THE NUMERICAL RANGE OF AN ELEMENT OF A NORMED ALGEBRA

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1. Introduction. Given a normed linear space X, let $S(X)$, X', $B(X)$ denote respectively the unit sphere $\{x: \|x\| = 1\}$ of X, the dual space of X, and the algebra of all bounded linear mappings of X into X. For each $x \in S(X)$ and $T \in B(X)$, let $D_X(x) = \{f \in X': \|f\| = f(x) = 1\}$, and $V(T; x) = \{f(Tx): f \in D_X(x)\}$. The *numerical range* $V(T)$ is then defined by

$$V(T) = \bigcup\{V(T; x): x \in S(X)\}. \tag{1}$$

Similarly, given an element a of a normed algebra A, the numerical range $V_A(a)$ is defined by

$$V_A(a) = \bigcup\{V_A(a; x): x \in S(A)\},$$

where $V_A(a; x) = \{f(ax): f \in D_A(x)\}$. In words, $V_A(a)$ is the numerical range in the sense (1) of the left regular representation of a on A.

When X is a Hilbert space, $V(T)$ coincides with the usual numerical range $W(T)$, and it is a well known theorem of Toeplitz, Hausdorff, and M. H. Stone **[8]** that $W(T)$ is convex and that its closure contains the spectrum $\mathrm{Sp}(T)$ of T. With an arbitrary normed linear space X, $V(T)$ is the union of the numerical ranges $W(T)$ in the sense of Lumer **[7]** corresponding to all choices of semi-inner-product on X that yield the given norm of X. The numerical range $W(T)$ corresponding to a semi-inner-product need not be convex, in fact need not be connected. $V(T)$ need not be convex, but it is proved in **[2]** that it is connected. Williams **[9]** has proved that the closure of $V(T)$ contains $\mathrm{Sp}(T)$ if X is a Banach space over **C**.

In this note we show that if A is a normed algebra with unit element e and $\|e\| = 1$, then $V_A(a)$ has very simple properties. In particular, $V_A(a)$ is compact and convex, and it contains the spectrum $\mathrm{Sp}_A(a)$ of a if A is a complex Banach algebra.

These results are applicable to a bounded linear operator $T \in B(X)$ by taking any subalgebra $\mathfrak{A}$ of $B(X)$ such that $I, T \in \mathfrak{A}$. We show that $V_{\mathfrak{A}}(T)$ is then the closed convex hull of $V(T)$.

2. Normed algebras. Let **F** denote **R** or **C**, and let $(A, \|\,.\,\|)$ be a normed algebra over **F**; i.e. A is a linear associative algebra over **F** and $\|\,.\,\|$ is an algebra-norm on A (a norm on the linear space A such that $\|xy\| \leqq \|x\| . \|y\|$ $(x, y \in A)$). Suppose also that A has a unit element e and that $\|e\| = 1$.

LEMMA. $V_A(a) = V_A(a; e)$ $(a \in A)$.

Proof. Given $x_0 \in S(A)$ and $f_0 \in D_A(x_0)$, let f be defined by

$$f(x) = f_0(xx_0) \quad (x \in A).$$

Then $f \in D_A(e)$, and so $f_0(ax_0) \in V_A(a; e)$. This proves that $V_A(a) \subset V_A(a; e)$, and the opposite inclusion is obvious.

THEOREM 1. *For each* $a \in A$, $V_A(a)$ *is a compact convex subset of* $\mathbf{F}$.

Proof. Since $D_A(e) = \{f \in A' : \|f\| \leqq 1 \text{ and } f(e) = 1\}$, $D_A(e)$ is a weak* compact convex set. Since $V_A(a; e)$ is the image of $D_A(e)$ under the weak* continuous linear mapping $f \to f(a)$, it follows that $V_A(a; e)$, and therefore $V_A(a)$, is a compact convex subset of $\mathbf{F}$.

THEOREM 2. *Let B be a subalgebra of A such that* $e \in B$. *Then, for each* $b \in B$,

$$V_B(b) = V_A(b).$$

Proof. By the Hahn–Banach theorem, the restriction mapping

$$f \to f|_B$$

maps $D_A(e)$ onto $D_B(e)$. Therefore $V_B(b; e) = V_A(b; e)$, and the lemma completes the proof.

Remark. No such simple invariance holds for $V(T)$ or $W(T)$ with respect to linear subspaces.

THEOREM 3. *Let A be complete and* $\mathbf{F} = \mathbf{C}$. *Then, for each* $a \in A$,

$$\mathrm{Sp}_A(a) \subset V_A(a).$$

Proof. Let $\lambda \in \mathrm{Sp}_A(a)$. Then $\lambda e - a$ has no inverse in A. Suppose that it has no right inverse. Then $(\lambda e - a)A$ is a proper right ideal J of A. Since A is complete, it follows that

$$\|x - e\| \geqq 1 \qquad (x \in J),$$

and therefore, by the Hahn–Banach theorem, there exists $f \in A'$ such that $f(e) = \|f\| = 1$ and $f(J) = 0$. Thus $f \in D_A(e)$ and $f(\lambda e - a) = 0$, from which $\lambda = f(a) \in V_A(a; e)$. A similar proof is available if $\lambda e - a$ has no left inverse.

An alternative proof, suggested by the referee, applies Theorem 2 to a closed commutative subalgebra B of A containing a and e, and uses the fact that the non-zero multiplicative linear functionals on B belong to $D_B(e)$.

Let N denote the set of all algebra-norms p on A equivalent to the given algebra-norm $\|.\|$ and with $p(e) = 1$. For each $p \in N$, let $V_{A,p}(a)$ denote the numerical range $V_A(a)$ computed in terms of p in place of $\|.\|$. Let $\mathrm{co}(E)$ denote the convex hull of E.

THEOREM 4. *Let A be complete and* $\mathbf{F} = \mathbf{C}$. *Then, for each* $a \in A$,

$$\mathrm{co}(\mathrm{Sp}_A(a)) = \bigcap \{V_{A,p}(a) : p \in N\}.$$

Proof. It is immediate from Theorems 1 and 3 that

$$\mathrm{co}(\mathrm{Sp}_A(a)) \subset \bigcap \{V_{A,p}(a) : p \in N\}.$$

To prove the opposite inclusion, it is enough, since $\mathrm{Sp}_A(a)$ is compact, to prove that every open circular disc containing $\mathrm{Sp}_A(a)$ also contains $V_{A,p}(a)$ for some $p \in N$. Suppose then that

$$|\lambda - \alpha| < r \qquad (\lambda \in \mathrm{Sp}_A(a)).$$

Then

$$\rho(a - \alpha e) < r,$$

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where $\rho(x)$ denotes the spectral radius of x. It is proved in [6] that, for each $x \in A$,

$$\rho(x) = \inf\{p(x): p \in N\}.$$

Therefore there exists $p \in N$ such that

$$p(a - \alpha e) < r.$$

But then it follows that

$$|\lambda - \alpha| < r \qquad (\lambda \in V_{A,p}(a)).$$

Remark. If A is complete and $\mathbf{F} = \mathbf{R}$, Theorems 3 and 4 remain valid provided that $\mathrm{Sp}(a)$ is replaced by $\mathrm{Sp}_A(a) \cap \mathbf{R}$.

Some important applications of the numerical range to normed algebras depend on an inequality relating the norm to the numerical radius $\sup\{|\lambda|: \lambda \in V_A(a)\}$. Such an inequality was proved for complex Banach algebras by Bohnenblust and Karlin [1, p. 219], and for complex semi-inner-product spaces by Lumer [7]. We give an elementary proof of the inequality, for complex normed algebras, which is in part derived from Lumer's proof.

THEOREM 5. *Let* $\mathbf{F} = \mathbf{C}$. *Then, for all* $a \in A$,

$$\| a \| \leqq 4 \sup\{|\lambda|: \lambda \in V_A(a)\}.$$

Proof. By Theorem 2, we may suppose that A is complete, for replacement of A by its completion does not alter $V_A(a)$. Let $a \in A$ and $\sup\{|\lambda|: \lambda \in V_A(a)\} \leqq \mu < 1$. Given $x \in S(A)$, there exists $f \in D_A(x)$, and we have, for all complex numbers λ with $|\lambda| \leqq 1$,

$$\|(e - \lambda a)x\| \geqq |f((e - \lambda a)x)| = |1 - \lambda f(ax)| \geqq 1 - \mu.$$

Therefore

$$\|(e - \lambda a)x\| \geqq (1 - \mu)\|x\| \qquad (x \in A, |\lambda| \leqq 1). \tag{1}$$

By Theorem 3, $\mathrm{Sp}_A(a) \subset V_A(a)$, and so $\rho(a) \leqq \mu < 1$, and $e - \lambda a$ is therefore invertible whenever $|\lambda| \leqq 1$. Therefore (1) gives

$$\|(e - \lambda a)^{-1}\| \leqq (1 - \mu)^{-1} \qquad (|\lambda| \leqq 1). \tag{2}$$

With $\omega_1, \ldots, \omega_n$ denoting the nth roots of unity, we have

$$a(e - a^n)^{-1} = \frac{1}{n} \sum_{k=1}^{n} \omega_k^{-1}(e - \omega_k a)^{-1},$$

and so, by (2),

$$\|a(e - a^n)^{-1}\| \leqq (1 - \mu)^{-1} \qquad (n = 1, 2, \ldots).$$

Since $\rho(a) < 1$, $e - a^n \to e$ as $n \to \infty$, and therefore

$$\|a\| \leqq (1 - \mu)^{-1}. \tag{3}$$

Given arbitrary $b \in A$ and $\delta > \sup\{|\lambda|: \lambda \in V_A(b)\}$, (3) holds with $a = (1/2\delta)b$ and $\mu = \frac{1}{2}$, and gives $\|b\| \leqq 4\delta$.

Remarks. (i) The constant 4 is not best possible. Bohnenblust and Karlin established the inequality with exp (1) in place of 4, and Glickfeld [5] has proved that this is best possible. An elaboration of the present proof also gives the sharp inequality.

(ii) Theorem 5 is false for algebras over **R**, for which it is possible to have $V_A(a) = \{0\}$ with $a \neq 0$. However, it is proved in [3] for Banach algebras A over **R**, that $a = 0$ whenever $V_A(a) = V_A(a^2) = \{0\}$. Theorem 2 now shows that this holds for all normed algebras over **R**.

3. Linear operators. The results of §2 are applicable to the algebra $B(X)$ with the operator norm $|T| = \sup\{\|Tx\| : \|x\| \leqq 1\}$, and to subalgebras of $B(X)$ that contain the identity operator I. Let $\mathfrak{A}$ be any such subalgebra of $B(X)$. Given $T \in \mathfrak{A}$, we then have two numerical ranges available for T, $V(T)$ computed in terms of X, and $V_{\mathfrak{A}}(T)$ computed in terms of $\mathfrak{A}$. By Theorem 2, $V_{\mathfrak{A}}(T)$ is independent of the choice of $\mathfrak{A}$. We consider briefly the relationship between $V(T)$ and $V_{\mathfrak{A}}(T)$.

Let $P = \{(x, f) : x \in S(X), f \in D_X(x)\}$, and, given $(x, f) \in P$, let $\Phi_{(x,f)}$ be the functional defined on $\mathfrak{A}$ by

$$\Phi_{(x,f)}(T) = f(Tx) \qquad (T \in \mathfrak{A}).$$

It is clear that $\Phi_{(x,f)} \in D_{\mathfrak{A}}(I)$, and so $V(T) \subset V_{\mathfrak{A}}(T)$.

THEOREM 6. *$V_{\mathfrak{A}}(T)$ is the closed convex hull of $V(T)$.*

Proof. By a lemma proved for $W(T)$ by Lumer [7, Lemma 12], we have

$$\sup\{\operatorname{Re}\lambda : \lambda \in V(T)\} = \inf\left\{\frac{1}{\alpha}[\,|I + \alpha T| - 1] : \alpha > 0\right\}. \tag{4}$$

Since $I \in \mathfrak{A}$, we have

$$|T| = \sup\{\,|TA| : A \in \mathfrak{A}, |A| \leqq 1\} \qquad (T \in \mathfrak{A}).$$

Therefore, by (4) applied to the left regular representation of T on $\mathfrak{A}$,

$$\sup\{\operatorname{Re}\lambda : \lambda \in V_{\mathfrak{A}}(T)\} = \inf\left\{\frac{1}{\alpha}[\,|I + \alpha T| - 1] : \alpha > 0\right\}. \tag{5}$$

It follows from (4) and (5) that $V(T)$ and $V_{\mathfrak{A}}(T)$ have the same closed convex hull, and so Theorem 1 completes the proof.

Remarks. (i) Let $\Pi = \{\Phi_{(x,f)} : (x, f) \in P\}$. The above proof also shows that $D_{\mathfrak{A}}(I)$ is the weak* closed convex hull of Π, which is essentially Lumer's Theorem 11 in [7].

(ii) It is proved in [2] that P is connected in the norm × weak* topology, i.e. the product of the norm topology on X and the weak* topology on X'. It is easy to prove that the mapping $(x, f) \to \Phi_{(x,f)}$ is continuous from P with the norm × weak* topology into $\mathfrak{A}'$ with the weak* topology. Therefore Π is a weak* connected subset of $\mathfrak{A}'$. It is also easy to prove that P is a closed subset of $X \times X'$ in the norm × weak* topology, and so the question arises whether Π is closed in $\mathfrak{A}'$. Duncan [4] has proved that Π is norm closed provided that X is complete and that the algebra $\mathfrak{A}$ is not too small, but that it need not be weak* closed.

REFERENCES

1. H. F. Bohnenblust and S. Karlin, Geometrical properties of the unit sphere of Banach algebras, *Ann. of Math.* **62** (1955), 217–229.

2. F. F. Bonsall, B. E. Cain, and H. Schneider, The numerical range of a continuous mapping of a normed space, *Aequationes Mathematicae*; to appear.

3. F. F. Bonsall and J. Duncan, Dually irreducible representations of Banach algebras, *Quart. J. Math. Oxford Ser.* (2) **19** (1968), 97–110.

4. J. Duncan, The evaluation functionals associated with an algebra of bounded operators, *Glasgow Math. J.* **10** (1969), 73–76.

5. B. W. Glickfeld, On an inequality in Banach algebra geometry and semi-inner product spaces, *Notices Amer. Math. Soc.* **15** (1968), 339–340.

6. R. B. Holmes, A formula for the spectral radius of an operator, *Amer. Math. Monthly* **75** (1968), 163–166.

7. G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.

8. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Col. Publ. XV (New York, 1932).

9. J. P. Williams, Spectra of products and numerical ranges, *J. Math. Anal. Appl.* **17** (1967), 214–220.

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