mathfrak{M}'(f)$ and $\downarrow\mathfrak{M}'(f)$ say, and hence to a new class of integrable functions, $\mathfrak{J}'(U)$. Clearly $\mathfrak{J}(U) \subset \mathfrak{J}'(U)$ and we have

LEMMA 10. $\mathfrak{J}(U) = \mathfrak{J}'(U)$.

Proof. This is clearly true if $\mathfrak{Z}(U)$ only contains the empty set; so we can assume it to contain other sets. From the above remark it is sufficient to prove $\mathfrak{J}'(U) \subset \mathfrak{J}(U)$.

If $f \in \mathfrak{J}'(U)$ and $a > 0$, choose $m' \in \downarrow\mathfrak{M}'(f)$ and $M' \in \uparrow\mathfrak{M}'(f)$ so that $0 \leqslant m'(x) - M'(x) \leqslant \frac{1}{2}a$ for all $x \in \bar{U}$. Let Z be the union of the exceptional sets from $\mathfrak{Z}(U)$, and E the union of the two enumerable sets, associated with these two functions. Let v_Z, the function associated with Z, be so chosen that $\frac{1}{2}a \geqslant v_Z \geqslant 0$, as can clearly be done using Axiom T. Define

$$m(x) = m'(x) + v_Z(x), \qquad M(x) = M'(x) - v_Z(x) \quad \text{for all } x \in \bar{U}.$$

Then $0 \leqslant m(x) - M(x) \leqslant a$ and if it can be shown that $m \in \downarrow\mathfrak{M}(f)$ and $M \in \uparrow\mathfrak{M}(f)$, the proof will be complete. The conditions (i), (iv), and (v) are immediate. As $v_Z \in \mathfrak{H}^*(U)$, $\uparrow Dv_Z(x) \leqslant 0$ for all $x \in U$ and so since $\uparrow Dm \leqslant \uparrow Dm' + \uparrow Dv_Z$, Condition (ii) is also satisfied. The same inequality also shows that if $x \notin Z - E$, then Condition (iii) holds. If, however, $x \in Z - E$, then $\uparrow Dm(x) = -\infty \leqslant f(x)$, and so (iii) holds for all $x \in U - E$. A similar argument can easily be given for M.

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As a result of this lemma Condition (iii) will be replaced by (iii)′ and the class of integrable functions $\mathfrak{J}(U)$ by the class $\mathfrak{J}'(U)$. The prime in the notation will, however, be omitted.

LEMMA 11. *If $\mathfrak{H}'(U)$ and $\mathfrak{H}''(U)$ are two classes of harmonic functions on U and if $\mathfrak{H}'(U) \subset \mathfrak{H}''(U)$ and $t \in \mathfrak{H}'(U)$, then $\mathfrak{J}'(U) \subset \mathfrak{J}''(U)$.*

The proof is immediate from the definitions. However, note that if the regular sets associated with $\mathfrak{H}'(U)$ form a proper subset of those associated with $\mathfrak{H}''(U)$, the derivatives used must be defined using an $\mathfrak{N}$ for which all the $\mathfrak{N}(x)$ consist of sets from this proper subset, and wherever such an $\mathfrak{N}$ is used, it must always be so chosen.

4. The $\mathfrak{H}$-integral. In this section, X is assumed to satisfy the same conditions as those stated at the beginning of Section 3. The fixed $\mathfrak{N}$ is again assumed to be convex and the set U is now assumed to have nearly all of the points of U^* regular.

If $f \in \mathfrak{J}(U)$, and F as usual denotes the function defined by Lemma 9, then the *$\mathfrak{H}_0$-integral of f over U* is defined to be

$$F_0(x) = -F(x) = \int_{U,x} f, \qquad \text{for all } x \in U.$$

If Φ is a function on U^* that is nearly everywhere equal to a continuous function, the *$\mathfrak{H}_\Phi$-integral of f over U* is defined to be

$$F_\Phi(x) = F_0(x) + H_\Phi(x) = \int_{U,\Phi,x} f, \qquad \text{for all } x \in U.$$

THEOREM 12. *If f and g are $\mathfrak{H}$-integrable on U, then*

(i) *$f \geqslant g$ implies that $F_\Phi \geqslant G_\Phi$;*

(ii) *for all real numbers a, b, $af + bg \in \mathfrak{J}(U)$ and*

$$\int_{U,\Phi,x} (af + bg) = aF_\Phi(x) + bG_\Phi(x), \qquad \textit{for all } x \in \uparrow U;$$

(iii) *if also $|f| \in \mathfrak{J}(U)$, then*

$$\left|\int_{U,\Phi,x} f\right| \leqslant \int_{U,\Phi,x} |f|, \qquad \textit{for all } x \in \uparrow U;$$

(iv) *$F_\Phi \in \mathfrak{C}(U)$ and*

$$\lim_{x \to z,\ x \in U} F_\Phi(x) = \Phi(z),$$

at all the regular points of z of U^*, in particular therefore nearly everywhere.

Proof. These results are immediate consequences of the definitions, the assumed properties of U and U^*, and the results quoted in Section 1.

THEOREM 13. *Let $U_1 \subset$ be a relatively compact open set with nearly all the points of U_1^* regular. Then if $f \in \mathfrak{J}(U)$, this implies that $f_1 \in \mathfrak{J}(U_1)$, where f_1 is the restriction of f to U_1. Further, if $x \in U_1$,*

$$\int_{U_1,\Phi,x} f_1 = \int_{U,x} f + H_{\Phi+\phi}(x),$$

where ϕ is the restriction of F to U_1^*.

Proof. Since U_1 is relatively compact and $F \in \mathfrak{C}(\bar{U})$, H_ϕ is defined and harmonic. Let m, M be any minor and major function of f on U respectively. For all $x \in U_1$ define

$$m_1(x) = m(x) - H_\phi(x), \qquad M_1(x) = M(x) - H_\phi(x).$$

Then m_1 is a minor function of f_1 on U_1 and M_1 is a major function. The proof of this is immediate using Lemma 9 and the properties of regular points. Further, given any $a > 0$ if m and M are so chosen that $0 \leqslant m - M \leqslant a$, then $0 \leqslant m_1 - M_1 \leqslant a$. Hence

$$\begin{aligned} \int_{U_1,x} f = -\inf\{m_1; m \in \downarrow\mathfrak{M}(f)\} &= -\sup\{M_1; M \in \uparrow\mathfrak{M}(f)\} \\ &= \int_{U,x} f + H_\phi(x), \qquad \text{for all } x \in U_1. \end{aligned}$$

The general case follows from this.

THEOREM 14. *If $f \in \mathfrak{I}(U)$ and $f(x) = g(x)$, $x \in U - Z$, $Z \in \mathfrak{Z}(U)$, then $g \in \mathfrak{I}(U)$ and $F_\Phi = G_\Phi$.*

The proof is immediate.

THEOREM 15. (i) *If $f \in \mathfrak{I}(U)$ and $f \geqslant 0$, then $F_\Phi \in \mathfrak{H}^*(U)$.*

(ii) *If both f and $|f|$ are $\mathfrak{H}$-integrable, then F_Φ is the difference of two functions hyperharmonic in U.*

Proof. Part (ii) is an easy consequence of (i) and **(6)**; and in (i) it is sufficient to show that $F_0 \in \mathfrak{H}^*(U)$. By Theorem 13, if $V \in \mathfrak{B}(U)$ and $x \in V$, and if f_1 is the restriction of f to V and ϕ the restriction of F_0 to V^*, then

$$\int_{V,x} f_1 = F_0(x) - H_\phi(x) \qquad \text{for all } x \in V.$$

Since $f_1 \geqslant 0$, it follows from Theorem 12 that for all $x \in V$, $F_0(x) \geqslant H_\phi(x)$. This, by Theorems 1 and 3 and the definition of hyperharmonic functions, proves that $F_0 \in \mathfrak{H}^*(U)$.

THEOREM 16. *Suppose that $F \in \mathfrak{C}(\bar{U})$ and that DF exists on $U - Z$, $Z \in \mathfrak{Z}(U)$ and that $\uparrow DF$ and $\downarrow DF$ are finite except on an enumerable set where*

$$\lim_{\mathfrak{N}(x)} \frac{\Delta F(x;\, V)}{s(x,\, V)} = 0.$$

Then if $f = DF$ where DF is defined and is zero elsewhere,

$$f \in \mathfrak{I}(U) \quad \textit{and} \quad \int_{U,x} f = -F(x) + H_F(x).$$

This results immediately from the fact that $F - H_F$ is both a major and a minor function for f on U.

COROLLARY 17. *If F_1 and F_2 both satisfy the conditions of Theorem 16 and if $DF_1 = DF_2$ on $U - Z$, $Z \in \mathfrak{Z}(U)$, then $F_1 - F_2 \in \mathfrak{H}(U)$.*

In order to prove differentiability properties of the integral two further axioms are needed.

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AXIOM 5. (i) *If* $Z_n \in \mathfrak{Z}(U)$, $n = 1, 2, \ldots$, *then*

$$\bigcup_{n=1}^{\infty} Z_n \in \mathfrak{Z}(U).$$

(ii) *If E is an enumerable subset of U, then $E \in \mathfrak{Z}(U)$.*

AXIOM 6. *Suppose given a set $Z \subset U$ there exists a sequence u_n of real continuous hyperharmonic functions on U such that* (i) $\lim u_n = 0$, *uniformly in* U, (ii) *for some* $a > 0$, $\downarrow Du_n(x) < -a$ *for all* $x \in Z$ *and all* $n = 1, 2, \ldots$. *Then* $Z \in \mathfrak{Z}(U)$.

An easy implication of this last axiom is that if u is a real continuous hyperharmonic function on U, then the set of points x at which $\downarrow Du(x) = -\infty$ is in $\mathfrak{Z}(U)$.

THEOREM 18. *If Axioms 5 and 6 are assumed and $f \in \mathfrak{J}(U)$, then $DF_\Phi(x)$ exists and equals $-f(x)$ for all x in U except perhaps in a set of $\mathfrak{Z}(U)$.*

Proof. It is sufficient to consider the case F_0.

For all $n \geqslant 1$, choose a major function, M_n, of f on U so that $0 \leqslant u_n = F - M_n \leqslant 1/n$. By Lemma 9, u_n is a real continuous hyperharmonic function on U. Let $U_1 = U - Z_1$, $Z_1 \in \mathfrak{Z}(U)$ be the set of points at which $\downarrow Du_n$ is finite for all n.

Since $\uparrow DF_0 = \uparrow D(-F) \leqslant -\downarrow DM_n - \downarrow Du_n$, it follows that $\uparrow DF_0 < \infty$ on a set $U_2 = U_1 - Z_2$, $Z_2 \in \mathfrak{Z}(U)$. Further, $\uparrow DF_0 \leqslant -f - \downarrow Du_n$, on a set $U_3 = U_2 - Z_3$, $Z_3 \in \mathfrak{Z}(U)$.

The functions u_n, $n = 1, 2, \ldots$, satisfy the conditions of Axiom 6 and so let $Z_m \in \mathfrak{Z}(U)$, $m \geqslant 4$, be the set of x such that $\downarrow Du_n(x) \leqslant -1/m$ for all n. Then

$$\uparrow DF_0(x) \leqslant -f(x) + 1/m \qquad \text{for all } x \in U_3 - Z_m.$$

Hence in

$$U_3 - \bigcup_{m=4}^{\infty} Z_m = U - Z,$$

$Z \in \mathfrak{Z}(U)$, $\uparrow DF_0 \leqslant -f$ and $\uparrow DF_0 < \infty$.

A similar argument shows that $\downarrow DF_0 \geqslant -f$ and $\downarrow DF_0 > -\infty$ in a set $U - Z'$, $Z' \in \mathfrak{Z}(U)$. This completes the proof of the theorem.

5. Some examples. In this section a few examples of the preceding theory are given. It is intended to develop these further and to discuss the Axioms 1–6 in a later paper.

Let X be the real line and $\mathfrak{H}(X)$ the class of linear functions. The $\mathfrak{B}$ consists of all bounded open sets. If $V = (a, b)$ and $f(a) = c$, $f(b) = d$, then

$$H_f(x) = \frac{(x-a)d + (b-x)c}{b-a},$$

$$\mu(V; x) = \frac{x-a}{b-a}\epsilon_b + \frac{b-x}{b-a}\epsilon_a,$$

where ϵ_x denotes the unit mass supported by the set $\{x\}$. Since all relatively compact U are regular, the extension to $\uparrow H_f$ is unnecessary and all real functions on U^* are harmonically resolutive. The empty set is the only negligible set. The class $\mathfrak{H}^*(U)$ is just the collection of continuous concave functions. The validity of Axioms H, B, T, and K is either immediate or classical. If

$$\mathfrak{N}(x) = \{(x - a, x + b); a > 0, b > 0\},$$

then

$$Df(x; \mathfrak{N}) = \lim_{a\to 0+,\ b\to 0+} \left\{\frac{bf(x - a) + af(x + b) - (a + b)f(x)}{(a + b)^3}\right\}$$

and the existence of this limit is equivalent to the existence of the Peano second derivative. If, however, $\mathfrak{N}(x) = \{(x - a, x + a); a > 0\}$, then

$$Df(x; \mathfrak{N}) = \lim_{a\to 0+} \left\{\frac{f(x + a) + f(x - a) - 2f(x)}{8a^2}\right\},$$

which is the symmetric Riemann (or Schwarz) derivative. Using the latter example, the validity of the remaining axioms is readily checked. Axiom 1 is seen to hold by taking $v(x) = -x^2$. Since the differentiability used in Axiom 2 is the ordinary first-order differentiability, its validity is immediate. The non-constant harmonic functions all satisfy Axiom 3. Finally since for these V, $\mu(V; x) = \frac{1}{2}\{\epsilon_a + \epsilon_b\}$, Axiom 4 holds. The sets $\mathfrak{Z}(U)$ are the sets of Lebesgue measure zero **(6**, p. 299**)**, which implies Axiom 5. Axiom 6 is implied by another known result **(5**, Theorem 3**)**. Finally, the integral defined above with this derivative and harmonic structure is just the James P^2-integral **(5, 6)**.

The above example can be generalized to $\mathfrak{H}(X)$ being the set of functions of the form $l + mh(x)$, where h is some fixed twice-differentiable strictly monotonic function and l, m are real numbers. If then $V = (a, b)$ and $f(a) = c$, $f(b) = d$,

$$H_f(x) = \frac{d(h(x) - h(a)) + c(h(b) - h(x))}{h(b) - h(a)},$$

$$\mu(V; x) = \frac{h(x) - h(a)}{h(b) - h(a)}\epsilon_b + \frac{h(b) - h(x)}{h(b) - h(a)}\epsilon_a.$$

If f is assumed to be differentiable a suitable number of times, then

$$Df(x; \mathfrak{N}) = \frac{1}{8}\left(f'' - \frac{h''f'}{h'}\right)(x),$$

which is a special case of the operators considered by Rudin **(11)**. It corresponds to the case, using his notation, of $q(t) = 0$; in the definition of his operator **(11**, (2.3)**)**, the functions u and v are

$$u(t) = 1 \quad \text{and} \quad v(t) = \frac{h(t) - h(x)}{h'(x)}.$$

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All the axioms H, B, T, and K hold, and constants are harmonic and the only negligible set is the empty set. Axiom 1 holds with

$$v(x) = -xh(x) + \int h(x)dx,$$

and Axioms 2 and 3 are immediate. Further discussion will be left for a later paper.

There are several other situations to which the theory developed here can be applied. If X is R^n, $n \geqslant 2$, and $\mathfrak{H}(X)$ is the set of classical harmonic functions, the properties of the harmonic structure are well known **(3)**. The differential operator defined above is, in this case, the Blaschke operator **(1, 3)**. All the axioms H, B, T, and K hold as well as Axioms 1, 2, and 3. Theorem 7 becomes a result due to Rudin **(10)**. This will extend to $\mathfrak{H}(X)$ being the solutions of general second-order elliptic differential equations satisfying certain conditions **(2)**. A further extension to X being considered a Green space **(2, 6)** is also possible.

Again if $X = R^n$, $n \geqslant 1$, the possibility of $\mathfrak{H}(X)$ being the solutions of a general parabolic differential equation could be considered **(2)**. Finally, since the definition of the derivative does not require X to be more than locally compact, the problem of defining the integral over non-compact sets could be discussed. This would be important for instance if X is the real line and $\mathfrak{H}(X)$ the solutions of $y'' - (a^2 + 1)y = 0$ **(9)**; the generalized derivative then coincides with another operator considered by Rudin **(9)**.

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