

## A UNIFYING APPROACH TO BRANCHING PROCESSES IN A VARYING ENVIRONMENT

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### Abstract

Branching processes  $(Z_n)_{n \geq 0}$  in a varying environment generalize the Galton–Watson process, in that they allow time dependence of the offspring distribution. Our main results concern general criteria for almost sure extinction, square integrability of the martingale  $(Z_n/E[Z_n])_{n \geq 0}$ , properties of the martingale limit  $W$  and a Yaglom-type result stating convergence to an exponential limit distribution of the suitably normalized population size  $Z_n$ , conditioned on the event  $Z_n > 0$ . The theorems generalize/unify diverse results from the literature and lead to a classification of the processes.

*Keywords:* Branching process; varying environment; Galton–Watson process; exponential distribution

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### 1. Introduction and main results

Branching processes  $(Z_n)_{n \geq 0}$  in a varying environment generalize the classical Galton–Watson processes, in that they allow time dependence of the offspring distribution. This natural setting promises relevant applications (e.g. to random walks on trees, as in [18]) and has recently received a renewal of interest, see e.g. [2, 4, 13, 20]. Former research on branching processes in a varying environment was temporarily affected by the appearance of certain exotic properties, and one could get the impression that it is difficult to grasp some kind of generic behaviour of these processes. Even so, steps in this direction were taken by Peter Jagers [15]; in particular, he aimed for a classification into supercritical, critical and subcritical regimes in the spirit of ordinary Galton–Watson processes. In this paper we take up this line of research. To this end we prove several theorems reaching from criteria for almost sure extinction up to Yaglom-type results. We require only mild regularity assumptions, and in particular we set no restrictions on the sequence of expectations  $E[Z_n]$ ,  $n \geq 0$ , thereby generalizing and unifying a number of individual results from the literature.

In order to define a branching process in a varying environment (BPVE), let the sequence  $Y_1, Y_2, \dots$  denote random variables with values in  $\mathbb{N}_0$ , and  $f_1, f_2, \dots$  their distributions. Let  $Y_{ni}$ ,  $n, i \in \mathbb{N}$ , be independent random variables such that  $Y_{ni}$  and  $Y_n$  coincide in distribution for all  $n, i \geq 1$ . Define the random variables  $Z_n$ ,  $n \geq 0$ , with values in  $\mathbb{N}_0$  recursively as

$$Z_0 := 1, \quad Z_n := \sum_{i=1}^{Z_{n-1}} Y_{ni}, \quad n \geq 1.$$

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Then the process  $(Z_n)_{n \geq 0}$  is called a *branching process in the varying environment*  $v = (f_1, f_2, \dots)$  with initial value  $Z_0 = 1$ . These processes may be considered as a model for the development of the size of a population where individuals reproduce independently with offspring distributions  $f_n$  potentially changing among generations. Without further mention we always require that  $0 < E[Y_n] < \infty$  for all  $n \geq 1$ .

There is one non-trivial statement on BPVEs requiring no extra assumption. It says that  $Z_n$  is almost surely (a.s.) convergent to a random variable  $Z_\infty$  with values in  $\mathbb{N}_0 \cup \{\infty\}$ . This result is due to Lindvall [17] and extends results of Church [5] (for a comparatively short proof see [14, Theorem 1.4]). It also clarifies under which conditions  $(Z_n)_{n \geq 0}$  may ‘fall asleep’ at a positive state, meaning that the event  $\{0 < Z_\infty < \infty\}$  occurs with positive probability. Let us call such a branching process *asymptotically degenerate*. Thus, for a BPVE it is no longer true that the process either gets extinct a.s. or else converges a.s. to infinity.

As mentioned above, a BPVE may exhibit extraordinary properties that do not show up for ordinary Galton–Watson processes. Thus a BPVE may possess different growth rates, as detected by MacPhee and Schuh [19]. Here we establish a framework which excludes such exceptional phenomena and elucidates the generic behaviour. As we shall see, this is naturally done in an  $\mathcal{L}^2$  setting.

Our main assumption is a uniformity requirement which reads as follows. There is a constant  $c < \infty$  such that for all natural numbers  $n \geq 1$  we have

$$E[Y_n^2; Y_n \geq 2] \leq c E[Y_n; Y_n \geq 2] \cdot E[Y_n | Y_n \geq 1]. \tag{A}$$

This regularity assumption is notably mild. As we shall explain in the next section, it is fulfilled for distributions  $f_n$ ,  $n \geq 1$ , belonging to any common class of probability measures, like Poisson, binomial, hypergeometric, geometric, linear fractional or negative binomial distributions, without any restriction on the parameters. It is also satisfied in the case that the random variables  $Y_n$ ,  $n \geq 1$ , are a.s. uniformly bounded by a constant  $c < \infty$ . To see this, take into account that we have  $E[Y_n | Y_n \geq 1] \geq 1$ . Since direct verification of (A) may be tedious in examples, we shall present in the next section a third moment condition which implies (A) and which can often be easily checked.

Let us call a BPVE *regular* if it fulfils condition (A).

**Remark 1.** (*A property of consistency*) Observe that together with a BPVE  $(Z_n)_{n \geq 0}$ , any subsequence  $(Z_{n_i})_{i \geq 0}$  with  $n_0 := 0 < n_1 < n_2 < \dots$  is a BPVE, too. We note that the condition (A) is then transmitted, i.e. any subsequence of a regular BPVE is regular, too. The proof will be given after Lemma 6 below.

Before presenting our results, let us agree on the following notational conventions. Let  $\mathcal{P}$  be the set of all probability measures on  $\mathbb{N}_0$ . The weights of  $f \in \mathcal{P}$  are named  $f[k]$ ,  $k \in \mathbb{N}_0$ . We set

$$f(s) := \sum_{k=0}^{\infty} s^k f[k], \quad 0 \leq s \leq 1.$$

Thus, we denote the probability measure  $f$  and its generating function by one and the same symbol. This facilitates presentation and will cause no confusion. Keep in mind that each operation applied to these measures has to be understood as an operation applied to their generating functions. Thus,  $f_1 f_2$  stands not only for the multiplication of the generating functions  $f_1, f_2$  but also for the convolution of the respective measures. Also,  $f_1 \circ f_2$  expresses the composition of

generating functions as well as the resulting probability measure. We shall consider the mean and second factorial moment of a random variable  $Y$  with distribution  $f$ ,

$$E[Y] = f'(1), \quad E[Y(Y - 1)] = f''(1),$$

and its normalized second factorial moment and normalized variance

$$\nu := \frac{E[Y(Y - 1)]}{E[Y]^2} = \frac{f''(1)}{f'(1)^2}, \quad \rho := \frac{\text{Var}[Y]}{E[Y]^2} = \nu + \frac{1}{E[Y]} - 1.$$

We shall discuss branching processes in a varying environment along the lines of ordinary Galton–Watson processes. For  $n \geq 1$ , let

$$q := P(Z_\infty = 0), \quad \mu_n := f'_1(1) \cdots f'_n(1), \quad \nu_n := \frac{f''_n(1)}{f'_n(1)^2}, \quad \rho_n := \nu_n + \frac{1}{f'_n(1)} - 1,$$

and also  $\mu_0 := 1$ . Thus,  $q$  is the probability of extinction and  $\mu_n = E[Z_n]$ ,  $n \geq 0$ . Note that for the standardized factorial moments  $\nu_n$  we have  $\nu_n < \infty$  under Assumption (A). This implies  $E[Z_n^2] < \infty$  for all  $n \geq 0$  (see Lemma 4 below).

Assumption (A) is a mild requirement with substantial consequences, as seen from the following diverse necessary and sufficient criteria for almost sure extinction.

**Theorem 1.** *Assume (A). Then the conditions*

- (i)  $q = 1$ ,
- (ii)  $E[Z_n]^2 = o(E[Z_n^2])$  as  $n \rightarrow \infty$ ,
- (iii)  $\sum_{k=1}^\infty \frac{\rho_k}{\mu_{k-1}} = \infty$ ,
- (iv)  $\sum_{k=1}^\infty \frac{\nu_k}{\mu_{k-1}} = \infty$  or  $\mu_n \rightarrow 0$

*are equivalent. Moreover, the conditions*

- (v)  $q < 1$ ,
- (vi)  $E[Z_n^2] = O(E[Z_n]^2)$  as  $n \rightarrow \infty$ ,
- (vii)  $\sum_{k=1}^\infty \frac{\rho_k}{\mu_{k-1}} < \infty$ ,
- (viii)  $\sum_{k=1}^\infty \frac{\nu_k}{\mu_{k-1}} < \infty$  and there exists  $0 < r \leq \infty$  such that  $\mu_n \rightarrow r$

*are equivalent.*

These conditions are useful in different ways. Condition (iii)/(vii) appears to be a particular suitable criterion for almost sure extinction, whereas conditions (iv) and (viii) will prove helpful for the classification of BPVEs. Condition (vi) will allow us to determine the growth rate of  $Z_n$  (see Theorem 2). Observe that (ii) can be rewritten as  $E[Z_n] = o(\sqrt{\text{Var}[Z_n]})$ . Briefly speaking, this means that under (A) we have almost sure extinction if and only if the noise dominates the average growth in the long run.

We point out that conditions (iii), (iv), (vii) and (viii) access not only the expectations  $\mu_n$  but also the second moments. This is a novel aspect in comparison to ordinary Galton–Watson processes and also to Agresti’s classical criterion on BPVEs [1, Theorem 2]. Agresti’s result

provides almost sure extinction if and only if  $\sum_{k \geq 1} 1/\mu_{k-1} = \infty$ . He could do so by virtue of his stronger assumptions, which exclude, e.g., asymptotically degenerate processes. In our setting there is the possibility that we have both  $\sum_{k \geq 1} \rho_k/\mu_{k-1} = \infty$  and  $\sum_{k \geq 1} 1/\mu_{k-1} < \infty$ , and also the other way round. This is shown by the following examples.

**Example 1.** Let  $Y_n$  take just the values  $n + 2$  and 0, with  $P(Y_n = n + 2) = n^{-1}$ . Then  $E[Y_n(Y_n - 1)] \sim n$ ,  $E[Y_n] = 1 + 2/n$ ,  $E[Y_n - 1 | Y_n \geq 1] \sim n$ , and thus (A) is fulfilled. Also,  $\mu_n \sim n^2/2$  and  $\rho_n \sim n$ , hence  $\sum_{k \geq 1} 1/\mu_{k-1} < \infty$  and  $\sum_{k \geq 1} \rho_k/\mu_{k-1} = \infty$ .

**Example 2.** Let  $Y_n$  take just the values 0, 1 and 2, with  $P(Y_n = 0) = P(Y_n = 2) = 1/(2n^2)$ . Then  $E[Y_n(Y_n - 1)] \sim n^{-2}$ ,  $E[Y_n] = 1$  and  $E[Y_n - 1 | Y_n \geq 1] \sim 1/(2n^2)$ , and thus (A) is fulfilled. Also,  $\mu_n = 1$  and  $\rho_n \sim n^{-2}$ , hence  $\sum_{k \geq 1} 1/\mu_{k-1} = \infty$  and  $\sum_{k \geq 1} \rho_k/\mu_{k-1} < \infty$ .

The last example exhibits an asymptotically degenerate branching process, as seen from Corollary 1 below.

Next we turn to the normalized population sizes

$$W_n := \frac{Z_n}{\mu_n}, \quad n \geq 0.$$

Clearly,  $(W_n)_{n \geq 0}$  constitutes a non-negative martingale, and thus there exists an integrable random variable  $W \geq 0$  such that we have  $W_n \rightarrow W$  a.s. as  $n \rightarrow \infty$ . With (A), the random variable  $W$  exhibits the dichotomy known for Galton–Watson processes.

**Theorem 2.** For a regular BPVE we have:

- (i) If  $q = 1$  then  $W = 0$  a.s.
- (ii) If  $q < 1$  then  $E[W] = 1$ ,  $E[W^2] < \infty$  and  $P(W = 0) = q$ .

In particular, in the case of  $q < 1$  the martingale  $(W_n)_{n \geq 0}$  is convergent in  $\mathcal{L}^2$ , implying

$$\text{Var}[W] = \sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}}. \tag{1}$$

This formula goes back to Fearn [10]. We point out that Assumption (A) excludes the possibility of  $P(W = 0) > q$  and, in particular, of different rates of growth as in the examples constructed by MacPhee and Schuh [19] (see also [6, 7]). By means of Theorem 2(ii) we also gain further insight into asymptotically degenerate processes. Under Assumption (A) they are just those processes which fulfil the properties  $q < 1$  and  $0 < \lim_{n \rightarrow \infty} \mu_n < \infty$ . Also, taking Theorem 1(v) and (viii) into account we obtain the following corollary.

**Corollary 1.** A regular BPVE is asymptotically degenerate if and only if both  $\sum_{k=1}^{\infty} \nu_k < \infty$  and the sequence  $(\mu_n)_{n \geq 0}$  has a positive, finite limit. Then  $Z_{\infty} < \infty$  a.s.

Now we address the behaviour of the random variables  $Z_n$  conditioned on the events that  $Z_n > 0$ . The next theorem shows that their values largely follow the corresponding conditional expectations  $E[Z_n | Z_n > 0]$ . For  $n \geq 0$  let

$$a_n := 1 + \mu_n \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

**Theorem 3.** For a regular BPVE, the sequence of random variables  $Z_n/a_n$  conditioned on  $Z_n > 0, n \geq 0$ , is tight, i.e. for any  $\varepsilon > 0$  there is a  $u < \infty$  such that, for all  $n \geq 0$ ,

$$P\left(\frac{Z_n}{a_n} > u \mid Z_n > 0\right) \leq \varepsilon; \tag{2}$$

moreover, there exist numbers  $\theta > 0$  and  $u > 0$  such that, for all  $n \geq 0$ ,

$$P\left(\frac{Z_n}{a_n} > u \mid Z_n > 0\right) \geq \theta. \tag{3}$$

Also, we have

$$\gamma a_n \leq E[Z_n \mid Z_n > 0] \leq a_n \tag{4}$$

with some constant  $\gamma > 0$ , so that we may replace  $a_n$  by  $E[Z_n \mid Z_n > 0]$  in (2) and in (3).

For  $q < 1$  we do not learn anything new from this theorem; here, Theorem 2(ii) gives much preciser information. Thus, let us focus on the case  $q = 1$ , the situation of almost sure extinction. At first sight one might expect that the constant  $\theta$  in (3) can be chosen arbitrarily close to 1, if only  $u$  gets sufficiently small. This will apply to many interesting cases, but it is not always true. The following example gives an illustration.

**Example 3.** For  $n \geq 1$  let

$$f_{2n-1}[1] = 2^{-n}, \quad f_{2n-1}[0] = 1 - 2^{-n} \quad \text{and} \quad f_{2n}[2^{n+1} - 1] = f_{2n}[1] = \frac{1}{2}.$$

It is easy to check that (A) is valid (as well as the conditions (B) and (C) below). We have  $f'_{2n-1}(1) = 2^{-n}$  and  $f'_{2n}(1) = 2^n$ , hence

$$\mu_{2n-1} = 2^{-n} \quad \text{and} \quad \mu_{2n} = 1$$

for all  $n \geq 1$ . In particular, we have  $Z_{2n-1} \rightarrow 0$  in probability, which entails  $q = 1$ . Also,  $v_{2n-1} = 0$  and  $v_{2n} \sim 2$  as  $n \rightarrow \infty$ , implying

$$\sum_{k=1}^{2n} \frac{v_k}{\mu_{k-1}} \sim \sum_{k=1}^n 2^{k+1} \sim 2^{n+2} \quad \text{and} \quad \sum_{k=1}^{2n-1} \frac{v_k}{\mu_{k-1}} = \sum_{k=1}^{2n-2} \frac{v_k}{\mu_{k-1}} \sim 2^{n+1},$$

and

$$a_{2n-1} \sim 3 \quad \text{and} \quad a_{2n} \sim 2^{n+2}.$$

From Theorem 3 it follows that there is a  $z < \infty$  such that

$$P(Z_{2n-1} > z \mid Z_{2n-1} > 0) \leq \frac{1}{2}$$

for all  $n \geq 1$ . Therefore,

$$\begin{aligned} P(Z_{2n} \leq z \mid Z_{2n} > 0) &= P(Z_{2n} \leq z \mid Z_{2n-1} > 0) \\ &\geq P(Z_{2n-1} \leq z \mid Z_{2n-1} > 0) f_{2n}[1]^z \\ &\geq 2^{-z-1} \end{aligned}$$

for all  $n \geq 1$ , and for any  $u > 0$ ,

$$P(Z_{2n}/a_{2n} > u \mid Z_{2n} > 0) \leq 1 - 2^{-z-1} \tag{5}$$

if  $a_{2n} \geq z/u$ . Since  $a_{2n} \rightarrow \infty$ , the constant  $\theta$  from (3) cannot take a value above  $1 - 2^{-z-1}$  in this example.

This example suggests that quite different scenarios may occur for BPVEs with  $q = 1$ , and that their behaviour may abruptly change from one subsequence to the next. We point out that Assumption (A) does not put (e.g. for Poisson distributions) any restrictions on the expectation  $\mu_n, n \geq 1$ , allowing a variety of examples. Of special interest is the case that the numbers  $a_n$  are uniformly bounded. Here, Theorem 3 reads as follows.

**Corollary 2.** *Under Assumption (A) the conditions*

- (i) *the sequence of random variables  $Z_n$  conditioned on the events that  $Z_n > 0, n \geq 0$ , is tight,*
- (ii)  $\sup_{n \geq 0} E[Z_n | Z_n > 0] < \infty,$
- (iii)  $\sum_{k=1}^n \frac{v_k}{\mu_{k-1}} = O\left(\frac{1}{\mu_n}\right)$  as  $n \rightarrow \infty,$

*are equivalent.*

For an ordinary Galton–Watson process these three conditions apply just in the subcritical regime, and then the conditioned random variables  $Z_n$  have a limiting distribution. It is easy to see that such a feature will not hold in general for a BPVE. Indeed, there are two offspring distributions  $\hat{f}$  and  $\tilde{f}$  such that the limiting distributions  $\hat{g}$  and  $\tilde{g}$  for the corresponding conditional Galton–Watson processes differ from each other. Choose an increasing sequence  $0 = n_0 < n_1 < n_2 < \dots$  of natural numbers and consider the BPVE  $(Z_n)_{n \geq 0}$  in the varying environment  $v = (f_1, f_2, \dots)$ , where  $f_n = \hat{f}$  for  $n_{2k} < n \leq n_{2k+1}, k \in \mathbb{N}_0$ , and  $f_n = \tilde{f}$  otherwise. Then it is obvious that  $Z_{n_{2k+1}}$  given the event  $Z_{n_{2k+1}} > 0$  converges in distribution to  $\hat{g}$ , and  $Z_{n_{2k}}$  given the event  $Z_{n_{2k}} > 0$  converges in distribution to  $\tilde{g}$ , provided that the sequence  $(n_k)_{k \geq 0}$  is increasing sufficiently fast.

Thus, it may come as a surprise that in the opposite situation,  $\mu_n^{-1} = o(\sum_{k=1}^n \frac{v_k}{\mu_{k-1}})$ , we encounter a distinctive behaviour of the conditional-limit distributions of  $Z_n$ , which is in accordance with Yaglom’s theorem for ordinary Galton–Watson processes. For technical reasons we have to somewhat strengthen Assumption (A). We require that for every  $\varepsilon > 0$  there is a constant  $c_\varepsilon < \infty$  such that, for all natural numbers  $n \geq 1$ ,

$$E[Y_n^2; Y_n > c_\varepsilon(1 + E[Y_n])] \leq \varepsilon E[Y_n^2; Y_n \geq 2]. \tag{B}$$

This condition is again widely satisfied, as we shall explain in the next section. It implies Assumption (A). Namely, for  $\varepsilon = 1/2$  we have

$$\begin{aligned} E[Y_n^2; Y_n \geq 2] &\leq 2E[Y_n^2; 2 \leq Y_n \leq c_{1/2}(1 + E[Y_n])] \\ &\leq 2c_{1/2}(1 + E[Y_n])E[Y_n; Y_n \geq 2]. \end{aligned} \tag{6}$$

Since  $1 + E[Y_n] \leq 2E[Y_n | Y_n \geq 1]$ , we obtain (A) with  $c = 4c_{1/2}$ .

**Theorem 4.** *Let (B) be satisfied and let  $q = 1$ . Then the following conditions are equivalent:*

- (i) *There is a sequence  $b_n, n \geq 0$ , of positive numbers such that  $Z_n/b_n$  conditioned on the event  $Z_n > 0$  converges in distribution to a standard exponential distribution as  $n \rightarrow \infty$ ;*

(ii)  $E[Z_n | Z_n > 0] \rightarrow \infty$  as  $n \rightarrow \infty$ ;

(iii)  $\frac{1}{\mu_n} = o\left(\sum_{k=1}^n \frac{v_k}{\mu_{k-1}}\right)$  as  $n \rightarrow \infty$ .

Under these conditions we may set  $b_n := E[Z_n | Z_n > 0]$ , and we have

$$E[Z_n | Z_n > 0] \sim \frac{\mu_n}{2} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}},$$

or equivalently

$$P(Z_n > 0) \sim 2\left(\sum_{k=1}^n \frac{v_k}{\mu_{k-1}}\right)^{-1}$$

as  $n \rightarrow \infty$ .

This theorem covers the classical results of Kolmogorov and Yaglom for critical Galton–Watson processes in the finite variance case (without further moment restrictions), since then (B) is trivially satisfied.

Our results show how to implement a classification of regular BPVEs which connects to the notions used for classical Galton–Watson processes. If  $q < 1$ , then in view of Theorem 2 and Corollary 1 we distinguish two regimes. There is the *supercritical regime* in the case of  $E[Z_n] \rightarrow \infty$ , and the *asymptotically degenerate regime* otherwise. If, on the other hand, we have  $q = 1$ , then Theorem 4 suggests characterizing the *critical regime* by the condition  $E[Z_n | Z_n > 0] \rightarrow \infty$  (and not just by some condition on the limiting behaviour of  $\mu_n$ , as one might do in a first attempt), and to allocate the other BPVEs to the *subcritical regime*. In this way we differentiate the clear-cut limiting property of critical BPVEs from the indeterminacy of the remaining processes. In this classification a subcritical BPVE  $(Z_n)_{n \geq 0}$  exhibits subcritical behaviour in the sense that according to Theorem 3 the random variables  $Z_n$  conditioned on  $Z_n > 0$  are tight, at least along some subsequence in which the  $a_n$  stay bounded. The  $Z_n$  may diverge with positive probability along some other subsequence, yet this does not in general imply critical behaviour in the sense that along that subsequence the random variables  $Z_n$ , conditioned on  $Z_n > 0$  and suitably scaled, have asymptotically an exponential distribution. For a counterexample we refer to Example 3 and (5).

By means of Theorems 1 and 3 we may streamline the determining conditions of the four regimes, as summarized in the subsequent overview.

**Proposition 1.** *A regular BPVE is*

$$\text{supercritical if and only if } \lim_{n \rightarrow \infty} \mu_n = \infty \text{ and } \sum_{k=1}^{\infty} \frac{v_k}{\mu_{k-1}} < \infty,$$

$$\text{asymptotically degenerate if and only if } 0 < \lim_{n \rightarrow \infty} \mu_n < \infty \text{ and } \sum_{k=1}^{\infty} \frac{v_k}{\mu_{k-1}} < \infty,$$

$$\text{critical if and only if } \lim_{n \rightarrow \infty} \mu_n \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} = \infty \text{ and } \sum_{k=1}^{\infty} \frac{v_k}{\mu_{k-1}} = \infty,$$

$$\text{subcritical if and only if } \liminf_{n \rightarrow \infty} \mu_n = 0 \text{ and } \liminf_{n \rightarrow \infty} \mu_n \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} < \infty.$$

Note that convergence of the means  $\mu_n$  is not enforced in the critical case; they may diverge, converge to zero or even oscillate in between.

**Example 4.** In the case  $0 < \inf_n v_n \leq \sup_n v_n < \infty$  (as e.g. for Poisson variables) the classification simplifies. Here, we are in the supercritical regime if and only if  $\sum_{k \geq 0} 1/\mu_k < \infty$  (enforcing  $\mu_n \rightarrow \infty$ ). Asymptotically degenerate behaviour is excluded, and there is plenty of room for critical processes, i.e. for processes which conform to the conditions  $\sum_{k \geq 0} 1/\mu_k = \infty$  and  $1/\mu_n = o(\sum_{k=0}^{n-1} 1/\mu_k)$ . The second requirement is e.g. fulfilled if we have  $\mu_n/\mu_{n-1} \rightarrow 1$  as  $n \rightarrow \infty$ . This latter condition covers a variety of scenarios for  $\mu_n$  below exponential growth and above exponential decay.

**Example 5.** In the binary case  $P(Y_n = 2) = p_n, P(Y_n = 0) = 1 - p_n$  we get  $f'_n(1) = f''_n(1) = 2p_n$ . Therefore  $v_k/\mu_{k-1} = 1/\mu_k$ , so that the situation conforms to the previous example.

**Example 6.** In the symmetric case  $P(Y_n = 0) = P(Y_n = 2) = p_n/2$  and  $P(Y_n = 1) = 1 - p_n$  we have  $\mu_n = 1$  and  $v_n = p_n$ . Here, we find critical or asymptotically degenerate behaviour, according to whether  $\sum_{k=1}^{\infty} p_n$  is divergent or convergent.

**Example 7.** If the  $Y_n$  take only the values 0 and 1, then all  $v_n$  vanish. Now the BPVE is subcritical or asymptotically degenerate, according to whether  $\mu_n$  converges to zero or to a positive value.

Our proofs rely largely on analytic considerations. The task is to get a grip on the probability measures  $f_1 \circ \dots \circ f_n$ , which are the distributions of the random variables  $Z_n$ . In order to handle such iterated compositions of generating functions we resort to a device which has been applied from the beginning in the theory of branching processes. For a probability distribution  $f$  on  $\mathbb{N}_0$  with positive, finite mean  $m$  we define a function  $\varphi : [0, 1) \rightarrow \mathbb{R}$  by the equation

$$\frac{1}{1 - f(s)} = \frac{1}{m(1 - s)} + \varphi(s), \quad 0 \leq s < 1.$$

In this way the mean and the ‘shape’ of  $f$  are separated to a certain extent. Indeed, Lemma 1 below shows that  $\varphi$  takes values which are of the size of the *standardized* second factorial moment  $v$ . Therefore we briefly name  $\varphi$  the *shape function* of  $f$ . As we shall see, these functions are useful to dissolve the generating function  $f_1 \circ \dots \circ f_n$  into a sum (see Lemma 4 below). Here, our contribution consists in obtaining sharp upper and lower bounds for the function  $\varphi$  and its derivative. The interaction of these bounds then allows for precise estimates e.g. of the survival probabilities  $P(Z_n > 0)$ . The role of Assumption (A) in this interplay is to keep both bounds together uniformly in  $n$ .

Concluding this introduction, let us comment on the literature. Agresti in his paper [1] on almost sure extinction already derived the sharp upper bound for  $\varphi$  which we give below in (8). We note that this bound is related to the well-known Paley–Zygmund inequality (compare the proof of Lemma 7). Agresti also obtained a lower bound for the survival probabilities, which, however, in general is away from our sharp bound. Lyons [18] obtained the equivalence of conditions (v), (vi), (vii) and (somewhat disguised) (viii) from Theorem 1 under the assumption that the random variables  $Y_n$  are a.s. bounded by a constant, with methods completely different from ours. He also proved Theorem 2, again under the assumption that the offspring numbers are a.s. uniformly bounded by a constant. D’Souza and Biggins [7] derived Theorem 2 under a different set of assumptions. They required that there are numbers  $a > 0, b > 1$  such that  $\mu_{m+n}/\mu_m \geq ab^n$  for all  $m, n \geq 1$  (called the uniform supercritical case). They

did not need finite second moments, but assumed instead that the random variables  $Y_n$  are uniformly dominated by a random variable  $Y$  with  $E[Y \log^+ Y] < \infty$ . Goettge [12] obtained  $E[W] = 1$  under the condition  $\mu_n \geq an^b$  with  $a > 0$ ,  $b > 1$  (together with a uniform domination assumption), but did not consider the validity of the equation  $P(W = 0) = q$ . In order to prove the conditional limit law from Theorem 4, Jagers [15] drew attention to uniform estimates due to Sevast'yanov [21] (see also [9, Lemma 3]). This approach demands, amongst others, the strong assumption that the sequence  $E[Z_n]$ ,  $n \geq 0$ , is bounded from above and away from zero. Independently, and in parallel to our work, Bhattacharya and Perlman [3] have presented a considerable generalization of Jager's result, on a different route and under assumptions which are stronger than ours. For recent results on almost sure extinction and asymptotic exponentiality of multitype BPVEs we refer to [8].

The remainder of this paper is organized as follows. In Section 2 we discuss the assumptions and several examples. In Section 3 we analyze the shape function  $\varphi$ . Section 4 contains the proofs of our theorems.

## 2. Examples

The following example illustrates the difference in range of the conditions (A) and (B).

**Example 8.** Let  $Y$  have a *linear fractional distribution*, meaning that

$$P(Y = y | Y \geq 1) = (1 - p)^{y-1} p, \quad y \geq 1,$$

with some  $0 < p < 1$  and some probability  $P(Y \geq 1)$ . Then, from properties of geometric distributions, we have

$$E[Y | Y \geq 1] = \frac{1}{p}, \quad E[Y - 1 | Y \geq 1] = \frac{(1 - p)}{p}, \quad E[Y(Y - 1) | Y \geq 1] = \frac{2(1 - p)}{p^2},$$

and it follows that

$$\begin{aligned} E[Y^2; Y \geq 2] &\leq 2E[Y(Y - 1)] = \frac{4(1 - p)}{p^2} P(Y \geq 1) \\ &= 4E[Y - 1; Y \geq 1] \cdot E[Y | Y \geq 1] \leq 4E[(Y; Y \geq 2)] \cdot E[Y | Y \geq 1]. \end{aligned}$$

Thus, for *any* sequence  $Y_n$  of linear fractional random variables, Assumption (A) is fulfilled with  $c = 4$ , whatever the parameters  $p_n$  and  $P(Y_n \geq 1)$  are.

However, for condition (B) the corresponding statement fails. To see this we resort for linear fractional distributions to the formula

$$\frac{2(1 - p_n)}{p_n^2} P(Y_n \geq 1) = E[Y_n(Y_n - 1)] \leq E[Y_n^2; Y_n \geq 2].$$

If we assume (B), then the inequality (6) is also valid, yielding

$$\frac{2(1 - p_n)}{p_n^2} P(Y_n \geq 1) \leq 4c_{1/2} E[Y_n - 1; Y_n \geq 1] \cdot (1 + E[Y_n]).$$

For linear fractional distributions this estimate may be rewritten as

$$\frac{2(1 - p_n)}{p_n^2} P(Y_n \geq 1) \leq 4c_{1/2} \frac{(1 - p_n)}{p_n} P(Y_n \geq 1) \left( 1 + \frac{1}{p_n} P(Y_n \geq 1) \right),$$

which simplifies to

$$\frac{1}{2c_{1/2}} \leq p_n + P(Y_n \geq 1).$$

Thus, condition (B) implies  $\inf_n (p_n + P(Y_n \geq 1)) > 0$ , and a sequence of linear fractional random variables satisfying  $p_n + P(Y_n \geq 1) \leq 1/n$  does not meet (B).

Incidentally, Theorem 4 still holds true for linear fractional  $Y_n, n \geq 1$ , regardless of the validity of (B). Then, as is well known,  $Z_n$  is also linear fractional for any  $n \geq 1$ , and consequently the sequence  $Z_n/E[Z_n | Z_n \geq 1]$  given the events that  $Z_n \geq 1$  converges in distribution to a standard exponential distribution provided that we have  $E[Z_n | Z_n \geq 1] \rightarrow \infty$ .

In other examples, a direct verification of Assumptions (A) or (B) can be cumbersome. Therefore we introduce another assumption, which is often easier to handle: there is a constant  $\bar{c} < \infty$  such that, for all natural numbers  $n \geq 1$ ,

$$E[Y_n(Y_n - 1)(Y_n - 2)] \leq \bar{c} E[Y_n(Y_n - 1)] \cdot (1 + E[Y_n]). \tag{C}$$

Condition (C) implies (A) and (B), as seen from the following proposition.

**Proposition 2.** *If condition (C) is fulfilled, then (B) holds with  $c_\varepsilon := \max(3, 5\bar{c}/\varepsilon)$  and (A) holds with  $c := \max(12, 40\bar{c})$ .*

*Proof.* From  $c_\varepsilon \geq 3$  and (C) we obtain

$$\begin{aligned} E[Y_n^2; Y_n > c_\varepsilon(1 + E[Y_n])] &\leq 5 E[(Y_n - 1)(Y_n - 2); Y_n > c_\varepsilon(1 + E[Y_n])] \\ &\leq 5 \frac{E[Y_n(Y_n - 1)(Y_n - 2)]}{c_\varepsilon(1 + E[Y_n])} \\ &\leq \frac{5\bar{c}}{c_\varepsilon} E[Y_n(Y_n - 1)]. \end{aligned}$$

It follows that

$$E[Y_n^2; Y_n > c_\varepsilon(1 + E[Y_n])] \leq \varepsilon E[Y_n^2; Y_n \geq 2],$$

which is our first claim. The second one follows by means of (6). □

Condition (C) can be easily handled by means of generating functions and their derivatives. Here are some examples.

**Example 9.** If the  $Y_n$  are a.s. uniformly bounded by a constant  $c$ , then (C) is satisfied with  $\bar{c} = c$ .

**Example 10.** Let  $Y$  be Poisson with parameter  $\lambda > 0$ . Then

$$E[Y(Y - 1)(Y - 2)] = \lambda^3 \leq \lambda^2(\lambda + 1) = E[Y(Y - 1)](1 + E[Y]).$$

Here, (C) is fulfilled with  $\bar{c} = 1$ .

**Example 11.** For binomial  $Y$  with parameters  $m \geq 1$  and  $0 < p < 1$  the situation is analogous, and here

$$E[Y(Y - 1)(Y - 2)] = m(m - 1)(m - 2)p^3 \leq m(m - 1)p^2mp \leq E[Y(Y - 1)](1 + E[Y]).$$

**Example 12.** For a *hypergeometric* distribution with parameters  $(N, K, m)$  we have, for  $N \geq 3$ ,

$$\begin{aligned} E[Y(Y-1)(Y-2)] &= \frac{m(m-1)(m-2)K(K-1)(K-2)}{N(N-1)(N-2)} \\ &\leq 3 \frac{m(m-1)K(K-1)}{N(N-1)} \frac{mK}{N} \leq 3E[Y(Y-1)](1+E[Y]), \end{aligned}$$

and (C) is satisfied with  $\bar{c} = 3$ . The case  $N \leq 2$  can immediately be included.

**Example 13.** For *negative binomial* distributions the generating function is given by

$$f(s) = \left( \frac{p}{1-s(1-p)} \right)^\alpha$$

with  $0 < p < 1$  and a positive integer  $\alpha$ . Now,

$$\begin{aligned} E[Y] &= \alpha \frac{1-p}{p}, \\ E[Y(Y-1)] &= \alpha(\alpha+1) \frac{(1-p)^2}{p^2}, \\ E[Y(Y-1)(Y-2)] &= \alpha(\alpha+1)(\alpha+2) \frac{(1-p)^3}{p^3}. \end{aligned}$$

Thus,

$$E[Y(Y-1)(Y-2)] \leq 3E[Y(Y-1)](1+E[Y]).$$

Again, (C) is fulfilled with  $\bar{c} = 3$ .

### 3. Bounds for the shape function

For  $f \in \mathcal{P}$  with mean  $0 < m = f'(1) < \infty$ , define the *shape function* as the function  $\varphi = \varphi_f : [0, 1) \rightarrow \mathbb{R}$  given by the equation

$$\frac{1}{1-f(s)} = \frac{1}{m(1-s)} + \varphi(s), \quad 0 \leq s < 1.$$

Due to convexity of  $f(s)$  we have  $\varphi(s) \geq 0$  for all  $0 \leq s < 1$ . By means of a Taylor expansion of  $f$  around 1, one obtains  $\lim_{s \uparrow 1} \varphi(s) = f''(1)/(2f'(1)^2)$ , and thus we extend  $\varphi$  by setting

$$\varphi(1) := \frac{\nu}{2} \quad \text{with} \quad \nu := \frac{f''(1)}{f'(1)^2}. \quad (7)$$

In this section we prove the following sharp bounds.

**Lemma 1.** *Assume  $f''(1) < \infty$ . Then, for  $0 \leq s \leq 1$ ,*

$$\frac{1}{2}\varphi(0) \leq \varphi(s) \leq 2\varphi(1). \quad (8)$$

Note that  $\varphi$  is identically zero if  $f[z] = 0$  for all  $z \geq 2$ . Otherwise,  $\varphi(0) > 0$  and the lower bound of  $\varphi$  becomes strictly positive. Choosing  $s = 1$  and  $s = 0$  in (8), we obtain  $\varphi(0)/2 \leq \varphi(1)$

and  $\varphi(0) \leq 2\varphi(1)$ . Note that for  $f = \delta_k$  (Dirac measure at point  $k$ ) and  $k \geq 2$  we have  $\varphi(1) = \varphi(0)/2$ , implying that the constants  $1/2$  and  $2$  in (8) cannot be improved. The upper bound was derived in [11] using a different method of proof.

The next lemma is based on a close investigation of the derivative of  $\varphi(s)$ .

**Lemma 2.** *Let  $Y$  be a random variable with distribution  $f$  and assume  $f''(1) < \infty$ . Then, for  $0 \leq s \leq 1$  and natural numbers  $a \geq 1$ ,*

$$\sup_{s \leq t \leq 1} |\varphi(1) - \varphi(t)| \leq 2mv^2(1 - s) + 2av(1 - s) + \frac{2}{m^2} E[Y^2; Y > a].$$

Uniform estimates of  $\varphi(1) - \varphi(s)$  based on third moments have already been obtained by Sevast'yanov [21] and others (see [9, Lemma 3]). Our lemma implies and generalizes these estimates. For the proof of these lemmas we use the following result.

**Lemma 3.** *Let  $g_1, g_2$  be elements of  $\mathcal{P}$  with the same support and satisfying the following property. For any  $y \in \mathbb{N}_0$  with  $g_1[y] > 0$  we have*

$$\frac{g_1[z]}{g_1[y]} \leq \frac{g_2[z]}{g_2[y]} \quad \text{for all } z > y.$$

Also, let  $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$  be a non-decreasing function. Then

$$\sum_{y=0}^{\infty} \alpha(y)g_1[y] \leq \sum_{y=0}^{\infty} \alpha(y)g_2[y].$$

*Proof.* The lemma's assumption is called the 'monotone likelihood ratio property', which is known to imply our claim. For convenience, we give a short proof. By assumption there is a non-decreasing function  $h(y)$ ,  $y \in \mathbb{N}_0$ , such that  $h(y) = g_2(y)/g_1(y)$  for all elements  $y$  of the support of  $g_1$ . Then, for any real number  $c$ ,

$$\begin{aligned} \sum_{y=0}^{\infty} \alpha(y)g_2[y] - \sum_{y=0}^{\infty} \alpha(y)g_1[y] &= \sum_{y=0}^{\infty} (\alpha(y) - c)(g_2[y] - g_1[y]) \\ &= \sum_{y=0}^{\infty} (\alpha(y) - c)(h(y) - 1)g_1[y]. \end{aligned}$$

For  $c := \min\{\alpha(y) : h(y) \geq 1\}$  we have  $\alpha(0) \leq c < \infty$ . For this choice of  $c$ , since  $h$  and  $\alpha$  are non-decreasing, every summand of the right-hand sum is non-negative. Thus, the whole sum is non-negative, too, and our assertion follows. □

*Proof of Lemma 1.*

(i) First, we examine a special case of Lemma 3. Consider for  $0 < s \leq 1$  and  $r \in \mathbb{N}_0$  the probability measures

$$g_s[y] = \frac{s^{r-y}}{1 + s + \dots + s^r}, \quad 0 \leq y \leq r.$$

Then, for  $0 < s \leq t \leq 1$ ,  $0 \leq y < z \leq r$  we have  $g_s[z]/g_s[y] = s^{y-z} \geq t^{y-z} = g_t[z]/g_t[y]$ . Hence, we obtain that

$$\sum_{y=0}^r yg_s[y] = \frac{s^{r-1} + 2s^{r-2} + \dots + r}{1 + s + \dots + s^r}$$

is a decreasing function in  $s$ . Also,  $\sum_{y=0}^r yg_0[y] = r$  and  $\sum_{y=0}^r yg_1[y] = r/2$ , and it follows for  $0 \leq s \leq 1$  that

$$\frac{r}{2} \leq \frac{r + (r-1)s + \dots + s^{r-1}}{1 + s + \dots + s^r} \leq r. \quad (9)$$

(ii) Next, we derive a second representation for  $\varphi$ . We have

$$1 - f(s) = \sum_{z=1}^{\infty} f[z](1 - s^z) = (1 - s) \sum_{z=1}^{\infty} f[z] \sum_{k=0}^{z-1} s^k$$

and

$$\begin{aligned} f'(1)(1 - s) - (1 - f(s)) &= (1 - s) \sum_{z=1}^{\infty} f[z] \sum_{k=0}^{z-1} (1 - s^k) \\ &= (1 - s)^2 \sum_{z=1}^{\infty} f[z] \sum_{k=1}^{z-1} \sum_{j=0}^{k-1} s^j \\ &= (1 - s)^2 \sum_{z=1}^{\infty} f[z]((z-1) + (z-2)s + \dots + s^{z-2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(s) &= \frac{m(1 - s) - (1 - f(s))}{m(1 - s)(1 - f(s))} \\ &= \frac{\sum_{y=1}^{\infty} f[y]((y-1) + (y-2)s + \dots + s^{y-2})}{m \cdot \sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}. \end{aligned}$$

From (9) it follows that

$$\varphi(s) \leq \frac{\psi(s)}{m} \leq 2\varphi(s), \quad (10)$$

with

$$\psi(s) := \frac{\sum_{y=1}^{\infty} f[y](y-1)(1 + s + \dots + s^{y-1})}{\sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}.$$

Now consider the probability measures  $g_s \in \mathcal{P}$ ,  $0 \leq s \leq 1$ , given by

$$g_s[y] := \frac{f[y](1 + s + \dots + s^{y-1})}{\sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}, \quad y \geq 1. \quad (11)$$

Then, for  $f[y] > 0$  and  $z > y$ , after some algebra,

$$\frac{g_s[z]}{g_s[y]} = \frac{f[z]}{f[y]} \prod_{v=1}^{z-y} \left( 1 + \frac{1}{s^{-1} + \dots + s^{-y-v+1}} \right),$$

which is an increasing function in  $s$ . Therefore, by Lemma 3, the function  $\psi(s)$  is increasing in  $s$ . In combination with (10) we get

$$\varphi(s) \leq \frac{\psi(s)}{m} \leq \frac{\psi(1)}{m} \leq 2\varphi(1), \quad 2\varphi(s) \geq \frac{\psi(s)}{m} \geq \frac{\psi(0)}{m} \geq \varphi(0).$$

This gives the claim of the lemma.  $\square$

*Proof of Lemma 2.* First, we estimate the derivative of  $\varphi$ , which is given by

$$\varphi'(s) = \frac{1}{m} \frac{mf'(s)}{(1-f(s))^2} - \frac{1}{m(1-s)^2}.$$

It turns out that this expression becomes more manageable if we replace the squared geometric mean  $\sqrt{mf'(s)}$  on the right-hand side by the square of the arithmetic mean  $(m + f'(s))/2$ . Therefore, we split the derivative into parts according to

$$\varphi'(s) = \psi_1(s) - \psi_2(s), \tag{12}$$

with

$$\psi_1(s) = \frac{1}{4m} \frac{(m + f'(s))^2}{(1-f(s))^2} - \frac{1}{m(1-s)^2}, \quad \psi_2(s) = \frac{1}{4m} \frac{(m + f'(s))^2}{(1-f(s))^2} - \frac{f'(s)}{(1-f(s))^2}.$$

We show that both  $\psi_1$  and  $\psi_2$  are non-negative functions, and estimate them from above.

For  $\psi_1$  we accomplish this task by introducing the function

$$\begin{aligned} \zeta(s) &:= (m + f'(s)) - 2 \frac{1-f(s)}{1-s} \\ &= \sum_{y=1}^{\infty} y(1 + s^{y-1})f[y] - 2 \sum_{y=1}^{\infty} \frac{1-s^y}{1-s} f[y] \\ &= \sum_{y=3}^{\infty} (y(1 + s^{y-1}) - 2(1 + s + \dots + s^{y-1}))f[y]. \end{aligned}$$

Since

$$\begin{aligned} \frac{d}{ds} (y(1 + s^{y-1}) - 2(1 + s + \dots + s^{y-1})) \\ &= y(y-1)s^{y-2} - 2(1 + 2s + \dots + (y-1)s^{y-2}) \\ &\leq y(y-1)s^{y-2} - 2s^{y-2}(1 + 2 + \dots + (y-1)) = 0 \end{aligned}$$

for all  $0 \leq s \leq 1$ , and since  $\zeta(1) = 0$ , we see that  $\zeta$  is a non-negative, decreasing function. Thus,  $\psi_1$  is a non-negative function, too. Also,  $\zeta(0) \leq m$ .

Moreover, we have for  $y \geq 3$  the polynomial identity

$$y(1 + s^{y-1}) - 2(1 + s + \dots + s^{y-1}) = (1-s)^2 \sum_{z=1}^{y-2} z(y-z-1)s^{z-1},$$

and consequently

$$\zeta(s) = (1-s)^2 \xi(s)$$

with

$$\xi(s) := \sum_{y=3}^{\infty} \sum_{z=1}^{y-2} z(y-z-1)s^{z-1} f[y].$$

The function  $\xi$  is non-negative and increasing.

Coming back to  $\psi_1$ , we rewrite it as

$$\psi_1(s) = \frac{\frac{1}{2}(m+f'(s))(1-s) - (1-f(s))}{(1-f(s))(1-s)} \cdot \frac{\frac{1}{2}(m+f'(s))(1-s) + (1-f(s))}{m(1-f(s))(1-s)}.$$

Using  $f'(s) \leq m$ , it follows that

$$\begin{aligned} \psi_1(s) &\leq \frac{\zeta(s)}{2(1-f(s))} \left( \frac{1}{1-f(s)} + \frac{1}{m(1-s)} \right) \\ &= \frac{\zeta(s)}{2} \left( \frac{1}{m(1-s)} + \varphi(s) \right) \left( \frac{2}{m(1-s)} + \varphi(s) \right) \\ &\leq 2\zeta(s) \left( \frac{1}{m^2(1-s)^2} + \varphi(s)^2 \right) \\ &= \frac{2\xi(s)}{m^2} + 2\zeta(s)\varphi(s)^2. \end{aligned}$$

By means of Lemma 1, by the monotonicity properties of  $\xi$  and  $\zeta$ , and by  $\varphi(1) = v/2$ ,  $\zeta(0) \leq m$ , we obtain

$$0 \leq \psi_1(s) \leq \frac{2\xi(s)}{m^2} + 2mv^2. \quad (13)$$

Now we investigate the function  $\psi_2$ , which we rewrite as

$$\psi_2(s) = \frac{1}{4m} \left( \frac{m-f'(s)}{1-f(s)} \right)^2.$$

We have

$$1-f(s) = \sum_{z=1}^{\infty} (1-s^z)f[z] = (1-s) \sum_{z=1}^{\infty} (1+s+\dots+s^{z-1})f[z]$$

and

$$m-f'(s) = \sum_{y=1}^{\infty} (1-s^{y-1})yf[y] = (1-s) \sum_{y=2}^{\infty} y(1+\dots+s^{y-2})f[y].$$

Using the notation from (11) it follows that

$$\frac{m-f'(s)}{1-f(s)} = \sum_{y=2}^{\infty} \frac{1+\dots+s^{y-2}}{1+\dots+s^{y-1}} yg_s[y] \leq \sum_{y=2}^{\infty} yg_s[y].$$

As above, we may apply Lemma 3 to the probability measures  $g_s$  and conclude that the right-hand term is increasing with  $s$ . Therefore,

$$0 \leq \frac{m-f'(s)}{1-f(s)} \leq \sum_{y=2}^{\infty} yg_1[y] = \frac{\sum_{y=2}^{\infty} y^2 f[y]}{\sum_{z=1}^{\infty} zf[z]} \leq \frac{2 \sum_{y=1}^{\infty} y(y-1)f[y]}{\sum_{z=1}^{\infty} zf[z]} = 2mv,$$

and hence

$$0 \leq \psi_2(s) \leq mv^2. \quad (14)$$

Coming to our claim, note first that owing to the non-negativity of  $\psi_1$  and  $\psi_2$  we obtain from (12), for any  $s \leq u \leq 1$ ,

$$-\int_s^1 \psi_2(t) dt \leq \varphi(1) - \varphi(u) \leq \int_s^1 \psi_1(t) dt.$$

Equations (13) and (14) entail

$$-mv^2(1-s) \leq \varphi(1) - \varphi(u) \leq \frac{2}{m^2} \int_s^1 \xi(t) dt + 2mv^2(1-s). \tag{15}$$

It remains to estimate the right-hand integral. We have, for  $0 \leq s < 1$ ,

$$\begin{aligned} \int_s^1 \xi(t) dt &= \sum_{y=3}^{\infty} \sum_{z=1}^{y-2} (y-z-1)(1-s^z)f[y] \\ &\leq (1-s) \sum_{y=3}^{\infty} (y-2)f[y] \sum_{z=1}^{y-2} \sum_{u=0}^{z-1} s^u \\ &= (1-s) \sum_{y=3}^{\infty} (y-2)f[y] \sum_{u=0}^{y-3} (y-2-u)s^u \\ &= (1-s) \sum_{u=0}^{\infty} s^u \sum_{y=u+3}^{\infty} (y-2)^2 f[y]. \end{aligned}$$

The right-hand sum is monotonically decreasing in  $u$ , and therefore for natural numbers  $a$  we end up with the estimate

$$\begin{aligned} \int_s^1 \xi(t) dt &\leq \sum_{y=3}^{\infty} (y-2)^2 f[y] (1-s) \sum_{u=0}^{a-1} s^u + \sum_{y=a+3}^{\infty} (y-2)^2 f[y] (1-s) \sum_{u=a}^{\infty} s^u \\ &\leq f''(1)a(1-s) + E[Y^2; Y > a]. \end{aligned}$$

Combining this estimate with (15), our claim follows. □

**Remark 2.** We have

$$\xi(1) = \sum_{y=3}^{\infty} \sum_{z=1}^{y-2} z(y-z-1)f[y] = \frac{1}{3} \sum_{y=3}^{\infty} z(z-1)(z-2)f[z] = \frac{f'''(1)}{3},$$

and hence from (12), (13), (14) and the monotonicity of  $\xi$  for  $0 \leq s \leq 1$ ,

$$-\frac{f''(1)^2}{f'(1)^3} \leq \varphi'(s) \leq \frac{2f'''(1)}{3f'(1)^2} + 2\frac{f''(1)^2}{f'(1)^3}.$$

The quality of these bounds becomes evident from the observation that

$$\varphi'(1) = \frac{1}{6} \frac{f'''(1)}{f'(1)^2} - \frac{1}{4} \frac{f''(1)^2}{f'(1)^3},$$

as follows by means of Taylor expansions of  $f$  and  $f'$  about 1.

#### 4. Proofs of the theorems

First let us consider some formulas for moments. There exists a clear-cut expression for the variance of  $Z_n$  due to Fearn [10]. It seems to be less noticed that there is a similar appealing formula for the second factorial moment of  $Z_n$ , which turns out to be more useful for our purpose.

**Lemma 4.** For a BPVE  $(Z_n)_{n \geq 0}$  we have

$$E[Z_n] = \mu_n, \quad \frac{E[Z_n(Z_n - 1)]}{E[Z_n]^2} = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

*Proof.* The proof follows a standard pattern. Let  $v = (f_1, f_2, \dots)$  denote a varying environment. For non-negative integers  $k \leq n$  let us define the probability measures

$$f_{k,n} := f_{k+1} \circ \dots \circ f_n$$

with the convention  $f_{n,n} = \delta_1$  (the Dirac measure at point 1). We have

$$f'_{k,n}(s) = \prod_{l=k+1}^n f'_l(f_{l,n}(s)),$$

in particular  $f'_{n,n}(s) = 1$ , and after some rearrangements we have

$$f''_{k,n}(s) = f'_{k,n}(s)^2 \sum_{l=k+1}^n \frac{f''_l(f_{l,n}(s))}{f'_l(f_{l,n}(s))^2 \prod_{j=k+1}^{l-1} f'_j(f_{j,n}(s))},$$

in particular  $f''_{n,n}(s) = 0$ . Since the distribution of  $Z_n$  is given by  $f_{0,n}$ , by choosing  $k = 0$  and  $s = 1$  Lemma 4 is proved.  $\square$

Next, we recall an expansion of the generating function of  $Z_n$  taken from [16] and [11]. This kind of formula has been used in many investigations of branching processes. Let  $\varphi_n$ ,  $n \geq 1$ , be the shape functions of  $f_n$ ,  $n \geq 1$ . Then, since  $f_{k,n} = f_{k+1} \circ f_{k+1,n}$  for  $k < n$ , we have

$$\frac{1}{1 - f_{k,n}(s)} = \frac{1}{f'_{k+1}(1)(1 - f_{k+1,n}(s))} + \varphi_1(f_{k+1,n}(s)).$$

Iterating the formula we end up with the following identity.

**Lemma 5.** For  $0 \leq s < 1$ ,  $0 \leq k < n$ ,

$$\frac{1}{1 - f_{k,n}(s)} = \frac{\mu_k}{\mu_n(1 - s)} + \varphi_{k,n}(s) \quad \text{with} \quad \varphi_{k,n}(s) := \mu_k \sum_{l=k+1}^n \frac{\varphi_l(f_{l,n}(s))}{\mu_{l-1}},$$

i.e.  $\varphi_{k,n}$  is the shape function of  $f_{k,n}$ .

In order to estimate survival probabilities, Assumption (A) now comes into play. The next lemma reveals its role.

**Lemma 6.** Condition (A) is fulfilled if and only if there is a constant  $c' < \infty$  such that we have  $\varphi_n(1) \leq c' \varphi_n(0)$  for all  $n \geq 1$ .

*Proof.* Recall that  $Y_n$  denotes a random variable with distribution  $f_n$ . We have  $P(Y_n \geq 2) = 0$  if and only if  $\varphi_n(1) = E[Y_n(Y_n - 1)] / (2E[Y_n]^2) = 0$ . Then both inequalities from (A) and from our lemma are valid for all  $c > 0$  and  $c' > 0$ , respectively. Therefore we may, without loss of generality, assume that  $P(Y_n \geq 2) > 0$  for all  $n \geq 1$ . Then we have

$$\varphi_n(0) = \frac{1}{1 - f_n[0]} - \frac{1}{f'_n(1)} = \frac{E[(Y_n - 1); Y_n \geq 1]}{E[Y_n]P(Y_n \geq 1)},$$

and therefore, because of (7),

$$\frac{\varphi_n(1)}{\varphi_n(0)} = \frac{E[Y_n(Y_n - 1)]P(Y_n \geq 1)}{2E[(Y_n - 1); Y_n \geq 1]E[Y_n]}.$$

It is not difficult to see that these expressions are bounded uniformly in  $n$  if and only if the same holds true for the terms

$$\frac{E[Y_n^2; Y_n \geq 2]P(Y_n \geq 1)}{E[Y_n; Y_n \geq 2]E[Y_n]},$$

which in turn is equivalent to condition (A). This gives our claim. □

In particular, if  $\varphi_n(1) \leq c' \varphi_n(0)$  for all  $n \geq 1$  then we obtain for the shape functions  $\varphi_{k,n}$  of the generating functions  $f_{k,n}$  from Lemma 5, by means of Lemmas 6 and 1,

$$\varphi_{k,n}(1) = \mu_n \sum_{l=k+1}^n \frac{\varphi_l(1)}{\mu_{l-1}} \leq c' \mu_n \sum_{l=k+1}^n \frac{\varphi_l(0)}{\mu_{l-1}} \leq 2c' \mu_n \sum_{l=k+1}^n \frac{\varphi_l(f_{l,n}(0))}{\mu_{l-1}} = 2c' \varphi_{k,n}(0)$$

for all  $1 \leq k \leq n$ . This estimate, together with Lemma 4, proves Remark 1 from Section 1, namely that any subsequence of a regular BPVE is regular, too.

The next lemma has a forerunner in Agresti’s estimate [1, Theorem 1].

**Lemma 7.** *Under Assumption (A) there is a  $\gamma > 0$  such that, for all  $n \geq 0$ ,*

$$\frac{E[Z_n]^2}{E[Z_n^2]} \leq P(Z_n > 0) \leq \frac{1}{\gamma} \frac{E[Z_n]^2}{E[Z_n^2]}.$$

*Proof.* The left-hand estimate is just the standard Paley–Zygmund inequality. For the right-hand estimate observe that  $P(Z_n > 0) = 1 - f_{0,n}[0] = 1 - f_{0,n}(0)$ . Using Lemma 4 with  $s = 0$  we get the representation

$$\frac{1}{P(Z_n > 0)} = \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}}, \tag{16}$$

and hence, by means of Lemma 1,

$$\frac{1}{P(Z_n > 0)} \geq \frac{1}{\mu_n} + \frac{1}{2} \sum_{k=1}^n \frac{\varphi_k(0)}{\mu_{k-1}}, \tag{17}$$

and, by Assumption (A), Lemma 6 and (7),

$$\frac{1}{P(Z_n > 0)} \geq \frac{1}{\mu_n} + \frac{1}{2c'} \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} = \frac{1}{\mu_n} + \frac{1}{4c'} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

Letting  $\gamma := \min(1, (4c')^{-1})$ , we obtain

$$\frac{1}{\mathbb{P}(Z_n > 0)} \geq \gamma \left( \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right).$$

On the other hand, Lemma 4 implies that

$$\frac{\mathbb{E}[Z_n^2]}{\mathbb{E}[Z_n]^2} = \frac{\mathbb{E}[Z_n(Z_n - 1)]}{\mathbb{E}[Z_n]^2} + \frac{1}{\mathbb{E}[Z_n]} = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{1}{\mu_n}. \quad (18)$$

Combining the last two formulas, our claim follows.  $\square$

*Proof of Theorem 1.*

(i) if and only if (ii): Since  $\lim_{n \rightarrow \infty} \mathbb{P}(Z_n > 0) = 1 - q$ , the equivalence follows from Lemma 7.

(ii) if and only if (iii): We have

$$\begin{aligned} \sum_{k=1}^n \frac{\rho_k}{\mu_{k-1}} &= \sum_{k=1}^n \frac{\nu_k + f_k(1)^{-1} - 1}{\mu_{k-1}} \\ &= \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \sum_{k=1}^n \left( \frac{1}{\mu_k} - \frac{1}{\mu_{k-1}} \right) = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{1}{\mu_n} - 1; \end{aligned} \quad (19)$$

thus, because of (18),

$$\frac{\mathbb{E}[Z_n^2]}{\mathbb{E}[Z_n]^2} = \sum_{k=1}^n \frac{\rho_k}{\mu_{k-1}} + 1. \quad (20)$$

This gives the claim.

(iii) if and only if (iv): This equivalence is an immediate consequence of (19).

(v) if and only if (vi): This implication follows again from Lemma 7.

(vi) if and only if (vii): This is a consequence of (20).

(vii) if and only if (viii): Again, this claim follows from (19).  $\square$

**Remark 3.** From (17) it follows that a sufficient condition for almost sure extinction is given by the single requirement  $\sum_{k \geq 1} \varphi_k(0)/\mu_{k-1} = \infty$  (without (A)). This confirms a conjecture of Jirina [16].

*Proof of Theorem 2.* Statement (i) is obviously valid. For the first part of statement (ii), note that, from Theorem 1(vi) it follows that  $\sup_{n \geq 0} \mathbb{E}[W_n^2] < \infty$ . Therefore the martingale  $(W_n)_{n \geq 0}$  is bounded in  $\mathcal{L}^2$ , implying  $\mathbb{E}[W] = \mathbb{E}[W_0] = 1$  and  $\mathbb{E}[W^2] < \infty$ . From (20) it follows that

$$\mathbb{E}[W^2] = \sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}} + 1.$$

This implies (1).

For the proof of the last claim we distinguish two cases. Either  $\mu_n \rightarrow r$  with  $0 < r < \infty$ , in which case  $W_n = Z_n/\mu_n \rightarrow Z_\infty/r$  a.s. and consequently  $W = Z_\infty/r$  a.s. and  $P(W = 0) = P(Z_\infty = 0) = q$ , or we may assume  $\mu_n \rightarrow \infty$  in view of Theorem 1(viii). Also,  $\{Z_\infty = 0\} \subset \{W = 0\}$  a.s., and thus it is sufficient to show that  $P(Z_\infty > 0, W = 0) = 0$ . First, we estimate  $P(Z_\infty = 0 | Z_k = 1)$  from below. From Lemmas 5 and 1, for  $k < n$ ,

$$\frac{1}{1 - P(Z_n = 0 | Z_k = 1)} = \frac{1}{1 - f_{k,n}(0)} \geq \frac{1}{2} \mu_k \sum_{l=k+1}^n \frac{\varphi_l(0)}{\mu_{l-1}},$$

as well as

$$\begin{aligned} \frac{1}{1 - E[e^{-\lambda W_n} | Z_k = 1]} &= \frac{1}{1 - f_{k,n}(e^{-\lambda/\mu_n})} \\ &\leq \frac{\mu_k}{\mu_n(1 - e^{-\lambda/\mu_n})} + 2\mu_k \sum_{l=k+1}^n \frac{\varphi_l(1)}{\mu_{l-1}} \end{aligned}$$

with  $\lambda > 0$ . By means of Lemma 4 this entails

$$\frac{1}{1 - E[e^{-\lambda W_n} | Z_k = 1]} \leq \frac{\mu_k}{\mu_n(1 - e^{-\lambda/\mu_n})} + \frac{4c'}{1 - P(Z_n = 0 | Z_k = 1)}.$$

Letting  $n \rightarrow \infty$  we get

$$\frac{1}{1 - E[e^{-\lambda W} | Z_k = 1]} \leq \frac{\mu_k}{\lambda} + \frac{4c'}{1 - P(Z_\infty = 0 | Z_k = 1)},$$

and with  $\lambda \rightarrow \infty$ ,

$$\frac{1}{P(W > 0 | Z_k = 1)} \leq \frac{4c'}{P(Z_\infty > 0 | Z_k = 1)}.$$

Using  $e^{-2x} \leq 1 - x$  for  $0 \leq x \leq 1/2$ , it follows for  $P(W > 0 | Z_k = 1) \leq (8c')^{-1}$  that

$$\begin{aligned} P(Z_\infty = 0 | Z_k = 1) &= 1 - P(Z_\infty > 0 | Z_k = 1) \geq 1 - 4c'P(W > 0 | Z_k = 1) \\ &\geq e^{-8c'P(W > 0 | Z_k = 1)} \geq (1 - P(W > 0 | Z_k = 1))^{8c'} \\ &= P(W = 0 | Z_k = 1)^{8c'}. \end{aligned} \tag{21}$$

Now we draw on a martingale which already appears in the work of D'Souza and Biggins [7]. For  $n \geq 0$ , let

$$M_n := P(W = 0 | Z_0, \dots, Z_n) = P(W = 0 | Z_n = 1)^{Z_n} \text{ a.s.}$$

From standard martingale theory  $M_n \rightarrow I\{W = 0\}$  a.s. In particular, we have

$$P(W = 0 | Z_n = 1)^{Z_n} \rightarrow 1 \text{ a.s. on the event that } W = 0, \tag{22}$$

a result which has already been exploited by D'Souza [6].

We distinguish two cases. Either there is an infinite sequence of natural numbers such that  $P(W > 0 | Z_n = 1) > (8c')^{-1}$  along this sequence, so (22) implies that  $Z_n \rightarrow 0$  a.s. on the event  $W = 0$ , or we may apply our estimate (21) to obtain from (22) that

$$P(Z_\infty = 0 | Z_n = 1)^{Z_n} \rightarrow 1 \text{ a.s. on the event that } W = 0.$$

Therefore, given  $\varepsilon > 0$ , we have, for  $n$  sufficiently large,

$$\begin{aligned} P(Z_\infty > 0, W = 0) &\leq \varepsilon + P(Z_n > 0, P(Z_\infty = 0 \mid Z_n = 1)^{Z_n} \geq 1 - \varepsilon) \\ &\leq \varepsilon + \frac{1}{1 - \varepsilon} E[P(Z_\infty = 0 \mid Z_n); Z_n > 0] \\ &= \varepsilon + \frac{1}{1 - \varepsilon} P(Z_\infty = 0, Z_n > 0). \end{aligned}$$

Letting  $n \rightarrow \infty$  we thus obtain  $P(Z_\infty > 0, W = 0) \leq \varepsilon$ ; the claim then follows with  $\varepsilon \rightarrow 0$ .  $\square$

*Proof of Theorem 3.* We begin with the proof of the last claim. Note that the assertion from Lemma 7 can be rewritten as

$$\gamma \frac{E[Z_n^2]}{E[Z_n]} \leq E[Z_n \mid Z_n > 0] \leq \frac{E[Z_n^2]}{E[Z_n]},$$

and (18) gives  $E[Z_n^2]/E[Z_n] = 1 + \mu_n \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} = a_n$ . This implies (4).

Consequently, by means of Markov’s inequality we obtain

$$P(Z_n/a_n > u \mid Z_n > 0) \leq \frac{1}{ua_n} E[Z_n \mid Z_n > 0] \leq \frac{1}{u},$$

which implies the theorem’s first claim.

Concerning the second claim we remark that for  $a_n < 2$  we may set  $u = 1/2$ . For  $a_n \geq 2$  we have, by means of Lemma 5, the estimate

$$\begin{aligned} 1 - s^u + P(Z_n/a_n > u) &\geq E[1 - s^{Z_n/a_n} \mid Z_n > 0]E[1 - s^{Z_n/a_n} \mid Z_n > 0] \\ &= \frac{1 - f_{0,n}(s^{1/a_n})}{1 - f_{0,n}(0)} \\ &= \left( \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}} \right) / \left( \frac{1}{\mu_n(1 - s^{1/a_n})} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s^{1/a_n}))}{\mu_{k-1}} \right) \end{aligned}$$

with  $0 < s < 1$  and  $u > 0$ . Lemmas 1 and 6 along with (7) yield the bound

$$\begin{aligned} 1 - s^u + P(Z_n/a_n > u) &\geq \sum_{k=1}^n \frac{\varphi_k(0)}{2\mu_{k-1}} / \left( \frac{1}{\mu_n(1 - s^{1/a_n})} + 2 \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} \right) \\ &\geq \frac{1}{4c'} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} / \left( \frac{1}{\mu_n(1 - s^{1/a_n})} + \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} \right). \end{aligned}$$

Moreover,  $1 - s^{1/a_n} \geq a_n^{-1}(1 - s)$ , since  $1/a_n \leq 1$ . Hence, choosing  $s = 1/2$  we get

$$1 - 2^{-u} + P(Z_n/a_n > u) \geq \frac{1}{4c'} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} / \left( \frac{2a_n}{\mu_n} + \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} \right).$$

Finally, from  $a_n \geq 2$  it follows that  $a_n \leq 2\mu_n \sum_{k=1}^n v_k/\mu_{k-1}$ , and consequently

$$1 - 2^{-u} + P(Z_n/a_n > u) \geq \frac{1}{20c'}$$

for all  $u > 0$ . If we now set  $\theta = 1/(40c')$  and choose  $u > 0$  so small that  $1 - 2^{-u} \leq \theta$  we obtain  $P(Z_n/a_n > u) \geq \theta$ , which is our second claim.  $\square$

The next lemma prepares the proof of Theorem 4. It clarifies the role of (B).

**Lemma 8.** *Assume condition (B) and let  $q = 1$ . Then the condition*

$$\frac{1}{\mu_n} = o\left(\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}\right)$$

implies

$$\sup_{0 \leq s \leq 1} \left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} \right| = o\left(\sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}}\right)$$

as  $n \rightarrow \infty$ .

*Proof.* Fix  $\varepsilon > 0$  and choose  $c_{\varepsilon/9}$  according to Assumption (B). Let

$$s_k := 1 - \frac{\eta}{1 + f'_k(1)}$$

with some  $0 < \eta < 1$ . Then, from Lemma 3 with  $a = \lfloor c_{\varepsilon/9} \rfloor$ ,

$$\sup_{s_k \leq t \leq 1} |\varphi_k(1) - \varphi_k(t)| \leq 2\nu_k \frac{f''_k(1)}{f'_k(1)} \frac{\eta}{1 + f'_k(1)} + 2c_{\varepsilon/9} \nu_k \eta + \frac{\varepsilon}{9} 4\nu_k.$$

From the estimate in (6) it follows that

$$f''_k(1) \leq 2c_{1/2} f'_k(1) (1 + f'_k(1)). \tag{23}$$

Therefore there is an  $\eta = \eta_\varepsilon > 0$  such that

$$\sup_{s_k \leq t \leq 1} |\varphi_k(1) - \varphi_k(t)| \leq \frac{\varepsilon}{2} \nu_k = \varepsilon \varphi_k(1). \tag{24}$$

Now set

$$r = r_{\varepsilon,n} := \min\{k \leq n : f_{k,n}(0) \leq s_k\}.$$

Because of  $f_{n,n}(0) = 0$  this minimum is attained. In view of (24) and Lemma 1 it follows that

$$\left| \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} - \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} \right| \leq \varepsilon \sum_{k=1}^{r-1} \frac{\varphi_k(1)}{\mu_{k-1}} + 3 \frac{\varphi_r(1)}{\mu_{r-1}} + 3 \sum_{k=r+1}^n \frac{\varphi_k(1)}{\mu_{k-1}}.$$

From (23) we have

$$\frac{\varphi_r(1)}{\mu_{r-1}} = \frac{f''_r(1)}{2f'_r(1)^2 \mu_{r-1}} \leq \frac{c_{1/2}(f'_r(1) + 1)}{f'_r(1) \mu_{r-1}} = c_{1/2} \left( \frac{1}{\mu_{r-1}} + \frac{1}{\mu_r} \right),$$

and from Lemma 6,

$$\sum_{k=r+1}^n \frac{\varphi_k(1)}{\mu_{k-1}} \leq c' \sum_{k=r+1}^n \frac{\varphi_k(0)}{\mu_{k-1}} \leq 2c' \sum_{k=r+1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}}.$$

From (16) it follows that  $P(Z_n > 0 | Z_r = 1)^{-1} = \mu_r/\mu_n + \mu_r \sum_{k=r+1}^n \varphi_k(f_{k,n}(0))/\mu_{k-1}$  for  $n > r$ , and hence we may proceed to

$$\sum_{k=r+1}^n \frac{\varphi_k(1)}{\mu_{k-1}} \leq \frac{2c'}{\mu_r(1 - f_{r,n}(0))} \leq \frac{2c'}{\mu_r(1 - s_r)} = \frac{2c'(f'_r(1) + 1)}{\eta\mu_r} = \frac{2c'}{\eta} \left( \frac{1}{\mu_{r-1}} + \frac{1}{\mu_r} \right).$$

Putting our estimates together, we get

$$\left| \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} - \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} \right| \leq \varepsilon \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} + 3 \left( c_{1/2} + \frac{2c'}{\eta} \right) \left( \frac{1}{\mu_{r-1}} + \frac{1}{\mu_r} \right). \tag{25}$$

Now the assumption  $1/\mu_n = o(\sum_{k=1}^n v_k/\mu_{k-1})$  comes into play. It implies that there is a positive integer  $r_\varepsilon$  such that, for all  $r, n$  with  $r_\varepsilon < r \leq n$ ,

$$3 \left( \frac{2c'}{\eta} + c_{1/2} \right) \left( \frac{1}{\mu_{r-1}} + \frac{1}{\mu_r} \right) \leq \frac{\varepsilon}{2} \sum_{k=1}^{r-1} \frac{v_k}{\mu_{k-1}} + \frac{\varepsilon}{2} \sum_{k=1}^r \frac{v_k}{\mu_{k-1}} \leq \varepsilon \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}}. \tag{26}$$

Also, from the assumptions that  $q=1$  and  $1/\mu_n = o(\sum_{k=1}^n v_k/\mu_{k-1})$ , together with Theorem 1(iv) and (7), we have

$$\sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} = \frac{1}{2} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}} \rightarrow \infty$$

as  $n \rightarrow \infty$ , which implies that (26) holds for all  $r \leq r_\varepsilon$  and thus for all  $r \leq n$ , if only  $n$  is large enough. Thereby we may combine (25) and (26) to obtain

$$\left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} \right| \leq 2\varepsilon \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}}$$

for sufficiently large  $n$ . This proves our claim. □

*Proof of Theorem 4.*

(i) implies (ii): We argue by contradiction. If assertion (ii) fails, then there is an increasing sequence  $(n_i)_{i \geq 0}$  in  $\mathbb{N}$  fulfilling  $\sup_i E[Z_{n_i} | Z_{n_i} > 0] < \infty$ . From Theorem 3 it follows that the random variables  $Z_{n_i}, i \geq 0$ , conditioned on  $Z_{n_i} > 0$  are tight. This does not conform with assertion (i), which proves the implication.

(ii) implies (iii): This implication follows from Theorem 3, since assertion (iii) just states that  $a_n \rightarrow \infty$ .

(iii) implies (i): For the proof, let

$$b_n := \frac{\mu_n}{2} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}}.$$

From Lemma 5 we have

$$\begin{aligned} 1 - E[e^{-\lambda Z_n/b_n} | Z_n > 0] &= \frac{1 - f_{0,n}(e^{-\lambda/b_n})}{1 - f_{0,n}(0)} \\ &= \left( \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}} \right) / \left( \frac{1}{\mu_n(1 - e^{-\lambda/b_n})} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(e^{-\lambda/b_n}))}{\mu_{k-1}} \right). \end{aligned}$$

Since  $b_n \rightarrow \infty$ , from Lemma 8 and the theorem’s assumption we have

$$1 - E[e^{-\lambda Z_n/b_n} | Z_n > 0] = \left( (1 + o(1)) \sum_{k=1}^n \frac{v_k}{2\mu_{k-1}} \right) / \left( (1 + o(1)) \frac{b_n}{\lambda \mu_n} + (1 + o(1)) \sum_{k=1}^n \frac{v_k}{2\mu_{k-1}} \right)$$

as  $n \rightarrow \infty$ . From the definition of  $b_n$  we get

$$1 - E[e^{-\lambda Z_n/b_n} | Z_n > 0] = \frac{\lambda + o(1)}{1 + \lambda}.$$

This implies assertion (i).

Moreover, from (16), Lemma 8 and assertion (iii) it follows that

$$\frac{1}{P(Z_n > 0)} = \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}} \sim \frac{1}{2} \sum_{k=1}^n \frac{v_k}{\mu_{k-1}}.$$

This formula give the extra claims, which concludes the proof. □

*Proof of Proposition 1.* By Theorem 1(viii) the condition  $q < 1$  is equivalent to the requirements of both  $\sum_{k=1}^\infty v_k/\mu_{k-1} < \infty$  and  $0 < \lim_n \mu_n \leq \infty$ . As already explained, the division between the supercritical regime and the asymptotically non-degenerate regime corresponds to the cases  $\lim_n \mu_n = \infty$  and  $0 < \lim_n \mu_n < \infty$ . This gives the first two assertions of the proposition.

Next, the critical regime is given by the requirements that both  $E[Z_n | Z_n > 0] \rightarrow \infty$  and  $q = 1$ . By Theorems 3 and 1(iv) we may equivalently require that  $1/\mu_n = o(\sum_{k=1}^n v_k/\mu_{k-1})$  together with either  $\sum_{k=1}^n v_k/\mu_{k-1} = \infty$  or  $\mu_n \rightarrow 0$ . However, the third and the first of these conditions imply the second one, therefore the third condition can be skipped, and we end up with the requirements  $1/\mu_n = o(\sum_{k=1}^n v_k/\mu_{k-1})$  and  $\sum_{k=1}^n v_k/\mu_{k-1} = \infty$ , as stated in the proposition.

Finally, the subcritical regime is characterized by the conditions  $E[Z_n | Z_n > 0] \not\rightarrow \infty$  and  $q = 1$ . Because of Theorem 3, the first condition is equivalent to the requirement  $a_n \not\rightarrow \infty$ , respectively to  $\liminf_n \mu_n \sum_{k=1}^n v_k/\mu_{k-1} < \infty$ . Moreover,  $\liminf_n \mu_n = 0$  implies  $q = 1$ , and therefore the conditions stated in the proposition imply subcriticality. Conversely, if  $q = 1$  then by Theorem 1(iv) we have  $\lim_n \mu_n = 0$  or  $\sum_{k=1}^n v_k/\mu_{k-1} < \infty$ . The former of these conditions trivially yields  $\liminf_n \mu_n = 0$ , whereas the latter, together with  $\liminf_n \mu_n \sum_{k=1}^n v_k/\mu_{k-1} < \infty$ , implies  $\liminf_n \mu_n = 0$ . Therefore the two conditions stated in the proposition are also necessary for subcriticality. □

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