ON THE EXISTENCE OF SOLUTIONS OF THE EQUATION $Lx \in Nx$ AND A COINCIDENCE DEGREE THEORY

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Communicated by J. B. Miller

Abstract

The coincidence degree for the pair (L, N) developed by Mawhin (1972) provides a method for proving the existence of solutions of the equation $Lx = Nx$ where $L: \operatorname{dom} L \subset X \to Z$ is a linear Fredholm mapping of index zero and $N: \overline{\Omega} \to Z$ is a (possibly nonlinear) mapping and Ω is a bounded open subset of X, X and Z being normed linear spaces over the reals. In this paper we have extended the coincidence degree for the pair (L, N) to solve the equation

$$Lx \in Nx,$$

where $L: \operatorname{dom} L \subset X \to Z$ is a linear Fredholm mapping of index zero, $N: \overline{\Omega} \to CK(Z)$ and X, Z and Ω are as above, $CK(Z)$ being the set of compact convex subsets of Z.

Subject classification (Amer. Math. Soc. (MOS) 1970): primary 47 H 15, 47 A 50; secondary 47 H 10, 47 A 55.

Keywords and phrases: Set-valued compact vector field, set-valued ultimately compact vector field, k-φ-contractions, coincidence degree.

Introduction

Let X and Z be normed linear spaces over the reals. Extensive researches have been undertaken on the study of the operator equation

(0.1) $$Lx = Nx,$$

where $L: \operatorname{dom} L \subset X \to Z$ is a linear mapping and $N: \operatorname{dom} N \subset X \to Z$ is a (possibly nonlinear) mapping. The equation (0.1) represents a wide class of problems including nonlinear ordinary, partial and functional differential equations. When L^{-1} exists, the reduced equation $x = L^{-1} Nx$ is under the scope of fixed point theory. For extensive literature for this case we refer to the survey works of Dolph and Minty (1964) and Ehrmann (1965).

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When L^{-1} does not exist and X and Z are Banach spaces, the basic works on the study of the equation (0.1) are due to Cacciopoli (1946), Shimizu (1948), Cronin (1950), Bartle (1953), Vainberg and Trenogin (1962), Vainberg and Aizengender (1968) and Nirenberg (1960–1961). These works involve some smallness assumption on N. The method for finding solutions of the equation (0.1), initiated by Cesari (1963) and (1964) and further developed by Locker (1967), Bancroft *et al.* (1968) and Williams (1968) deals with a more general class of mappings. For application of Cesari's method to differential equation we refer to Cesari (1969, 1971) and Hale (1969, 1971).

Using an equivalence theorem which reduces the problem of existence of solutions of the equation (0.1) to that of fixed points of an auxiliary mapping and Leray–Schauder degree, Mawhin (1972) developed a degree called the coincidence degree for the pair (L, N) and applied to nonlinear differential equations (for example, see Gaines and Mawhin (1977). In essence, Mawhin's method preserves the spirit of the works of the authors mentioned above.

In the recent past the Leray–Schauder degree theory for a single-valued compact vector field has been extended to a larger class of single-valued mappings, namely to k-set contractive vector fields by Nussbaum (1969, 1971), ball condensing vector fields by Vainikko and Sadovskii (1968) and Borisovich and Sapronov (1968), ultimately compact vector fields by Sadovskii (1968) (see also Sadovskii (1972) and Danes (1974)). On the other hand, Leray–Schauder degree theory has been extended to set-valued compact vector fields by Granas (1959), Cellina and Lasota (1969), Ma (1972) and more recently to ultimately compact vector fields by Petryshyn and Fitzpatrick (1974).

The coincidence degree of Mawhin (1972) has been sharpened by Hetzer (1975a, b) by replacing the complete continuity assumption by k-set contraction with $k < 1$ and Leray–Schauder degree by the corresponding degree of k-set contractive vector field mentioned above.

The purpose of this paper is to consider the equation

$$(0.2) \qquad\qquad Lx \in N(x),$$

where L: dom $L \subset X \to Z$ is a single-valued linear Fredholm mapping of index zero and N: dom $N \subset X \to CK(Z)$ is a mapping, X and Z being normed linear spaces.

Like Mawhin (1972) we have proved equivalence theorems which reduce the problem of existence of solutions of the equation (0.2) to that of fixed points of an auxiliary set-valued vector fields given by Petryshyn and Fitzpatrick (1974), and we have built up the coincidence degree theory for the pair (L, N) appearing in the equation (0.2). We have proved that this degree has all the usual properties of a degree theory. We have also extended the Rouche's theorem and the Leray–Schauder continuation principle to our context.

1. Degree theory for set-valued ultimately compact vector fields

In this section, we shall recall the concept of an ultimately compact mapping and the degree theory of such mappings as introduced by Petryshyn and Fitzpatrick (1974). We shall also consider the definition of measure of non-compactness and the definition and properties of $k-\varphi$-contractions as a special class of ultimately compact mappings.

1A. Notations and definitions

Let X denote a separated locally convex topological vector space over the reals with the additional property that for each compact subset A of X, there is a retraction of X onto the convex closure of A. By virtue of a theorem due to Dugundji (1951), this property automatically holds when X is metrizable, especially when X is a normed linear space. For any $B \subset X$, let $\overline{CO}\, B$ denote the convex closure of B and let $\bar{B}$ and ∂B denote respectively the closure and boundary of B. Let $K(B)$ and $CK(B)$ denote respectively the set of nonempty closed convex subsets of B and the set of nonempty compact convex subsets of B. If F is a set-valued mapping, then $F(B) = \bigcup_{x \in B} F(x)$.

DEFINITION 1.1. A mapping F defined on a set $B \subset X$ and taking values in the set of subsets of X is said to be *upper-semicontinuous*, henceforth denoted u.s.c., if, given an open set V in X with $F(x) \subset V$, there exists an open subset W of X containing x such that $F(W) \subset V$. A u.s.c. mapping F is said to be a *compact vector field* if $(I-F)(B)$ is relatively compact.

CONSTRUCTION. Let $\Omega \subset X$ be an open set and let $F: \bar{\Omega} \to K(X)$ be u.s.c. We define a transfinite sequence $\{K_\alpha\}$ as follows:

$$K_0 = \overline{CO}\, F(\bar{\Omega})$$

$$K_\alpha = \begin{cases} \overline{CO}\, F(\bar{\Omega} \cap K_{\alpha-1}) & \text{if } \alpha \text{ is an ordinal such that } \alpha-1 \text{ exists,} \\ \bigcap_{\beta \leq \alpha} K_\beta & \text{if } \alpha \text{ is an ordinal such that } \alpha-1 \text{ does not exist.} \end{cases}$$

It is easily verified that the following properties hold:

(1.1)   each K_α is closed, convex and $K_\alpha \subset K_\beta$ for all $\alpha \geq \beta$,

(1.2) $F(K_\alpha \cap \bar{\Omega}) \subset K_\alpha$ for each ordinal α.

Since the transfinite sequence $\{K_\alpha\}$ is nonincreasing, there is an ordinal γ such that $K_\gamma = K_{\gamma+1}$ and hence, $K_\beta = K_\gamma$ for each $\beta \geq \gamma$. We define $K = K(F, \bar{\Omega}) = K_\gamma$. Clearly,

$$K_\gamma = K_{\gamma+1} = \overline{CO}\, F(\bar{\Omega} \cap K_\gamma) = \overline{CO}\, F(\bar{\Omega} \cap K).$$

We thus have

(1.3) $$K = \bigcap_\beta K_\beta = \overline{CO}\, F(\overline{\Omega} \cap K).$$

DEFINITION 1.2. A u.s.c. mapping $F: \overline{\Omega} \to K(X)$ is said to be *ultimately compact* if either $K \cap \overline{\Omega} = \varnothing$ or, if $K \cap \overline{\Omega} \neq \varnothing$, then $F(\overline{\Omega} \cap K)$ is relatively compact. If F is an ultimately compact mapping, we shall call $I - F$ an *ultimately compact vector field*, where I is the identity mapping on X.

DEFINITION 1.3. Let $\Omega \subset X$ be open and let $F: \overline{\Omega} \to K(X)$ be ultimately compact with $0 \notin x - F(x)$ for each $x \in \partial \Omega$. If $K \cap \Omega$ is empty, define *the degree of $I - F$ on Ω with respect to zero*, denoted by $d(I - F, \Omega, 0)$, to be zero. If $K \cap \Omega \neq \varnothing$, let ρ be a retraction of X onto K and define

(1.4) $$d(I - f, \Omega, 0) = d_c(I - F\rho, \rho^{-1}(\Omega), 0),$$

where the right-hand term is the degree for compact set-valued vector fields given by Ma (1972).

REMARK 1.1. To see that this degree is well defined and has all the usual properties of the Leray–Schauder degree, please refer to Petryshyn and Fitzpatrick (1974).

The following theorem which we shall use later in the proof of the Continuation Theorem seems to be new and has its own interest.

THEOREM 1.1. (Reduction Formula.) *Let X be a metrizable, locally convex topological vector space with the additional property that, for each closed subspace E and any compact subset B of E, there exists a retraction of E onto the convex closure of B. Let $F: \overline{\Omega} \to K(X)$ be an ultimately compact mapping such that $x \notin F(x)$ for each $x \in \partial \Omega$. Let E_0 be a finite dimensional subspace of X containing the closure of $F(\overline{\Omega})$. Then*

$$d(I - F, \Omega, 0) = d(I - F|_{\overline{\Omega} \cap E_0}, \Omega \cap E_0, 0).$$

PROOF. As K is a closed convex compact set, $K \cap E_0$ is also closed convex and compact. Let $\rho_1: E_0 \to K \cap E_0$ be a retraction of E_0 onto $K \cap E_0$. Now define $\rho_2: K \cup E_0 \to K$ by

$$\rho_2(x) = \begin{cases} \rho_1(x) & \text{if } x \in E_0, \\ x & \text{if } x \in K. \end{cases}$$

As ρ_1 is a retraction of E_0 onto $K \cap E_0$, $\rho_1(x) = x$ for all x in $K \cap E_0$ and it is clear that ρ_2 is well defined in $K \cup E_0$. Also, since $\partial K \cap E_0 \subset \partial(K \cap E_0)$, and ρ_1 is continuous on E_0, ρ_2 is a continuous mapping on the closed set $K \cup E_0$. By Dugundji's Extension of Tietze's Theorem (1951), there exists an extension ρ of ρ_2, $\rho: X \to \overline{CO}\, \rho_2(K \cup E_0)$ such that ρ is continuous. Now as $\rho_2(K \cup E_0) = K$

which is closed and convex, ρ is a retraction of X onto K and is an extension of ρ_2 and ρ_1. Now,

$$x \in \rho_1^{-1}(\Omega \cap E_0) \Leftrightarrow \rho_1 x \in \Omega \quad \text{and} \quad \rho_1 x \in E_0$$

$$\Leftrightarrow x \in E_0 \quad \text{and} \quad \rho x = \rho_1 x \in \Omega$$

$$\Leftrightarrow x \in E_0 \cap \rho^{-1}(\Omega).$$

Hence we have $\rho_1^{-1}(\Omega \cup E_0) = E_0 \cap \rho^{-1}(\Omega)$. By Definition 1.3,

$$d(I - F|_{\bar{\Omega} \cap E_0}, \Omega \cap E, 0) = d_c(I - F\rho_1|_{(\rho_1^{-1}(\Omega \cap E_0))^-}, \rho_1^{-1}(\Omega \cap E_0), 0)$$

$$= d_c(I - F\rho_1|_{(\rho^{-1}(\Omega) \cap E_0)^-}, \rho^{-1}(\Omega) \cap E_0, 0)$$

$$= d_c(I - F\rho|_{(\rho^{-1}(\Omega) \cap E_0)^-}, \rho^{-1}(\Omega) \cap E_0, 0).$$

By the continuity of ρ and the hypothesis of the theorem that $(F(\bar{\Omega}))^- \subset E_0$, we have

$$(F\rho(\rho^{-1}(\Omega))^-)^- = (F\rho(\rho^{-1}(\bar{\Omega})))^-$$

$$\subset (F(\bar{\Omega}))^-$$

$$\subset E_0.$$

Hence, we may apply Theorem 11.1 of Ma (1972) and we have

$$d_c(I - F\rho)|_{(\rho^{-1}(\Omega) \cap E_0)^-}, \rho^{-1}(\Omega) \cap E_0, 0)$$

$$= d_c(I - F\rho, \rho^{-1}(\Omega), 0)$$

$$= d(I - F, \Omega, 0),$$

the last equality holding by Definition 1.3 as ρ is a retraction of X onto K. Hence we obtain the required result,

$$d(I - F, \Omega, 0) = d(I - F|_{E_0 \cap \bar{\Omega}} \Omega \cap E_0, 0).$$

1C. $k - \varphi$-contractions

DEFINITION 1.4. Let C be a lattice with a minimal element which we denote by zero, 0. A mapping $\varphi: 2^X \to C$, where 2^X denotes the family of all subsets of X, is called a *measure of noncompactness* if, for any $A \subset X$, $B \subset X$, it satisfies the following properties:

(1.5) $\varphi(\overline{\text{CO}} A) = \varphi(A),$

(1.6) $\varphi(A) = 0$ if and only if A is precompact,

(1.7) $\varphi(A \cup B) = \max \{\varphi(A), \varphi(B)\}.$

It follows from (1.7) that $A \subset B \Rightarrow \varphi(A) \leqslant \varphi(B)$.

DEFINITION 1.5. Let φ be a measure of noncompactness and we additionally assume that the lattice C has the property that, for each $c \in C$ and $\lambda \in R$ with $\lambda > 0$, there is defined an element $\lambda c \in C$. An u.s.c. mapping $F: \overline{\Omega} \to CK(X)$ is called a $k - \varphi$-contraction or a $k - \varphi$-contractive mapping if there exists some $k > 0$ such that, for every subset A of $\overline{\Omega}$,

$$\varphi(F(A)) \leqslant k\varphi(A).$$

The following three propositions follow almost immediately from the definition of a $k - \varphi$-contraction and the proofs will be left to the reader.

PROPOSITION 1.1. Let φ be a measure of noncompactness as given in Definition 1.5, with the additional property that, for any $A \subset X$, $B \subset X$,

(1.8) $$\varphi(A+B) \leqslant \varphi(A)+\varphi(B)$$

If $F: \overline{\Omega} \to CK(X)$ and $G: \overline{\Omega} \to CK(X)$ are k_1- and k_2-φ-contractions respectively then $(F+G): \overline{\Omega} \to CK(X)$ defined by

$$(F+G)(x) = F(x)+G(x)$$

is a $(k_1+k_2) - \varphi$-contraction.

PROPOSITION 1.2. Let φ be a measure of noncompactness as in Definition 1.5. Let $F: \overline{\Omega} \to CK(X)$ be a $k_1 - \varphi$-contraction and let $G: X \to X$ be a linear, continuous, single-valued mapping such that, for each $A \subset X$, we have

$$\varphi(G(A)) \leqslant k_2\varphi(A)$$

Then $GF: \overline{\Omega} \to CK(X)$ defined by

$$GF(x) = \{G(y): y \in F(x)\}$$

is a $k_1 k_2 - \varphi$-contraction.

NOTE. Linearity and continuity of G ensures that $GF(x)$ is a compact, convex set for each $x \in \overline{\Omega}$.

PROPOSITION 1.3. Let φ be a measure of noncompactness as in Definition 1.5. If F and G are $k - \varphi$-contractions, then so is $\lambda F + (1-\lambda)G$, where $\lambda \in [0,1]$.

THEOREM 1.2. Let $\varphi: 2^X \to R^+ = \{t \in R: t \geqslant 0\} \cup \{\infty\}$ be a measure of noncompactness and suppose that $F: \overline{\Omega} \to CK(X)$ is a $k - \varphi$-contraction with $0 < k < 1$ and $\varphi(F(\Omega)) < \infty$. If either X is quasi-complete or $\overline{\Omega}$ is complete, then F is ultimately compact.

PROOF. This follows from Lemmas 3.2 and 3.4 of Petryshyn and Fitzpatrick (1974).

The $k-\varphi$-contractions as defined in Definition 1.5 are a generalization of k-ball-contractions and k-set-contractions for multivalued mappings and are an extension of the $k-\varphi$-contractions for single-valued mappings. Nussbaum (1971) and Sadovskii (1972) have made contributions in these cases and more generalized multivalued $k-\varphi$-contractions were introduced by Petryshyn and Fitzpatrick (1974). In the following we shall recall the χ and γ measures of noncompactness and restate some of the properties of the $k-\varphi$-contractions for such φ.

If $\{\rho_\alpha : \alpha \in A\}$ is a family of seminorms which define the topology on X, given $\alpha \in A$ and $\Omega \subset X$, we define

$$\chi_\alpha(\Omega) = \inf\left\{\varepsilon > 0 : \exists\{x_1, \ldots, x_n\} \subset X, \text{ with } \Omega \subset \bigcup_{i=1}^{n} \{y : \rho_\alpha(x_i - y) < \varepsilon\}\right\},$$

$\gamma_\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega$ can be contained in the union of a finite number of sets, each with ρ_α-diameter $< \varepsilon\}$.

Let $C = \{\varphi : A \to R^+$ be the set of all mappings from A into $R^+\}$ with the usual definitions of ordering, maximum, multiplication by a real number, etc. Then C forms a lattice and the two mappings $\chi : 2^X \to C$ and $\gamma : 2^X \to C$ are defined by

$$\chi(\Omega)(\alpha) = \chi_\alpha(\Omega) \quad \text{and} \quad \gamma(\Omega)(\alpha) = \gamma_\alpha(\Omega)$$

for every $\alpha \in A$, $\Omega \subset X$.

It can be verified that χ and γ are measures of noncompactness and, furthermore, they satisfy the following:

(1.9a) $B \subset X$ is bounded if and only if $\gamma_\alpha(B)$ or $\chi_\alpha(B)$ is finite for each $\alpha \in A$.

(1.9b) $B \subset X$ is precompact if and only if, for each $\alpha \in A$,

$$\gamma_\alpha(B) = \chi_\alpha(B) = 0,$$

(1.9c) $\chi(\lambda\Omega) = |\lambda| \chi(\Omega)$,

$\gamma(\lambda(\Omega)) = |\lambda| \gamma(\Omega)$ for any $\Omega \subset X$, $\lambda \in R$.

(1.9d) For any $\Omega_1 \subset X$, $\Omega_2 \subset X$,

$$\chi(\Omega_1 + \Omega_2) \leqslant \chi(\Omega_1) + \chi(\Omega_2),$$

$$\gamma(\Omega_1 + \Omega_2) \leqslant \gamma(\Omega_1) + \gamma(\Omega_2).$$

THEOREM 1.3. *Let $F: \Omega \to CK(X)$ be a $k-\varphi$-contraction where $0 < k < 1$ and $\varphi = \chi$ or γ. Suppose that either X is quasi-complete or Ω is complete and suppose that $F(\bar{\Omega})$ is bounded. Then F is ultimately compact.*

The proof follows immediately from Lemmas 3.2 and 3.5 of Petryshyn and Fitzpatrick (1974.)

Suppose X is a normed linear space with norm $\| \ \|$ and the metric $d\colon X \times X \to R^+$ is defined by $d(x, y) = \| x - y \|$. If we let the norm $\| \ \|$ be the only element of A, the set C is isomorphic to R^+ and χ and γ reduce to the ball- and set-measures of noncompactness. Let us denote these two measures of noncompactness by χ_d and γ_d respectively.

THEOREM 1.4. (a) *Let F be a $k - \varphi$-contraction where $\varphi = \chi$, γ, χ_d or γ_d. Then for $\lambda \in R$, λf is a $|\lambda| k - \varphi$-contraction.*
   (b) *Suppose F and G are $k_1 -$ and $k_2 - \varphi$-contractions respectively where φ is χ, γ, χ_d or γ_d. Then $(F+G)$ is a $(k_1 + k_2) - \varphi$-contraction.*
   (These results follow immediately from (1.9c) and (1.9d) and Proposition 1.1.)

2. Notations and algebraic preliminaries

We shall include in this section some preliminary results obtained by Mawhin (1972) (see also Gaines and Mawhin (1977) which we shall use in the section 3).
   Let L be a linear single-valued operator between X and Z, two vector spaces, where dom L, the domain of L, is a subspace of X. We shall denote the kernel or null-space of L, $L^{-1}(0)$, by ker L, the range space of L, $L(\text{dom } L)$, by Im L and the quotient space $Z/\text{Im } L$, the cokernel of L, by coker L.
   Given a vector subspace Y of a vector space E, there always exists a projection, a linear and indempotent operator, P of E onto Y and E is the direct sum of Im $P = Y$ and ker P. If E is a topological vector space, and P is a continuous projection, then E is the topological direct sum of Im P and ker P.

DEFINITION 2.1. If X, Z, L are as above, let P and Q be projections on X and Z respectively such that Im $P = $ ker L and ker $Q = $ Im L. Such a pair of projections (P, Q) will be called *exact with respect to L.*

DEFINITION 2.2. Let L_p be the restriction of L to ker $P \cap$ dom L. The L_p is an isomorphism from ker $P \cap$ dom L to Im L. Let $K_p\colon$ Im $L \to$ ker $P \cap$ dom L be the inverse of L_p. K_p is then called the *pseudo inverse of L associated with P.*
   Let $\pi\colon Z \to$ coker L be the canonical surgection, that is $\pi z = z + $ Im L for each $z \in Z$. Clearly, the restriction of π to Im Q is an algabraic isomorphism. If Z is a topological vector space and coker L is given the quotient topology, then π is continuous.
   The following results are almost immediate:

(2.1) $PK_p = 0,$

(2.2) $LK_p = L_p K_p = I,$

(2.3) $$K_p L = K_p L(I-P) = I-P,$$

(2.4) $$Qz = 0 \Leftrightarrow z \in \operatorname{Im} L \Leftrightarrow \pi z = 0,$$

where the zeros denote the null elements of the respective spaces.

The following two results are also easy consequences of the above.

PROPOSITION 2.1. *Let (P, Q) and (P', Q') be pairs of projections exact with respect to L. Then*

(2.5) $$K_{p'} = (I-P')K_p,$$

(2.6) $$PK_{p'} + P' K_p = 0,$$

where $K_p, K_{p'}$ denote the pseudo-inverses of L associated with P and P' respectively.

PROPOSITION 2.2. *Let P, P' be projections of X onto $\ker L$ and let $P'' = aP + bP'$ for some real numbers a, b. Then, P'' is a projection onto $\ker L$ if and only if $a + b = 1$. If this necessary and sufficient condition holds, the pseudo inverse of L associated with P'' is given by*

$$K_{p''} = aK_p + bK_{p'}.$$

3. Coincidence degree for set-valued $k - \varphi$-contractive perturbations of linear Fredholm mappings

In this section, we will extend the notion of coincidence degree as developed by Mawhin (1972) to the case where the second mapping is set-valued. Such a degree theory will provide a method for proving the existence of solutions to the equation

$$Lx \in Nx.$$

3A. An equivalence theorem

THEOREM 3.1. *Let X and Z be two vector spaces over the same scalar field. Let $L: \operatorname{dom} L \subset X \to Z$ be a linear mapping and $N: A \subset X \to 2^Z$ be a set-valued mapping. Further, assume that there is a linear injective (one-to-one) mapping*

$$\psi: \operatorname{coker} L \to \ker L.$$

Then $x_0 \in \operatorname{dom} L \cap A$ is a solution of the equation

(3.1) $$Lx \in Nx$$

if and only if x_0 is a fixed point of the set-valued mapping $M_\psi: A \to 2^X$ defined by

(3.2) $$M_\psi x = Px + [\psi\pi + K_p(I-Q)]Nx$$

for every pair (P, Q) of exact projections with respect to L, where π and K_p have their meaning as explained in Section 2. In other words,

$$(3.3) \qquad (L - N)^{-1}(0) = (I - M_\psi)^{-1}(0).$$

PROOF. Since the images under P and ψ are contained in $\ker L$ and that under K_p is in $X_{I-P} \cap \operatorname{dom} L$, it is clear that $M_\psi(A) \subset \operatorname{dom} L$. First, let us suppose that $x_0 \in A \cap \operatorname{dom} L$ with $Lx_0 \in Nx_0$. Then

$$[\psi\pi + K_p(I - Q)]Lx_0 \in [\psi\pi + K_p(I - Q)]Nx_0.$$

Hence using (2.3) and (2.4) we have

$$(I - P)x_0 \in [\psi\pi + K_p(I - Q]Nx_0.$$

Therefore

$$x_0 \in M_\psi x_0.$$

Next, let us suppose that $x_0 \in A \cap \operatorname{dom} L$ with $x_0 \in M_\psi x_0$, that is

$$(3.4) \qquad x_0 \in Px_0 + [\psi\pi + K_p(I - Q)]Nx_0.$$

Since the operator $\psi\pi + K_p(I - Q)$ is injective (see Lemma 3.3) we have

$$(3.5) \qquad [\psi\pi + K_p(I - Q)]^{-1}[\psi\pi + K_p(I - Q)]Nx_0 = Nx_0.$$

Hence it follows from (3.4) and (3.5) that

$$(3.6) \qquad [\psi\pi + K_p(I - Q)]^{-1}(I - P)x_0 \in Nx_0.$$

Thus

$$(3.7) \qquad [\psi\pi + K_p(I - Q)]^{-1} = [(\pi/\operatorname{Im} Q)^{-1}\psi^{-1}P + L]$$

yields $Lx_0 \in Nx_0$, where $\pi/\operatorname{Im} Q$ denotes the restriction of π to $\operatorname{Im} Q$. We now establish (3.7).

For each $z \in Z$ we have by using (2.2)

$$[(\pi/\operatorname{Im} Q)^{-1}\psi^{-1}P + L][\psi\pi + K_p(I - Q)]z$$
$$= (\pi/\operatorname{Im} Q)^{-1}\pi z + (I - Q)z = Qz + (I - Q)z = z.$$

Also if $x \in \operatorname{dom} L$, then using (2.3) and (2.4) we have

$$[\psi\pi + K_p(I - Q)][(\pi/\operatorname{Im} Q)^{-1}\psi^{-1}P + L]x = Px + (I - P)x = x.$$

3B. Basic assumptions

Before we define the coincidence degree for (L, N), we shall state the assumptions which we shall make on the mappings.

ASSUMPTIONS. (a) X is a real Banach space and Z is a real normed linear space.

(b) $L: \operatorname{dom}_L \subset X \to Z$ is a linear Fredholm mapping of index zero defined on a subspace $\operatorname{dom} L$ of X, that is L is linear, $\operatorname{Im} L$ is closed and

$$\dim \ker L = \dim \operatorname{coker} L < \infty,$$

where 'dim' denotes dimension.

(c) Ω is a bounded, open set in X and the set-valued mapping $N: \overline{\Omega} \to CK(Z)$ takes each x in the closure of Ω to a nonempty compact convex subset of Z.

(d) N is upper-semicontinuous with $\pi N(\overline{\Omega})$ bounded in $\operatorname{coker} L$.

(e) Let (P, Q) be an exact pair of projections with respect to L and let K_p be the pseudo-inverse of L associated with P. Let φ be a measure of noncompactness defined on 2^X such that either (i) φ satisfies the subadditivity condition of Proposition 1.1 and takes values in $R^+ = \{t \in R: t \geqslant 0\} \cup \{\infty\}$ or (ii) we additionally assume that Z is a Banach space and φ is one of χ, γ, χ_d and γ_d. We assume that with such a measure of noncompactness φ, $K_p(I-Q)N$ is a $k-\varphi$-contraction with $0 < k < 1$ and that $\varphi(K_p(I-Q)N(\overline{\Omega})) < \infty$. In this case we also assume that K_p is continuous.

(f) $0 \notin (L-N)(\operatorname{dom} L \cap \partial\Omega)$ where $\partial\Omega$ denotes the boundary of Ω.

REMARK 3.1. From assumption (b), the exact pair of projections (P, Q) may be assumed continuous and will hereafter be assumed continuous. Moreover, with the quotient norm topology $\operatorname{coker} L$ is a normed space and the canonical surjection π is continuous with respect to this topology. Also, (b) is sufficient condition for the existence of a linear isomorphism $\psi: \operatorname{coker} L \to \ker L$.

PROPOSITION 3.1. *Let assumptions* (a) *to* (d) *hold and let* (P, Q) *and* (P', Q') *be exact pairs of continuous projections with respect to L. Suppose that* (P, Q) *satisfy assumption* (e). *Then the pair* (P', Q') *also satisfies the assumption* (e).

PROOF. Writing $\pi_Q = \pi/\operatorname{Im} Q$ and $\pi_{Q'} = \pi/\operatorname{Im} Q'$ and using (2.5) we have

$$(3.8) \qquad K_{p'}(I-Q')N = (I-P')K_p(I-Q')N$$

$$= (I-P')K_p(I-Q)N + (I-P')K_p(Q-Q')N$$

$$= (I-P)K_p(I-Q)N + (I-P)\tilde{K}_p(\pi_Q^{-1} - \pi_{Q'}^{-1})\pi N,$$

where $\tilde{K}_p$ denotes the restriction of K_p to the finite dimensional subspace $(Q-Q')Z$. Thus $\tilde{K}_p$ is continuous. Since $\pi N(\overline{\Omega})$ is bounded in a finite dimensional subspace of X, it follows that $(I-P')\tilde{K}_p(\pi_Q^{-1} - \pi_{Q'}^{-1})\pi N$ is a $0-\varphi$-contraction. Hence from Propositions 1.1 and 1.2 it follows that $K_{p'}(I-Q')N$ is a $k-\varphi$-contraction. That $K_{p'}$ is continuous follows from (2.5) as K_p and $(I-Q')$ are continuous. Finally applying φ to both sides of (3.8) and using subadditivity of φ we can easily show that $\varphi(K_{p'}(I-Q')N(\overline{\Omega}) < \infty$.

DEFINITION 3.1. A mapping $N: \overline{\Omega} \to CK(Z)$ satisfying (c), (d) and (e) is said to be a $L-k-\varphi$-*contraction*. (We see that this is a proper definition as assumption (e) is independent of the choice of (P, Q).)

3C. Definition of coincidence degree

PROPOSITION 3.2. *Suppose assumptions* (a) *to* (e) *are satisfied and* M_ψ *is the mapping defined in Theorem* 3.1 *for some continuous isomorphism*

$$\psi: \operatorname{coker} L \to \ker L.$$

Then for each x in $\overline{\Omega}$, $M_\psi x$ *is a compact convex subset of X and* M_ψ *is a* $k-\varphi$-*contraction.*

PROOF. Since P, Q, K_p, ψ and π are all linear and $N(x)$ is convex for each $x \in \overline{\Omega}$, it follows that $M_\psi x$ is convex for each $x \in \overline{\Omega}$. Again since P, Q, K_p and ψ are continuous and Nx is compact, $M_\psi x = Px + [\psi\pi + K_p(I-Q)]Nx$ is compact for each $x \in \overline{\Omega}$.

Now P is linear continuous and has a finite dimensional range. Hence P is compact and is, therefore, a $0-\varphi$-contraction. Also $\psi\pi N(\overline{\Omega})$ being bounded subset of a finite-dimensional subspace is relatively compact.

We now prove that $[\psi\pi + K_p(I-Q)]N$ is a $k-\varphi$-contraction. Let $A \subseteq \overline{\Omega}$. Noting that

$$[\psi\pi + K_p(I-Q)]N(A) \subset \psi\pi N(A) + K_p(I-Q)N(A)$$

we have

$$\varphi([\psi\pi + K_p(I-Q)]N(A)) \leqslant \varphi(\psi\pi N(A)) + K_p(I-Q)N(A))$$

$$\leqslant \varphi(\psi\pi N(A)) + \varphi(K_p(I-Q)N(A))$$

$$\text{(by subadditivity of } \varphi\text{)}$$

$$\leqslant k\varphi(A)$$

as $\varphi(\psi\pi N(A) = 0$, $\psi\pi N(A)$ being relatively compact. Now from Proposition 1.1 it follows that M_ψ is a $k-\varphi$-contraction from $\overline{\Omega}$ to $CK(X)$.

REMARK 3.2. We note that assumption in (e) that K_p is continuous has been used to prove that $M_\psi x$ is a compact subset for each $x \in \overline{\Omega}$. This assumption is not unrealistic. For, if in addition to the assumption (b) $L: \operatorname{dom} L \; X \to Z$ is a closed operator and Z is a Banach space, then K_p is continuous. To see this let $y_n \to y$, $y_n \in \operatorname{Im} L$ and $K_p y_n = x_n \to x$. Since

$$Lx_n = LK_p y_n = y_n \quad \text{and} \quad x_n \in \operatorname{dom} L \cap X_{I-P},$$

we have by closedness of L that $Lx = y$ and $x \in \operatorname{dom} L$. Clearly $x \in X_{I-P}$ as X_{I-P} is closed. Hence $K_p y = K_p Lx = (I-P)x = x$ and obviously $y \in \operatorname{Im} L$ as $\operatorname{Im} L$ is

closed. Thus K_p is closed. Again since $\operatorname{Im} L$ is closed, the closed graph theorem yields that K_p is continuous.

REMARK 3.3. From Proposition 3.2, we see that if the assumptions (a) to (f) are satisfied M is an ultimately compact mapping (see Theorems 1.2 and 1.3). It follows from assumption (f) and Theorem 3.1 that $0 \notin (I - M_\psi)(\operatorname{dom} L \cap \partial\Omega)$. Thus the degree of the ultimately compact field $I - M_\psi$ with respect to zero is well defined.

DEFINITION 3.2. Let $\mathscr{L}_L$ denote the set of all continuous isomorphisms from $\operatorname{coker} L$ to $\ker L$. ψ, ψ' are said to be *homotopic in* $\mathscr{L}_L$ if there exists a continuous mapping $\bar{\psi}: \operatorname{coker} L \times [0, 1] \to \ker L$ such that $\bar{\psi}(\cdot, 0) = \psi$, $\bar{\psi}(\cdot, 1) = \psi'$ and, for each $\lambda \in [0, 1]$, $\bar{\psi}(\cdot, \lambda) \in \mathscr{L}_L$.

REMARK 3.4. To be homotopic is an equivalence relation which partitions $\mathscr{L}_L$ into equivalence classes called homotopy classes.

The following two propositions and corollary are quoted from Gaines and Mawhin (1977):

PROPOSITION 3.3. ψ and ψ' are homotropic in $\mathscr{L}_L$ if and only if $\det(\psi' \psi^{-1}) > 0$.

COROLLARY 3.1. $\mathscr{L}_L$ is partitioned into two homotopy classes.

DEFINITION 3.3. $\psi: \operatorname{coker} L \to \ker L$ is said to be *orientation preserving* if $\{\psi a_1, \psi a_2, \ldots, \psi a_n\}$ belongs to the orientation chosen in $\ker L$ where $\{a_1, a_2, \ldots, a_n\}$ is a basis for $\operatorname{coker} L$ belonging to a certain chosen orientation. Otherwise, ψ is said to be orientation reversing.

PROPOSITION 3.4. *If* $\operatorname{coker} L$ *and* $\ker L$ *are oriented then* ψ *and* ψ' *are homotopic in* $\mathscr{L}_L$ *if and only if they are simultaneously orientation preserving or orientation reversing.*

LEMMA 3.1. *Let* X *and* Z *be normed linear spaces and let* Ω *be a bounded open subset of* X. *Let* $\varphi: 2^X \to C$ *be a measure of non-compactness as given in assumption* (e). *Let* $F: \bar{\Omega} \times [0, 1] \to CK(X)$ *be an upper-semicontinuous mapping such that* $\varphi(F(\bar{\Omega} \times [0, 1])) < \infty$ *and, for some* $k \in (0, 1)$, *we have*

$$\varphi(F(A \times [0, 1])) \leqslant k\varphi(A) \quad \text{for every } A \subset \bar{\Omega}.$$

Then $F((K' \cap \bar{\Omega}) \times [0, 1])$ *is relatively compact where*

$$K' = K(F, \bar{\Omega} \times [0, 1]).$$

PROOF. As $K' = K(F, \overline{\Omega} \times [0,1]) = \overline{CO} \, F((\overline{\Omega} \cap K') \times [0,1])$,

$$\varphi(K' \cap \Omega) \leqslant \varphi(K') = \varphi(F(\overline{\Omega} \cap K') \times [0,1])$$
$$\leqslant k\varphi(\overline{\Omega} \cap K').$$

As $0 < k < 1$, and $\varphi(F(\overline{\Omega} \times [0,1])) < \infty$, we have

$$\varphi(K' \cap \overline{\Omega}) = \varphi(F((\overline{\Omega} \cap K') \times [0,1])) = 0.$$

Hence, $K' \cap \overline{\Omega}$ and $F((\overline{\Omega} \cap K') \times [0,1])$ are precompact and by the assumption that $\overline{\Omega}$ is complete, we conclude that $F((\overline{\Omega} \cap K') \times [0,1])$ is relatively compact.

THEOREM 3.2. *Let assumptions* (a) *to* (f) *be satisfied. Then* $d(I - M_\psi, \Omega, 0)$ *as defined in Definition* 1.3 *depends only on* L, N *and the homotopy class of* ψ *in* $\mathscr{L}_L$.

PROOF. Let (P, Q) and (P', Q') be exact pairs of projections with respect to L. Let $\psi, \psi' \in \mathscr{L}_L$ be in the same homotopy class and let $\overline{\psi}$: coker $L \times [0,1] \to \ker L$ be the mapping in Definition 3.2. Let

$$M = P + [\psi\pi + K_p(I - Q)]N,$$
$$M' = P' + [\psi'\pi + K_{p'}(I - Q')]N.$$

For each $\lambda \in [0,1]$, define

$$P_\lambda = (1-\lambda)P + \lambda P',$$
$$Q_\lambda = (1-\lambda)Q + \lambda Q'.$$

By Proposition 2.2, (P_λ, Q_λ) is an exact pair of projections with respect to L. Moreover, $P_0 = P$, $P_1 = P'$, $Q_0 = Q$ and $Q_1 = Q'$ and $KP_\lambda = (1-\lambda)K_p + \lambda K_{p'}$. Define $M^*: \overline{\Omega} \times [0,1] \to CK(X)$ by

$$M^*(x, \lambda) = P_\lambda x + [\overline{\psi}(\pi(\cdot), \lambda) + KP_\lambda(I - Q_\lambda)]Nx.$$

By theorem 3.1 and assumption (f),

$$x \notin M^*(x, \lambda) \quad \text{for every } x \in \partial\Omega, \ \lambda \in [0,1].$$

Also, $M^*(\cdot, 1) = M'$ and $M^*(\cdot, 0) = M$.

We claim that $M^*((\overline{\Omega} \cap K') \times [0,1])$ is relatively compact, where

$$K' = K(M^*, \overline{\Omega} \times [0,1]).$$

Now, writing explicitly,

$$M^*(x, \lambda) = (1-\lambda)Px + \lambda P'x + [\overline{\psi}(\pi(\cdot), \lambda)$$
$$+ \{(1-\lambda)K_p + \lambda K_{K_{p'}}\}\{I - (1-\lambda)Q - \lambda Q'\}]Nx$$
$$= (1-\lambda)Px + \lambda P'x + [\overline{\psi}(\pi(\cdot), \lambda)$$
$$+ \{(1-\lambda)K_p + \lambda(I - P')K_p\}\{I - Q + \lambda(Q - Q')\}]Nx$$
$$= (1-\lambda)Px + \lambda P'x + [\overline{\psi}(\pi(\cdot), \lambda) + (I - \lambda P')K_p(I - Q)$$
$$+ \lambda(I - \lambda P')K_p(Q - Q')]Nx.$$

Using the same argument as in Proposition 3.1 we can show that for each $\lambda \in [0, 1]$, $\lambda(I - \lambda P') K_p(Q - Q')N$ is $0 - \varphi$-contraction. Now by using the assumption (e) and similar argument as in Proposition 3.2 we can show that for each $\lambda \in [0, 1]$.

$$[\bar{\psi}(\pi(\cdot), \lambda) + (I - \lambda P') K_p(1 - Q) + \lambda(I - \lambda P') K_p(Q - Q')]N$$

is $k - \varphi$-contraction (note that P and P' being compact maps are both $0 - \varphi$-contraction). Thus, it follows from Propositions 1.1 and 1.2 that for each $\lambda \in [0, 1]$, $M^*(\cdot, \lambda)$ is a $k - \varphi$-contraction.

Now,

$$\varphi(M^*(A \times [0, 1])) = \varphi \left(\bigcup_{\lambda \in [0,1]} M^*(A, \lambda) \right)$$

$$= \max_{\lambda \in [0,1]} \varphi(M^*(A, \lambda)).$$

Since for each $\lambda \in [0, 1]$, $M^*(\cdot, \lambda)$ is a $k - \varphi$-contraction

$$\varphi(M^*(A \times [0, 1])) \leqslant k\varphi(A).$$

From the preceding lemma, $M^*((\Omega \cap K') \times [0, 1])$ is relatively compact. By the Homotopy Invariance Theorem given in Petrysyn and Fitzpatrick (1974),

$$d(I - M^*(\cdot, 0), \Omega, 0) = d(I - M^*(\cdot, 1), \Omega, 0)$$

or

$$d(I - M, \Omega, 0) = d(I - M', \Omega, 0).$$

Thus the degree of $I - M_\psi$ on Ω with respect to zero is independent of the choice of P, Q and ψ within the same homotopy class.

DEFINITION 3.4. Suppose that assumptions (a) to (f) are satisfied and ψ is an orientation preserving continuous isomorphism from coker L to ker L. Then, the *coincidence degree of L and N in Ω, denoted by $d[(L, N), \Omega]$,* is defined by

(3.9) $$d[(L, N), \Omega] = d(I - M_\psi, \Omega, 0),$$

where $M_\psi : \Omega \to CK(X)$ is defined by

$$M_\psi = P + [\psi \pi + K_p(I - Q)]N$$

and the right-hand term is the degree for the set-valued ultimately compact field $I - M_\psi$ as defined in Definition 1.3.

REMARK 3.3. (a) If $X = Z$, $L = I$, then ker $L = \{0\}$ and thus coker $L = \{0\}$. This implies that Im $L = X$ and hence, $P = 0$, $Q = 0$ and $K_p(I - Q) = I$ and the only isomorphism between coker L and ker L is the trivial one $\psi(0) = 0$. The assumption (b) is trivially satisfied and (e) reduces to assuming that N is a $k - \varphi$-contraction for some k in $(0, 1)$ with $\varphi(N(\Omega)) < \infty$. Assumption (f) means that N has no fixed points on the boundary of Ω. As $M_\psi = N$, we have

$$d[(I, N), \Omega] = d(I - N, \Omega, 0).$$

We may, in fact, replace assumption (e) by the assumption that N is ultimately compact.

3D. Basic properties of the coincidence degree

In this section, unless otherwise specified, we shall assume that assumptions (a) to (f) are satisfied such that the Coincidence Degree is well defined.

THEOREM 3.3. (a) Existence Theorem.
If $d[(L, N), \Omega] \neq 0$, then $0 \in (L-N)(\operatorname{dom} L \cap \Omega)$.
(b) Additivity Property.
Let Ω_1, Ω_2 be disjoint open sets such that $\Omega_1 \cup \Omega_2 \subset \Omega$ $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$ and $0 \notin (L-N)(\partial \Omega_1 \cup \partial \Omega_2)$. Then,

$$d[(L, N), \Omega] = d[(L, N), \Omega_1] + d[(L, N), \Omega_2].$$

(c) Excision Property.
If $\Omega_1 \subset \Omega$ is an open set such that $(L-N)^{-1}(0) \subset \Omega_1$ then

$$d[(L, N), \Omega] = d[(L, N), \Omega_1].$$

PROOF. (a) and (b) follow from the Definition of Coincidence Degree and the corresponding properties of degree of an ultimately compact vector field given by Petryshyn and Fitzpatrick (1974). By taking $\Omega_2 = \Omega \setminus \overline{\Omega}_1$, that is

$$\Omega_2 = \{ x \in \Omega : x \notin \overline{\Omega}_1 \}.$$

The result (c) follows from (a) and (b).

THEOREM 3.4. *If Ω is a symmetric bounded neighbourhood of the origin and $N(-x) = -Nx$ for all $x \in \overline{\Omega}$, then $dL, N), \Omega]$ is odd.*

PROOF. Note that, as P, Q, K_p, ψ and π are all linear, the condition on N implies that $M_\psi(-x) = -M_\psi(x)$ for all $x \in \Omega$. Thus, by the corresponding property of degree of an ultimately compact vector field (Petryshyn and Fitzpatrick (1974)) and the definition of Coincidence Degree, $d[(L, N), \Omega]$ is odd.

THEOREM 3.5. (Homotopy Invariance.) *Let assumptions (a) and (b) be satisfied and let Ω be a bounded, open subset of X. Let φ, P, Q and K_p be as given in assumption (e) and suppose $\tilde{N} : \overline{\Omega} \times [0, 1] \to CK(Z)$ satisfy the following*
(i) *$\tilde{N}$ is upper-semicontinuous on $\overline{\Omega} \times [0, 1]$,*
(ii) *$\pi N(\overline{\Omega} \times [0, 1])$ is bounded,*
(iii) *$\varphi(K_p(I-Q)\tilde{N}(\overline{\Omega} \times [0, 1])) < \infty$,*
(iv) *there exists $k \in (0, 1)$ such that, for every $A \subset \Omega$,*

$$\varphi(K_p(I-Q)\tilde{N}(A \times [0, 1])) < k\varphi(A),$$

(v) *for each* $\lambda \in [0, 1]$,

(3.10) $0 \notin (L - N(\cdot, \lambda)) (\operatorname{dom} L \cap \partial\Omega)$.

Then, $d[(L, N(\cdot, \lambda)), \Omega]$ *is independent of* λ *in* $[0, 1]$.

PROOF. Let $\psi : \operatorname{coker} L \to \ker L$ be an orientation preserving continuous iso-morphism. Define $M_\psi : \overline{\Omega} \times [0, 1] \to CK(X)$ by

$$M_\psi(x, \lambda) = Px + [\psi\pi + K_p(I - Q)]\tilde{N}(x, \lambda).$$

Then, by Lemma 3.1 and (v), M_ψ satisfies the conditions of Theorem 2.2 of Petryshyn and Fitzpatrick (1974). Hence, by the definition of Coincidence Degree,

$$d[(L, N(\cdot, 0)), \Omega] = d[(L, N(\cdot, 1)), \Omega].$$

Now, for any $\lambda \in [0, 1]$, let $\lambda' = \lambda t$ and apply the above to $\tilde{N}'(\cdot, t)$, $t \in [0, 1]$ where $\tilde{N}'(\cdot, t) = \tilde{N}(\cdot, \lambda')$. Then,

$$\begin{aligned} d[(L, \tilde{N}(\cdot, \lambda)), \Omega] &= d[(L, \tilde{N}'(\cdot, 1)), \Omega] \\ &= d[(L, \tilde{N}'(\cdot, 0)), \Omega] \\ &= d[(L, \tilde{N}(\cdot, 0)), \Omega]. \end{aligned}$$

Hence, $d[(L, \tilde{N}(\cdot, \lambda)), \Omega]$ is independent of λ in $[0, 1]$.

COROLLARY 3.2. *Let assumptions* (a) *and* (b) *hold and let* Ω *be an open bounded subset of* X. *Let* N *and* N' *be two* $L - k - \varphi$-*contractions on* $\overline{\Omega}$ *satisfying* (f) *such that* $Nx = N'x$ *for each* $x \in \partial\Omega$. *Then* $d[(L, N), \Omega] = d[(L, N'), \Omega]$.

PROOF. Define

$$\tilde{N} : \overline{\Omega} \times [0, 1] \to CK(Z)$$

by

$$\tilde{N}(x, \lambda) = (1 - \lambda)Nx + \lambda N' x.$$

Then $\tilde{N}$ is clearly upper-semicontinuous and satisfies all the other conditions of Theorem 3.5. Hence by Theorem 3.5,

$$\begin{aligned} d[(L, \tilde{N}), \Omega] &= d[(L, \tilde{N}(\cdot, 0)), \Omega] \\ &= d[(L, \tilde{N}(\cdot, 1)), \Omega] \end{aligned}$$

and hence,

$$d[(L, N), \Omega] = d[(L, N'), \Omega].$$

DEFINITION 3.5. Let X and Z be normed linear spaces with norms denoted by $\| \|$. Let x be any point of X (or Z) and let A, B be subsets of X (or Z). Then $D^*(x, A) = \inf\{\| x - a \| : a \in A\}$ is the usual distance between x and A and we define

$$d^*(A, B) = \inf\{\| a - b \| : a \in A, b \in B\}$$

to be the distance between A and B. In fact, $d^*(x, A)$ is equivalent to the distance between A and the singleton $\{x\}$.

LEMMA 3.2.
$$d^*(x, A) + d^*(B, C) \geqslant d^*(x, A + C - B).$$

PROOF. If $a \in A$, $b \in B$ and $c \in C$, $a + c - b \in A + C - B$ and hence, for every $a \in A$, $b \in B$, $c \in C$,
$$\| x - (a + c - b) \| \geqslant d^*(x, A + C - B)$$
Now,
$$\| x - (a + c - b) \| \leqslant \| x - a \| + \| b - c \|$$
Hence, for every $a \in A$, $b \in B$ and $c \in C$, we have
$$d^*(x, A + C - B) \leqslant \| x - a \| + \| b - c \|$$
and so,
$$d^*(x, A + C - B) \leqslant \inf \{ \| x - a \| : a \in A \} + \inf \{ \| b - c \| : b \in B, c \in C \}$$
$$= d^*(x, A) + d^*(B, C).$$

LEMMA 3.3. *For each* $x \in \operatorname{dom} L \cap \overline{\Omega}$, *we have*
$$(I - M_\psi)x = [\psi\pi + K_p(I - Q)](L - N)x,$$
where $\psi\pi + K_p(I - Q)$ *is an algebraic isomorphism between* Z *and* $\operatorname{dom} L$.

PROOF.
$$[\psi\pi + K_p(I - Q)](L - N) = [\psi\pi + K_p(I - Q)]L - [\psi\pi + K_p(I - Q)]N$$
$$= K_p(I - Q)L - [\psi\pi + K_p(I - Q)]N \quad \text{by (2.4)}$$
$$= K_p L - [\psi\pi + K_p(I - Q)]N$$
$$= I - P - [\psi\pi + K_p(I - Q)]N \quad \text{by (2.3)}$$
$$= I - M_\psi.$$

To show that $\psi\pi + K_p(I - Q)$ is an isomorphism, consider the equation

(3.11) $$[\psi\pi + K_p(I - Q)]z = y$$

for some $y \in \operatorname{dom} L$.
  This is equivalent to

(3.12) $$\psi\pi z = Py,$$

(3.13) $$K_p(I - Q)z = (I - P)y.$$

Now, as $\ker \pi = \operatorname{Im} L = \operatorname{Im}(I - Q)$, $\psi\pi_Q$, the restriction of $\psi\pi$ to $\operatorname{Im} Q$, is an

isomorphism from $\operatorname{Im} Q$ to $\ker L$ and hence (3.12) is equivalent to

(3.14) $Qz = (\psi \pi_Q)^{-1} Py$

and since $LK_p = I$ and $LP = 0$, (3.13) is equivalent to

(3.15) $(I - Q)z = L(I - P)y = Ly$

Hence,

$$z = Qz + (I - Q)z$$
$$= (\psi \pi_Q)^{-1} Py + Ly.$$

This shows the existence and uniqueness of the solution z of equation (3.11) for each given y in $\operatorname{dom} L$. Hence $\psi \pi + Kp(I - Q)$ is an isomorphism from Z to $\operatorname{dom} L$.

LEMMA 3.4. *Let assumptions* (a) *to* (f) *be satisfied. If* $M_\psi(\partial \Omega)$ *is relatively compact, then there exists* $\mu > 0$ *such that*

(3.16) $\inf \{d^*(Lx, Nx): x \in \partial \Omega \cap \operatorname{dom} L\} \geqslant \mu.$

PROOF. By assumption (f), $d^*(Lx, Nx) > 0$ for all $x \in \partial \Omega \cap \operatorname{dom} L$. Now, suppose that for all $\mu > 0$, (3.16) does not hold. Then for each positive integer n, there exists $x_n \in \partial \Omega \cap \operatorname{dom} L$ such that

$$d^*(Lx_n, Nx_n) < \frac{1}{n}.$$

Now, $d^*(x_n, M_\psi x_n) < \| x_n - y \|$ for all $y \in M_\psi x_n$.

Using the preceding lemma and noting that $\psi \pi + K_p(I - Q)$ is a continuous linear operator from Z onto $\operatorname{dom} L$, we have for each $z_n \in Nx_n$,

$$(\psi \pi + K_p(I - Q)(Lx_n - z_n)) = x_n - y \quad \text{for some } y \in M_\psi x_n.$$

Hence, for all $z_n \in Nx_n$,

$$\|(\psi \pi + K_p(I - Q))(Lx_n - z_n)\| \geqslant d^*(x_n, M_\psi x_n).$$

If $\|\psi \pi + K_p(I - Q)\| = \alpha \geqslant 0$,

$$d^*(x_n, M_\psi x_n) \leqslant \|\psi \pi + K_p(I - Q)\| \| Lx_n - z_n \|$$
$$= \alpha \| Lx_n - z_n \| \quad \text{for all } z_n \in Nx_n$$

Hence,

$$d^*(x_n, M_\psi x_n) \leqslant \alpha d^*(Lx_n, Nx_n)$$
$$< \alpha \frac{1}{n}.$$

Thus for each integer n, there exists some $u_n \in M_\psi x_n$ such that

$$\| x_n - u_n \| < \frac{\alpha}{n}.$$

Now since $u_n \in M_\psi x_n \subset M_\psi(\partial\Omega)$ which is relatively compact, we can find a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$ and the triangle inequality

$$\|x_{n_k} - u_0\| \leqslant \|x_{n_k} - u_{n_k}\| + \|u_{n_k} - u_0\|$$

$$< \frac{\alpha}{n_k} + \|u_{n_k} - u_0\|$$

implies that $x_{n_k} \to u_0$ as $n_k \to \infty$. As $x_{n_k} \in \partial\Omega$ which is closed, $u_0 \in \partial\Omega$. By upper-semicontinuity of M_ψ, $u_{n_k} \in M_\psi x_{n_k}$ for each n_k implies that $u_0 \in M_\psi u_0$ which is a a contradiction as $u_0 \in \partial\Omega$. Hence (3.16) holds for some $\mu > 0$.

REMARK 3.5. In Gaines and Mawhin (1977), Rouché's Theorem was extended to the context of Coincidence Degree. The following theorem is a version of Rouché's Theorem in our situation.

THEOREM 3.6. *Let assumptions* (a) *to* (f) *be satisfied and assume that $M_\psi(\partial\Omega)$ is relatively compact. Let $\mu > 0$ be such that*

$$\inf\{d^*(Lx, Nx): x \in \partial\Omega \cap \operatorname{dom} L\} \geqslant \mu.$$

Then, for each $L - k - \varphi$-contraction $N': \bar{\Omega} \to CK(Z)$ satisfying assumption (f) *and the following condition:*

$$\sup\{d^*(Nx, N'x): x \in \partial\Omega\} < \mu$$

we have

$$d[(L, N), \Omega] = d[(L, N'), \Omega].$$

PROOF. *Let $\tilde{N}: \bar{\Omega} \times [0, 1] \to CK(Z)$ be defined by*

$$\tilde{N}(x, \lambda) = (1 - \lambda)Nx + \lambda N'x.$$

It can easily be verified that conditions (i) to (iv) of Theorem 3.5 are satisfied. Now,

$$d^*(Lx, \tilde{N}(x, \lambda)) = d^*(Lx, Nx - \lambda(Nx - N'x))$$

$$\geqslant d^*(Lx, Nx) - \lambda d^*(Nx, N'x)$$

the last inequality following from Lemma 3.2 by putting $B = \lambda N'x$, $C = \lambda Nx$ and $A = Nx - \lambda Nx + \lambda N'x$.

Hence, for each $(x, \lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times [0, 1]$,

$$d^*(Lx, \tilde{N}(x, \lambda)) > \mu - \lambda\mu \geqslant 0.$$

This shows that $\tilde{N}$ satisfies the last condition of Theorem 3.5 and hence,

$$d[(L, N), \Omega] = d[(L, \tilde{N}(\cdot, 0)), \Omega] = d[(L, \tilde{N}(\cdot, 1)), \Omega] = d[L, N'), \Omega].$$

Thus,

$$d[(L, N), \Omega] = d[(L, N), \Omega].$$

3E. A generalized continuation theorem and existence theorems

In Gaines and Mawhin (1977), the Leray–Schauder Continuation Theorem was extended to the context of Coincidence Degree. Here, we shall extend it to the set-valued situation. We shall also consider some existence Theorems for $Lx \in Nx$.

DEFINITION 3.6. Consider the mapping $F: X \to CK(X)$ where X is the zero-dimensional space $\{0\}$. As $CK(X)$ may only contain nonempty subsets of X, $CK(X) = \{\{0\}\}$ and hence F is the mapping $F(0) = \{0\}$. We define $d(F, \{0\}, 0) = 1$ and this degree agrees with the usual properties of the degree for an ultimately compact field F. We also set $d(F, \varphi, 0) = 0$.

DEFINITION 3.7. Let X and Z be normed linear spaces and let L be a linear Fredholm mapping of index zero. Let P, Q, K_P and φ be given as in assumption (e) and let Ω be an open bounded subset of X such that Ω is complete. Let $a > 0$ and let $N^*: \bar{\Omega} \times [0, a] \to CK(Z)$ be a set valued mapping. Let N^* satisfy the following conditions:

(i) N^* is upper-semicontinuous on $\bar{\Omega} \times [0, a]$,
(ii) $N^*(\bar{\Omega} \times [0, a])$ is bounded,
(iii) $\varphi(K_p(I - Q)N^*(\bar{\Omega} \times [0, a])) < \infty$,
(iv) there exists a positive $k < 1$ such that, for every $A \subset \bar{\Omega}$,

$$\varphi(K_p(I - Q)N^*(A \times [0, a])) \leqslant k\varphi(A).$$

Then N^* is said to be a $L - k - \varphi\text{-contraction on } \Omega \times [0, a]$.

REMARK 3.5. With N^* as defined above, it can be seen that for each $\lambda \in [0, a]$, $N^*(\cdot, \lambda)$ is $L - k - \varphi$-contraction as defined by assumptions (c), (d) and (e). Also note that for $a = 1$, N^* satisfies the first four conditions of the homotopy invariance theorem, Theorem 3.5.

Now, let assumptions (a) to (f) be satisfied for a pair of mappings $L: \text{dom} L \to Z$ and $N: \bar{\Omega} \to CK(Z)$ and let $N^*: \bar{\Omega} \times [0, 1] \to CK(Z)$ be a $L - k - \varphi$-contraction on $\bar{\Omega} \times [0, 1]$ such that $N^*(\cdot, 1) = N$.

Let $y \in \text{Im} L$ and consider the family of equations

(3.17) $Lx \in \lambda N^*(x, \lambda) + y$.

An element $(x, \lambda) \in \bar{\Omega} \times [0, 1]$ satisfying (3.17) is said to be a solution of (3.17). If λ is specified, any $x \in \bar{\Omega}$ satisfying the equation for that λ is also called a solution. It will be clear from the context whether a solution is an element of $\bar{\Omega}$ or $\bar{\Omega} \times [0, 1]$.

LEMMA 3.5. *For each $\lambda \in (0, 1]$, the set of solutions of* (3.17) *is equal to the set of solutions of the equation*

(3.18) $Lx \in [Q + \lambda(I - Q)]N^*(x, \lambda) + y$

and if $\lambda = 0$, every solution of (3.18) *is a solution of* (3.17).

PROOF. If $\lambda = 0$, (3.18) reduces to

$$Lx \in QN^*(x, 0) + y.$$

But $Lx = (I - Q)Lx$ which implies that

$$Lx \in (I - Q)[QN^*(x, 0) + y] = \{y\}.$$

This means that $Lx = y$ or x is a solution of (3.17) for $\lambda = 0$.

Let $\lambda \in (0, 1]$ and let x be a solution of (3.17). Then there exists $u \in N^*(x, \lambda)$ such that

$$Lx = \lambda u + y.$$

Hence $u = \lambda^{-1}(Lx - y) \in \operatorname{Im} L$. Therefore $Qu = 0$ and thus,

$$u = (I - Q)u \in (I - Q)N^*(x, \lambda).$$

Hence,

$$Lx = 0 + \lambda u + y = [Q + \lambda(I - Q)]u + y \in [Q + \lambda(I - Q)]N^*(x, \lambda) + y,$$

that is x is a solution of (3.18).

Conversely, let x be a solution of (3.18). Then there exists $v \in N^*(x, \lambda)$ such that

$$Lx = [Q + \lambda(I - Q)]v + y.$$

Hence $0 = QLx = Qv + \lambda Q(I - Q)v + Qy = Qv$. Thus,

$$Lx = Qv + \lambda(I - Q)v + y$$

$$= \lambda v + y \quad \text{as } Qv = 0$$

$$\in \lambda N^*(x, \lambda) + y,$$

that is x is a solution of (3.17).

THEOREM 3.7. (A Generalized Continuation Theorem.) *Let L and N be mappings satisfying assumptions* (a) *to* (f) *and let N^* be a $L - k - \varphi$-contraction on $\overline{\Omega} \times [0, 1]$ such that $N^*(\cdot, 1) = N$. Let $y \in \operatorname{Im} L$ and we assume the following conditions hold:*

(1) $Lx \notin \lambda N^*(x, \lambda) + y$ *for every $x \in \partial\Omega \cap \operatorname{dom} L$, $\lambda \in (0, 1)$.*

(2) $0 \notin \pi N^*(x, 0)$ *for every $x \in L^{-1}\{y\} \cap \partial\Omega$.*

(3) $d(g(\cdot)|_{L^{-1}\{y\}}, \Omega \cap \Omega_1 \cap L^{-1}\{y\}, 0) \neq 0$,

where the left-hand expression is the Brouwer degree for the single-valued compact field g restricted to the affine finite-dimensional space $L^{-1}\{y\}$ and g and Ω_1 are defined as follows: As $\ker L$ is a finite dimensional subspace of X, $-\psi\pi N^*(\cdot + K_p y, 0)$ defined on $((\Omega - K_p y))^- \cap \ker L$ is a set-valued compact field with respect to zero (the conclusion that $0 \notin -\psi\pi N^*(x + K_p y, 0)$ for every $x \in \partial(\Omega - K_p y) \cap \ker L$ follows from condition (2)). *In Ma* (1972), *Section 5.2, it has been shown that there exists a single-valued compact field g and an open bounded set $\Omega_1 \subset \ker L$ containing zero*

such that $g(\cdot + K_p y)$ *and* $-\psi \pi N^*(\cdot + K_p y, 0)$ *are homotopic and*

$$g(x + K_p y) = x + K_p y \quad \text{for all } x \text{ in } (((\Omega - K_p y))^- \setminus (\Omega_1 - K_p y)) \cap \ker L.$$

Ma has also defined the degree of the set-valued compact field $-\psi \pi N^*(\cdot + K_p y, 0)$ *by*

$$d(-\psi \pi N^*(\cdot + K_p y, 0)|_{\ker L}, (\Omega - K_p y) \cap \ker L, 0)$$
$$= d(g(\cdot + K_p y)|_{\ker L}, (\Omega - K_p y) \cap (\Omega_1 - K_p y) \cap \ker L, 0).$$

Then, for each $\lambda \in [0, 1)$, *equation* (3.17) *has at least one solution in* Ω *and for* $\lambda = 1$, *the equation*

(3.19) $Lx \in Nx + y$

has at least one solution in $\overline{\Omega}$.

PROOF. Let $\lambda \in [0, 1]$ be considered fixed. For each $x \in \overline{\Omega}$, $\mu \in [0, 1]$ we define

$$\tilde{N}(x, \mu) = [Q + \lambda \mu (I - Q)] N^*(x, \lambda \mu) + y.$$

Clearly $\tilde{N}$ is a $L - k - \varphi$-contraction in $\Omega \times [0, 1]$.

Let us now consider the case where $\lambda \in [0, 1)$. By condition (1) and Lemma 3.5 if $\lambda \neq 0$

$$Lx \notin \tilde{N}(x, \mu) \quad \text{for every } x \in \partial\Omega \cap \text{dom}\, L, \mu \in (0, 1].$$

Also, if $\mu = 0$ or $\lambda = 0$

$$\tilde{N}(x, \mu) = QN^*(x, 0) + y$$

and $Lx \in \tilde{N}(x, \mu)$ would imply that $Lx = y$ and $0 \in QN^*(x, 0)$ or $x \in L^{-1}\{y\}$ and $0 \in \pi N^*(x, 0)$. Thus, by assumption (2), $x \notin \partial\Omega$. Hence, for every $x \in \partial\Omega \cap \text{dom}\, L$, $\mu \in [0, 1]$,

(3.20) $Lx \notin \tilde{N}(x, \mu).$

By Theorem 3.5, $d[(L, \tilde{N}(\cdot, \mu)), \Omega]$ is independent of μ in $[0, 1]$ and hence,

$$d[(L, \tilde{N}(\cdot, 1)), \Omega] = d[(L, \tilde{N}(\cdot, 0)), \Omega]$$
$$= d[(L, QN^*(\cdot, 0) + y), \Omega]$$
$$= d(I - P - [\psi \pi + K_p(I - Q)][QN^*(\cdot, 0) + y], \Omega, 0)$$

that is

(3.21) $d[(L, \tilde{N}(\cdot, 1)), \Omega] = d(I - P - \psi \pi N^*(\cdot, 0) - K_p y, \Omega, 0).$

Let us now consider two cases. Firstly let us assume $\ker L = \{0\}$. Then $P = 0$, $Q = 0$, $\pi = 0$, $K_p = L^{-1}$ and hence, from (3.21), we have

(3.22) $d[(L, \tilde{N}(\cdot, 1)), \Omega] = d(I - L^{-1} y, \Omega, 0).$

Now, $L^{-1}\{y\} = \{L^{-1} y\}$ is a zero dimensional space and hence, for condition (3) to be satisfied, $L^{-1}\{y\} \cap \Omega \cap \Omega_1 \neq \varnothing$.

Hence, $L^{-1}y \in \Omega$ and so, as the right-hand term of (3.22) has reduced to the degree of a single-valued mapping $I - L^{-1}y$, we have

$$d[(L, \tilde{N}(\cdot, 1)), \Omega] = d(I - L^{-1}y, \Omega, 0)$$
$$= d(I, \Omega, L^{-1}y)$$
$$= 1.$$

From Theorem 3.3, there exists $x \in \Omega$ such that $Lx \in \tilde{N}(x, 1)$, that is, for some $x \in \Omega$

$$Lx \in [Q + \lambda(I - Q)]N^*(x, \lambda) + y.$$

and by Lemma 3.5, equation (3.17) has at least one solution in Ω. Now let us consider the case where $\ker L \neq \{0\}$. By a change of variables, we have

(3.23) $d(I - P - \psi\pi N^*(\cdot, 0) - K_p y, \Omega, 0) = d(I - P - \psi\pi N^*(\cdot + K_p y, 0), \Omega - K_p y, 0).$

As $\ker L$ is a finite-dimensional subspace containing the range of $P + \psi\pi N^*$, we may apply Theorem 1.1 and obtain

(3.24) $d(I - P - \psi\pi N^*(\cdot + K_p y, 0), \Omega - K_p y, 0)$

$$= d[(I - P - \psi\pi N^*(\cdot + K_p y, 0))|_{\ker L}, (\Omega - K_p y) \cap \ker L, 0)$$
$$= d(-\psi\pi N^*(\cdot + K_p y, 0)|_{\ker L}, (\Omega - K_p y) \cap \ker L, 0)$$
$$= d(g(\cdot + K_p y)|_{\ker L}, (\Omega - K_p y) \cap (\Omega_1 - K_p y) \cap \ker L, 0)$$

the last equality holding by definition.

By a change of variables again,

(3.25) $d(g(\cdot + K_p y)|_{\ker L}, (\Omega - K_p y) \cap (\Omega_1 - K_p y) \cap \ker L, 0)$

$$= d(g(\cdot)|_{L^{-1}\{y\}}, \Omega \cap \Omega_1 \cap L^{-1}\{y\}, 0)$$
$$\not\equiv 0 \quad \text{by condition (3)}.$$

Hence, from (3.21), (3.23), (3.24) and (3.26),

$$d[(L, N(\cdot, 1)), \Omega] \neq 0$$

and again, we conclude from Theorem 3.3 and 3.5 that equation (3.17) has at least one solution in Ω.

Now, for $\lambda = 1$, equation (3.17) becomes

(3.17) $Lx \in Nx + y = N^*(x, 1) + y.$

If, for every $x \in \partial\Omega \cap \operatorname{dom} L$, (3.17) does not hold, then $Lx \notin \tilde{N}(x, \mu)$ for each $x \in \partial\Omega \cap \operatorname{dom} L$ and each $\mu \in [0, 1]$ and the above proof can be repeated. If, however, there exists x in $\partial\Omega \cap \operatorname{dom} L$ such that $Lx \in Nx + y$, then a solution exists in $\partial\Omega \subset \bar{\Omega}$. Hence (3.17) always has a solution in $\bar{\Omega}$.

This completes the proof of the Theorem.

THEOREM 3.8. *Let X be a Banach space, Z normed linear spaces and let L be a linear Fredholm mapping of index zero from a subspace of X into Z.*

Let Ω be an open bounded subset of X and let $\tilde{N}: \bar{\Omega} \times [0, 1] \to CK(Z)$ be a $L-k-\varphi$-contraction on $\bar{\Omega} \times [0, 1]$. If for each $\lambda \in [0, 1]$ and $x \in \partial\Omega \cap \operatorname{dom} L$, we have

$$Lx \notin \tilde{N}(x, \lambda)$$

and if $d[L, \tilde{N}(\cdot, \lambda_0)), \Omega] \neq 0$ for some $\lambda_0 \in [0, 1]$, then for each $\lambda \in [0, 1]$, the equation

(3.26) $Lx \in \tilde{N}(x, \lambda)$

has at least one solution in Ω.

PROOF. By Theorem 3.5, for each $\lambda \in [0, 1]$,

$$d[(L, \tilde{N}(\cdot, \lambda)), \Omega] = d[(L, N(\cdot, \lambda_0)), \Omega] \neq 0$$

and hence by Theorem 3.3, the equation

$$Lx \in \hat{N}(x, \lambda)$$

has a solution in Ω.

COROLLARY 3.3. (A Generalized Borsuk's Theorem.) *Let X, Z and L be as in Theorem 3.8 and let Ω be a bounded open subset of X, symmetric with respect to the origin and containing it. Let $\tilde{N}: \bar{\Omega} \times [0, 1] \to CK(Z)$ be a $L-k-\varphi$-contraction on $\bar{\Omega} \times [0, 1]$. Also, suppose that $\tilde{N}(-x, 0) = -\tilde{N}(x, 0)$ for each $x \in \bar{\Omega}$.*

Then equation (3.16) has a solution in Ω for each $\lambda \in [0, 1]$.

PROOF. From Theorem 3.4, $d[(L, N(\cdot, 0)), \Omega]$ is odd and hence different from zero. The result follows from the preceding Theorem.

COROLLARY 3.4. (A Generalized Krasnoselskii Theorem.) *Let X, Z, L and Ω be as in Corollary 3.3. and let $N: \bar{\Omega} \to CK(Z)$ be a $L-k-\varphi$-contraction such that for each $\lambda \in [0, 1]$ and $x \in \partial\Omega \cap \operatorname{dom} L$, we have*

(3.27) $[(L-N)x] \cap [\lambda(L-N)(-x)] = \varnothing.$

Then the equation

(3.28) $Lx \in Nx$

has at least one solution in Ω.

PROOF. Define $N: \bar{\Omega} \times [0, 1] \to CK(Z)$ by

$$\tilde{N}(x, \lambda) = (1+\lambda)^{-1}[Nx - \lambda N(-x)].$$

It can be easily verified that $\tilde{N}$ is a $L-k-\varphi$-contraction on $\Omega \times [0, 1]$. Now,

$\tilde{N}(x, 0) = Nx$ and $N(x, 1) = \frac{1}{2}[Nx - N(-x)]$ which is odd. We claim that $Lx \notin N(x, \lambda)$ for each $\lambda \in [0, 1]$ and each $x \in \partial\Omega \cap \operatorname{dom} L$. Assuming otherwise, there exist $\lambda \in [0, 1]$, $x \in \partial\Omega \cap \operatorname{dom} L$ such that

$$(1 + \lambda)Lx \in Nx - \lambda N(-x);$$

that is, there exist $u \in Nx$, $v \in N(-x)$ such that

$$(1 + \lambda)Lx = u - \lambda v$$

or

$$Lx - u = \lambda(L(-x) - v)$$

which contradicts (3.27).

Hence the conditions of Theorem 3.8 are satisfied and thus, there is a $x \in \Omega$ such that

$$Lx \in \tilde{N}(x, 0) = Nx$$

and so equation (3.28) has a solution in Ω.

4. A different approach

In building up the coincidence degree for the pair (L, N) where N is a single-valued mapping, Mawhin (1972) (see also Gaines and Mawhin (1977)) has assumed continuity of the mappings πN and $K_p(I - Q)N$. It can be easily seen that if we replace the upper-semicontinuity of N by that of πN in our condition (d) in Section 3B, our degree theory built up in the previous section will still hold under the remaining assumptions. However, it is not clear if we can replace the continuity of K_p by that of $K_p(I - Q)N$ (see Remark 3.2).

The purpose of this section is to indicate that a coincidence degree theory under assumptions similar to those of Mawhin (1972) can be built up via an alternative equivalence theorem.

4A. Another equivalence theorem

The following equivalence theorem has its own interest.

THEOREM 4.1. *Let X and Z be two vector spaces over the same scalar field. Let $L: \operatorname{dom} L \subset X \to Z$ be a linear mapping and $N: A \subset X \to 2^Z$ be a set-valued mapping. Further, assume that there is a mapping $\psi: \operatorname{coker} L \to \ker L$ such that $\psi^{-1}(0) = \{0\}$. Then, $x_0 \in \operatorname{dom} L \cap A$ is a solution of the equation*

(4.1) $$Lx \in Nx$$

if and only if x_0 is a fixed point of the set-valued mapping $\hat{M}_\psi: A \to 2^X$ defined by

(4.2) $$\hat{M}_\psi(x) = Px + \psi\pi\hat{N}x + K_p(I - Q)\hat{N}x$$

for every pair (P, Q) of exact projections with respect to L, where π, K_p, Q have their meanings as explained in Section 2 and $\hat{N}: A \to 2$ is defined by

$$Nx = \begin{cases} Nx \cap \operatorname{Im} L & \text{if } Nx \cap \operatorname{Im} L \neq \varnothing, \\ Nx & \text{if } Nx \cap \operatorname{Im} L = \varnothing. \end{cases}$$

In other words,

(4.3) $(L-N)^{-1}(0) = (I - \hat{M}\psi)^{-1}(0).$

PROOF. Since the images under P and ψ are contained in $\ker L$ and those under K_p are in $X_{I-P} \cap \operatorname{dom} L$, it is clear that $\hat{M}_\psi(A) \subset \operatorname{dom} L$. Now, for each $x \in X$, we write

(4.4) $x = Px + K_p Lx$

as $K_p L = I - P$ from (2.3). Also, since $\psi^{-1}(0) = 0$, we have

$$(\psi \pi \hat{N})^{-1}(0) = (\pi \hat{N})^{-1}(0).$$

Now let us suppose that

$$Lx \in Nx.$$

Since $Nx \cap \operatorname{Im} L \neq \varnothing$, $\tilde{N}x = Nx \cap \operatorname{Im} L$ and $Lx \in \hat{N}x$. Hence, $0 \in \pi \hat{N}x$ or

$$x \in (\pi \hat{N})^{-1}(0) = (\psi \pi \hat{N})^{-1}(0)$$

which implies that

(4.5) $0 \in \psi \pi \hat{N} x.$

Since $Lx \in Nx$,

(4.6) $K_p Lx = K_p(I-Q)Lx \in K_p(I-Q)\hat{N}x.$

From (4.4), (4.5) and (4.6),

$$\begin{aligned} x &= Px + K_p Lx + 0 \\ &\in Px + K_p(I-Q)\hat{N}x + \psi \pi \hat{N}x \\ &= M_\psi x. \end{aligned}$$

Conversely, if $x \in M_\psi x$, then

$$x \in Px + \psi \pi \hat{N}x + K_p(I-Q)\hat{N}x.$$

Let $u \in \psi \pi \hat{N}x$ and $v \in \hat{N}x$ be such that

(4.7) $x = Px + u + K_p(I-Q)v.$

Now,

$$\begin{aligned} Px &= Px + Pu + PK_p(I-Q)v \\ &= Px + u \end{aligned}$$

and hence $u = 0$ from which $0 \in \psi \pi \hat{N}x$ and thus $0 \in \pi \hat{N}x$. This implies that

$\hat{N}x \cap \text{Im}\,L \neq \varnothing$ and, consequently, $Nx \cap \text{Im}\,L \neq \varnothing$. Thus,

$$\hat{N}x = Nx \cap \text{Im}\,L \subset \text{Im}\,L$$

and so, $Q\hat{N}x = \{0\}$. As $v \in \hat{N}x$, $Qv = 0$ and, with $u = 0$, (4.7) reduces to

$$x = Px + K_p v.$$

Thus,

$$Lx = LPx + LK_p v = v \in \hat{N}x.$$

Hence $Lx \in \hat{N}x$ and as $\hat{N}x \subset Nx$ for every x, we have $Lx \in Nx$ and hence the proof of the theorem.

REMARK 4.1. From Theorems 3.1 and 4.1 it is clear that each solution $x \in \Omega$ of the equation $Lx \in Nx$ is a fixed point of M_ψ as well as of $\hat{M}_\psi$ where M_ψ and $\hat{M}_\psi$ are respectively as defined in Theorems 3.1 and 4.1. We should point out that if we define the mapping $\overline{M}_\psi \colon \overline{\Omega} \to 2^X$ by

$$\overline{M}_\psi x = Px + \psi \pi Nx + K_p(I - Q)Nx, \quad x \in \overline{\Omega},$$

then it is clear that each solution in $\overline{\Omega}$ of $Lx \in Nx$ is a fixed point of $\overline{M}_\psi$ as for each $x \in \overline{\Omega}$, $M_\psi x = Px + [\psi \pi + K_p(I - Q)]Nx$ is always a subset of

$$\overline{M}_\psi x = Px + \psi \pi Nx + K_p(I - Q)Nx.$$

However, each fixed point of $\overline{M}_\psi$ is not necessarily a solution of the equation $Lx \in Nx$. In other words, the solution set in $\overline{\Omega}$ of the equation $Lx \in Nx$ may not coincide with fixed point set of $\overline{M}_\psi$. To show this we furnish the following example.

EXAMPLE 4.1. Let $L \colon R^2 \to R^2$ be defined by

$$L(x, y) = (x, 0) \quad \text{for all } (x, y) \in R^2.$$

Let $\Omega = (-1, 1) \times (-1, 1)$ and let $N \colon \overline{\Omega} \to CK(R^2)$ be defined by

$$N(x, y) = \begin{cases} \{(1, 1)\} & \text{if } (x, y) \neq (1, 1), \\ \{(t, t) \colon 0 \leqslant t \leqslant 1\} & \text{if } (x, y) = (1, 1). \end{cases}$$

Now, let $x = (1, 1)$. Then

$$Lx = L(1, 1) = (1, 0).$$

Since $(0, 0) \in N(1, 1)$ and $(0, 0) \in \text{Im}\,L$, we have $0 \in \pi N(1, 1)$ and hence $0 \in \psi \pi N(1, 1)$. Since $I - Q$ is the projection of R^2 on the x-axis

$$K_p(I - Q)N(1, 1) = K_p([0, 1] \times \{0\}).$$

But $\text{Im}\,P = \ker L = y$ axis. Hence $K_p \colon R \times \{0\} \to R \times \{0\}$ is the identity mapping.
    Hence $K_p(I - Q)N(1, 1) = [0, 1] \times \{0\}$ and so, $(1, 0) \in K_p(I - Q)N(1, 1)$. Also, $P(1, 1) = (0, 1)$. Hence

$$(1, 1) = (0, 1) + (0, 0) + (1, 0)$$
$$= P(1, 1) + (0, 0) + (1, 0)$$
$$\in P(1, 1) + \psi \pi N(1, 1) + K_p(I - Q)N(1, 1),$$

that is $(1, 1)$ is a fixed point of $\overline{M}_\psi$. But $L(1, 1) = (1, 0) \notin N(1, 1)$.

Thus the above example shows that we cannot replace $\hat{N}$ by N in our equivalence Theorem 4.1 nor could we remove the square bracket in the definition of M_ψ in our equivalence Theorem 3.1.

REMARK 4.2. (a) If N is a single-valued mapping, we can regard it as set-valued by considering the single point Nx as a singleton $\{Nx\}$ and it follows that in this case, $N = \hat{N}$.

(b) If, for every $x \in A$, either $Nx \cap \text{Im} L = \varnothing$ or $Nx \subset \text{Im} L$, then $N = \hat{N}$.

4B. Basic assumptions

For this section we will make the following assumptions.

ASSUMPTIONS. (a)′ Same as (a) of Section 3B.

(b)′ Same as (b) of Section 3B.

(c)′ Ω is a bounded, open set in X and the set-valued mapping $N: \overline{\Omega} \to CK(Z)$ takes each x in the closure of Ω to a nonempty compact, convex subset of Z.

(d)′ If $\hat{N}: \overline{\Omega} \to CK(Z)$ is the mapping associated with N as defined in Theorem 4.1, $\hat{N}$ is assumed to be upper-semicontinuous with $\pi \hat{N}(\Omega)$ bounded in $\text{coker} L$.

(e)′ Let (P, Q) be an exact pair of continuous projections with respect to L and let K_p be a pseudo-inverse of L associated with P. Let φ be a measure of noncompactness defined on 2^X such that either (i) φ satisfies the subadditivity condition of Proposition 1.1 and takes values $R^+ = \{t \in R: t \geqslant 0\} \cup \{\infty\}$ or (ii) we additionally assume that Z is a Banach space and φ is one of χ, γ, χ_d and γ_d. We assume that for such a measure of noncompactness, $K_p(I - Q)\hat{N}$ is a $k - \varphi$-contraction with $0 < k < 1$ and that $\varphi(K_p(I - Q)\hat{N}(\Omega)) < \infty$.

(f)′ Same as (f) of Section 3B.

PROPOSITION 4.1. *Let assumption* (a)′ *to* (d)′ *hold and let* (P, Q) *and* (P', Q') *be exact pairs of continuous projections with respect to* L. *Suppose that the pair* (P, Q) *satisfies the assumption* (e)′. *Then the pair* (P', Q') *also satisfies the assumption* (e)′.

PROOF. The proof follows similarly from that of Proposition III.1 in Gaines and Mawhin (1977) by noting that a compact mapping is a $0 - \varphi$-contraction.

4C. Definition of coincidence degree

PROPOSITION 4.2. *Suppose assumptions* (a)′ *to* (e)′ *are satisfied and* $\hat{M}_\psi$ *is the*

mapping defined in Theorem 4.1 *for some continuous isomorphism*

$$\psi : \operatorname{coker} L \to \ker L.$$

Then, for each x in $\bar{\Omega}$, $M_\psi x$ is a compact convex subset of X and $\hat{M}_\psi$ is a $k - \varphi$-contraction.

PROOF. Since P, Q, K_p, ψ and π are all linear, and since $\hat{N}x$ is convex for each $x \in \bar{\Omega}$, $\hat{M}_\psi x$ is convex. Now, as $K_p(I - Q)\hat{N}$ is a $k - \varphi$-contraction, $K_p(I - Q)\hat{N}x$ is compact by definition. By the continuity of ψ and π, $\psi\pi Nx$ is also compact and hence $\hat{M}_\psi x \in CK(X)$ for each $x \in \bar{\Omega}$.

Now, P is linear, continuous and has a finite dimensional range and is therefore compact and a $0 - \varphi$-contraction. Also, $\psi\pi N(\bar{\Omega})$ is bounded, closed and contained in a finite-dimensional subspace of X. Hence it is relatively compact and $\psi\pi\hat{N}$ is thus a $0 - \varphi$-contraction. From assumption (e)' and Proposition 1.1, $\hat{M}_\psi$ is a $k - \varphi$-contraction from $\bar{\Omega}$ to $CK(X)$.

REMARK 4.3. From Proposition 4.1, we see that if the assumptions (a)' to (e)' are satisfied M_ψ is an ultimately compact mapping. It follows from assumption (f)' and Theorem 4.1 that $0 \notin (I - M_\psi)(\operatorname{dom} L \cap \partial\Omega)$. Thus the degree of the ultimately compact field $I - \hat{M}_\psi$, with respect to zero, is well defined.

DEFINITION 4.1. Suppose that assumptions (a)' to (f)' are satisfied and ψ is an orientation preserving continuous isomorphism from $\operatorname{coker} L$ to $\ker L$ (see Definition 3.3). Then, the *coincidence degree of L and N in Ω, denoted by* $\partial[(L, N), \Omega]$, is defined by

(4.8) $$\partial[(L, N), \Omega] = d(I - \hat{M}_\psi, \Omega, 0)$$

where $\hat{M}_\psi : \Omega \to CK(X)$ is defined by

$$\hat{M}_\psi = P + \psi\pi\hat{N} + K_p(I - Q)\hat{N}$$

and the right-hand term is the degree for the set-valued ultimately compact field $I - \hat{M}_\psi$ as defined in Definition 1.3.

REMARK 4.4. (a) If $X = Z$, $L = I$, then $\ker L = \{0\}$ and thus $\operatorname{coker} L = \{0\}$. This implies that $\operatorname{Im} L = X$ and hence, $P = 0$, $Q = 0$ and $K_p(I - Q) = I$ and the only isomorphism between $\operatorname{coker} L$ and $\ker L$ is the trivial one $\psi(0) = 0$. The assumption (b)' is trivially satisfied and (e)' reduces to assuming that N is a $k - \varphi$-contraction for some k in $(0, 1)$ with $\varphi(\hat{N}(\Omega)) < \infty$. Assumption (f)' means that N and $\hat{N}$ have no fixed points on the boundary of Ω. As $\hat{M}_\psi = N$, we have

$$\partial[(I, N), \Omega] = d(I - \hat{N}, \Omega, 0).$$

We may, in fact, replace assumptions (d)′ and (e)′ by the assumption that $\hat{N}$ is ultimately compact.

(b) As $N = \hat{N}$, by definition,

$$\partial[(L, N), \Omega] = \partial[(L, \hat{N}), \Omega].$$

4D. Basic properties of the coincidence degree

Following the same argument as given in Section 3D, we can show that this degree $\partial[(L, N), \Omega]$ has all the basic properties of a degree. In other words, if (a)′ to (f)′ are satisfied, then Theorems 3.3–3.5 hold with $d[(L, N), \Omega]$ replaced by $\partial[(L, N), \Omega]$. Also Rouché's Theorem and Generalized Continuation Theorem can be obtained under suitable assumptions.

5. A general remark

The basic difference between the degree theory presented in Section 3 and that in Section 4 lies in the continuity conditions appearing in assumptions (d) and (e), and assumptions (d)′ and (e)′ respectively. At the beginning of Section 4 and Remark 3.2 we have already discussed assumptions (e) and (e)′. Assumptions (d) and (d)′ differ in the upper-semicontinuity of N (or πN) and $\hat{N}$ (or $\pi \hat{N}$). Thus in order to apply the degree $d[(L, N), \Omega]$ or $\partial[(L, N), \Omega]$ to the pair (L, N) we need respectively the upper-semicontinuity of N or $\hat{N}$. The following two examples show that the upper-semicontinuity of one does not, in general, follow from the upper-semicontinuity of the other.

EXAMPLE 5.1. This example gives a pair (L, N) where N is u.s.c. but $\hat{N}$ is not. Thus $\partial[(L, N), \Omega]$ cannot be defined.

$L: R \to R$ is the zero operator, that is $Lx = 0$ for all $x \in R$.

$$N: \bar{\Omega} = [-1, 1] \to CK(R)$$

is defined by

$$\hat{N}x = \begin{cases} \{\sin 1/x\} & \text{if } x \neq 0, \\ [-1, 1] & \text{if } x = 0. \end{cases}$$

Then,

$$\hat{N}x = \begin{cases} \{\sin 1/x\} & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0 \end{cases}$$

which is not u.s.c.

EXAMPLE 5.2. Here we give an example where $\hat{N}$ is u.s.c. but N is not. Let $L: R^2 \to R^2$ be defined by

$$L(x, y) = (x, 0) \quad \text{for } (x, y) \in R^2$$

and

$$N: \bar{\Omega} = [-1,1] \times [-1,1] = ((-1,1) \times (-1,1))^- \to CK(R^2)$$

be defined by

$$N(x,y) = \begin{cases} \{(x,t): 0 \leqslant t \leqslant 1\} & \text{if } y = 0 \quad \text{and} \quad x > 0, \\ \{(x,y)\} & \text{otherwise.} \end{cases}$$

Thus, $\hat{N}(x,y) = \{(x,y)\}$ for all $(x,y) \in \bar{\Omega}$. Hence $\hat{N}$ is u.s.c.

To see that N is not u.s.c., consider the sequences $\{u_n\}$ and $\{v_n\}$ with

$$u_n = \left(\frac{1}{n}, 0\right) \in \bar{\Omega} \quad \text{and} \quad v_n = \left(\frac{1}{n}, 1\right).$$

Now,

$$N(u_n) = N\left(\frac{1}{n}, 0\right) = \left\{\left(\frac{1}{n}, t\right): 0 \leqslant t \leqslant 1\right\}$$

and hence $v_n \in N(u_n)$ for each n. Now, $u_n \to (0,0)$ and $v_n \to (0,1)$. $N(0,0) = \{(0,0)\}$ and hence $(0,1) \notin N(0,0)$. Thus N is not u.s.c.

The following two propositions give sufficient conditions for $\hat{N}$ to be upper-semicontinuous when N is upper-semicontinuous. These propositions are included here for the sake of interest.

PROPOSITION 5.1. *Let* $N: \bar{\Omega} \to CK(Z)$ *be upper-semicontinuous and let*

$$\hat{N}: \bar{\Omega} \to CK(Z)$$

be defined by

$$\hat{N}x = \begin{cases} Nx \cap A & \text{if } Nx \cap A \neq \varnothing, \\ Nx & \text{if } Nx \cap A = \varnothing, \end{cases}$$

where A *is a closed subset of* Z.

   Suppose the set

$$S = \{y \in \bar{\Omega}: Ny \cap A = \varnothing\} \text{ is closed in } X.$$

Then $\hat{N}$ *is upper-semicontinuous.*

PROOF. If $A = Z$, $\hat{N} = N$ and there is nothing to prove. Now let us assume that $A \neq Z$, that is $A^c \neq \varnothing$, where A^c denotes the complement of A.

Let $x \in \bar{\Omega}$ and let V be an open set in Z containing $\hat{N}x$. To show that $\hat{N}$ is u.s.c. at x, we need to find an open set W of X containing x such that $\hat{N}(W) \subset V$.

Let us consider two cases:

(i) If $Nx \cap A = \varnothing$, then $Nx = \hat{N}x$ and by upper-semicontinuity of N, there exists an open set W containing x such that $N(W) \subset V$. But for every $y \in W$, $\hat{N}y \subset Ny$ and hence $\hat{N}(W) \subset N(W) \subset V$.

(ii) If $Nx \cap A \neq \varnothing$, then $\hat{N}x = Nx \cap A$. Let $U = V \cup A^c$. As A is closed, U is open and $Nx = \hat{N}x \cup (Nx \cap A^c) \subset V \cup A^c = U$, we have by upper-semicontinuity

N, that there exists an open set W' containing x such that $N(W') \subset U$. Now, let $W = W' \cap S^c$. Since $Nx \cap A \neq \varnothing$, $x \in S^c$ and so $x \in W$. By our assumption on S, W is open. Now, if $y \in W$,

$$\hat{N}y = Ny \cap A \subset N(W') \cap A \subset U \cap A = (V \cup A^c) \cap A = V \cap A \subset V.$$

Hence $\hat{N}(W) \subset V$. Thus, we have shown that $\hat{N} : \overline{\Omega} \to CK(Z)$ is upper-semi-continuous.

REMARK 5.1. In our case, $A = \operatorname{Im} L$ which is closed.

PROPOSITION 5.2. *Let* $N : \overline{\Omega} \to CK(Z)$ *be u.s.c. and let* $\hat{N} : \overline{\Omega} \to CK(Z)$ *be defined by*

$$\hat{N}x = \begin{cases} Nx \cap \operatorname{Im} L & \text{if } Nx \cap \operatorname{Im} L \neq \varnothing, \\ Nx & \text{if } Nx \cap \operatorname{Im} L = \varnothing. \end{cases}$$

Suppose that $\hat{N}(\overline{\Omega})$ *is closed and that, for any* $x, y \in \overline{\Omega}$ *with* $x \neq y$, *we have the condition*

$$(Nx \setminus \operatorname{Im} L) \cap (Ny \setminus \operatorname{Im} L) = \varnothing.$$

Then, $\hat{N}$ *is u.s.c.*

PROOF. Let $\{x_n\}$ be a sequence in $\overline{\Omega}$ converging to x and $\{y_n\}$ be a sequence in Z such that $y_n \in \hat{N}x_n$ for every positive integer n and $\{y_n\}$ converges to y. We wish to prove that $y \in Nx$. Now, $y_n \in \hat{N}x_n \subset Nx_n$ for each n. By the upper-semicontinuity of N, $y \in Nx$. If $y \in \operatorname{Im} L$, then $y \in \hat{N}x$ and the proof is complete.

Suppose that $y \notin \operatorname{Im} L$. Hence $y \in Nx \setminus \operatorname{Im} L$. Now, since $y_n \in \hat{N}(\overline{\Omega})$ for each n and $\hat{N}(\overline{\Omega})$ is closed, $y \in \hat{N}(\overline{\Omega})$. Let $\bar{x} \in \overline{\Omega}$ be such that $y \in \hat{N}\bar{x} \subset N\bar{x}$. Thus we have $y \in N\bar{x} \setminus \operatorname{Im} L$ and by the above condition, $x = \bar{x}$. Hence $y \in \hat{N}x$.

Acknowlegement

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