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EXTENSION OF A SEMIGROUP EMBEDDING THEOREM TO SEMIRINGS

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It is well known [1, 3] that a commutative semigroup $(S, +)$ can be embedded in a semigroup which is a union of groups if and only if S is *separative* ($2a=a+b=2b$ implies $a=b$). We extend this result to additively commutative semirings.

A *semiring* $(S, +, \cdot)$ is a set S with associative addition $(+)$ and multiplication $(\cdot)$, the latter distributing over addition from left and right. In what follows $(S, +, \cdot)$ will denote a semiring in which the additive semigroup $(S, +)$ is commutative. An element 0 can be adjoined, where $s=s+0$, $0=0\cdot s=s\cdot 0$ for all s in S, to form S^o. Then *a divides b*, written a/b, if $a+x=b$ for some x in S^o. The semiring congruence N is defined by aNb if a/mb and b/na for positive integers m and n. A semiring S is *archimedean* if aNb for each a and b in S. The following two results are now direct extensions of material from [1].

LEMMA 1. *Let S be a semiring, $S_a (a\in Y)$ the congruence classes of N.*

(1) *N is the smallest semiring congruence such that S/N is an additive semilattice: each $(S_a, +)(a \in Y)$ is an archimedean semigroup.*

(2) *The decomposition of $(S, +)$ into a semilattice of archimedean semigroups is unique and S is additively separative if and only if the archimedean components are additively cancellative.*

(3) *If S is an additively commutative semiring such that $x^2=x$ for all $x\in S$, then for each $a\in Y$, $(S_a, +, \cdot)$ is an archimedean semiring.*

In $S\times S$ let $T'=\{(a, b): aNb \text{ in } S\}$ and define the relation M on T' by $(a, b)M(c, d)$ if and only if both aNc and $a+d=b+c$.

THEOREM 2. *Let S be an additively separative semiring, T' and M as above. On T' define $(a, b)+(c, d)=(a+c, b+d)$ and $(a, b)(c, d)=(ac+bd, ad+bc)$. Then:*

(1) *$(T', +, \cdot)$ is an additively commutative semiring.*

(2) *M is a congruence and $T=T'/M$ a union of additive groups.*

(3) *Denoting elements of T by $[a, b]$, the map $F: x\to[2x, x]$ is an embedding of S into T.*

Proof. Clearly T' is a semiring with commutative addition. It is easily shown that M is reflexive and symmetric, while transitivity of M follows from additive cancellation in the classes of the congruence N on S. Similarly M is shown to be compatible with addition and multiplication in T'.

For $(a, b) \in T'$ we have also that (b, a) and (a, a) are in T' and obtain the result

$$[a, b]+[a, a] = [a+a, b+a] = [a, b]$$

since $(a)N(b)N(2a)N(a+b)$ and $2a+b=b+2a$. The inverse of $[a, b]$ in T is the element $[b, a]$.

If $F(x)=F(y)$ then $[2x, x]=[2y, y]$, hence $(2x)N(x)N(y)N(2y)$ and $2x+y=x+2y$. The N-congruence class containing x and y is cancellative, implying $x=y$: F is thus injective. Trivially F is an additive homomorphism. For x and y in S we obtain $(xy)N(2xy)N(4xy)N(5xy)$ and $2(xy)+4(xy)=xy+5xy$, thereby proving that $F(xy)=[2(xy), xy]=[5(xy), 4(xy)]=[2x, y]=F(x)\cdot F(y)$.

THEOREM 3. *Let $(S, +, \cdot)$ be an additively commutative semiring.*

(1) *S is embeddable in a semiring which is a union of additive groups if and only if S is additively separative.*

(2) *If $x=x^2$ for all x in S, and S is additively separative, then S is embeddable in a semiring which is a union of rings.*

Proof. We need only consider the case where S is embeddable in a semiring T, T being a union of additive groups and therefore the union of maximal additive groups, written $H(x)$ for x in S. Let $a, b \in S$, such that $2a=a+b=2b$. Then $H(a)$ contains the image of $2a$, $H(b)$ the image of $2b$ under the embedding, implying that $H(a)$ meets $H(b)$ and thus that $H(a)=H(b)$ [1]. Cancellation in $H(a)$ then implies $a=b$.

Recall that $F(x)=[2x, x]$ from Lemma 2. The element $[x, x]$ is both an additive idempotent and an additive identity for $[2x, x]$ and is contained in a maximal additive subgroup, denoted here by $H(x)$. From [2] $H(x)$ is a subring if and only if $[x, x]$ is also multiplicatively idempotent. Clearly $x=x^2$ implies $[x, x]=([x, x])^2$ and consequently that $H(x)$ is a subring. The archimedean components of the decomposition of S in Lemma 1 will be subsemirings under this condition.

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