ON THE THEORY OF A CASCADE OF STALLED AEROFOILS

L. C. WOODS

(A paper read at the Third Annual Meeting of the *Australian Mathematical Society* at Adelaide, August, 1958)

Summary

A mathematical theory of the separating flow past a cascade of aerofoils is developed. The flow is assumed to be inviscid, incompressible and two-dimensional. The wakes are represented by regions of stationary fluid, which could, in the general case, maintain a pressure gradient, although much of the theory is developed for the case of constant wake pressure. The theory is thus a generalization of the classical Helmholtz flow past isolated obstacles to a cascade of such bodies. There is no limit on the stagger of the cascade. Equations are given for the lift and drag on each body of the cascade, and for the flow deflexion and velocity distributions. The theory is shown to have a number of interesting applications in addition to that of the cascade of stalled aerofoils for which it was first developed.

1. Introduction

A reasonable first approximation to the flow through the blading of an axial compressor can be obtained by studying the flow through an infinite cascade of aerofoils. The theory for the design of cascades (see Rosenblat and Woods 1956, [6], where further references are given) and for the steady flow through given cascades (see Robinson and Laurmann 1956 [5], pp. 147—158) is well-established for the case of non-separating flows, but extensions of this theory to the case of separating flows have apparently not yet been made.* Yet the stall problem in the blades of axial compressors is still a very serious one.

* After completing this paper my attention was drawn to some work reported in p.p. 148—150 of a recent text by Birkhoff and Zarantonello (1957) [1]. The method and results given in the present paper appear to be more general than those published earlier, but I have not been able to obtain the references quoted by Birkhoff and Zarantonello to check this.

The phenomena is very complicated. There are two distinct aspects of the stall problem; first, and most serious, is the problem of "stall flutter", which occurs when a row of blades reaches the critical stall angle, and secondly the phenomena of the "rotating stall", which occurs at an angle of attack somewhat smaller than the critical stall angle (Pearson 1953 [4]). Both these flow problems are extremely difficult to represent by accurate theoretical models, amenable to mathematical analysis.

The aerodynamics of the stall flutter problem involves an unsteady flow problem akin to the unsteady non-separating flow past an isolated oscillating aerofoil, the theory of which is well-known. Theories for the non-separating flow through a cascade of oscillating aerofoils have been given in [2], [3] and [8]. A theory of the aerodynamics appropriate to the stall flutter of an isolated aerofoil has been developed in [10]; it is hoped at a later date to extend this work to cascade of aerofoils, and the theory given in this paper will form the basis for this extension.

In this paper we shall study the steady problem of the flow through a cascade of stalled or partially stalled aerofoils. In later papers, provided the mathematical analysis does not become prohibitively complicated, it is hoped to extend this work first to the stall flutter problem, and secondly to the rotating stall problem. In as much as the flow in the later problem can be regarded as being quasi-stationary, and provided the stalled region covers at least three or four blades, the theory given below has an immediate application, permitting an estimate in the "lift" force acting on the blading due to the stall.

Several other applications of the theory given in the paper are also indicated: they are (a) the flow past a slotted wall, (b) the flow through a slotted wall and (c) the flow through a series of tubes (see figure 5).

2. General Theory

Consider the cascade of aerofoils of chord length c, gap H and stagger angle α shown in figure 1. They are arranged along the Oy-axis at intervals of H, so that the flow conditions at points $(x, y + nH)$, $n = 0, \pm 1, \pm 2$, are identical.

Ley (q, θ) be the velocity vector in polar coordinates, then the inlet and outlet conditions are

$$(q, \theta) = (V, \alpha), \text{ and } (q, \theta) = (V, \beta), \tag{1}$$

respectively.

The flow is assumed to separate from the aerofoils at corresponding points D_1 on their upper surfaces, with the result that a relatively large wake of slowly moving, turbulent fluid extends behind each aerofoil. We shall represent these wakes by regions of stationary fluid, separated from the main stream by "free" streamlines along which, in general, the pressure could vary. The nature of this pressure variation is unfortunately not deducible from inviscid flow theory, and therefore, short of attempting to solve the almost prohibitively difficult equations of viscous, turbulent flow, we are forced to adopt some hypothesis about this wake pressure.

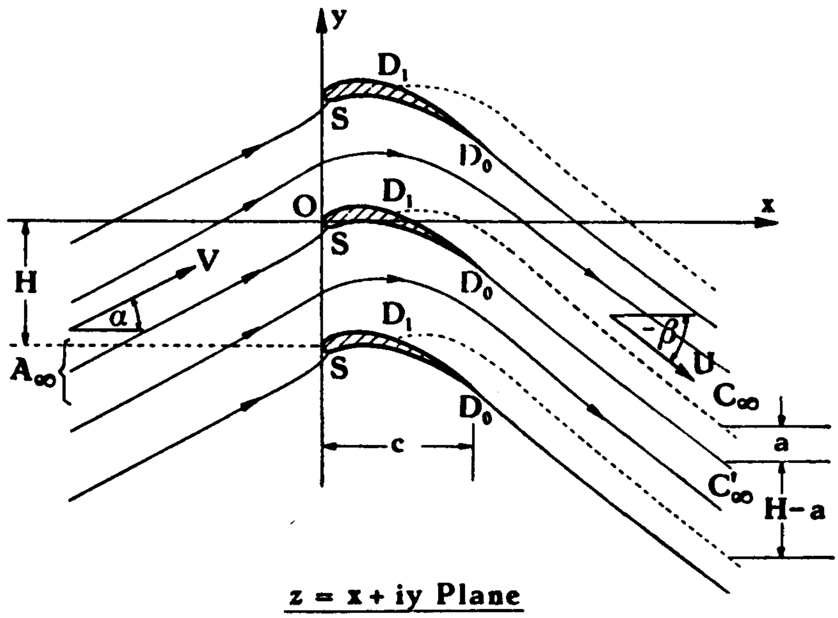


Figure 1

The simplest hypothesis is that of Helmholtz, namely that the wake pressure is constant, and therefore equal to its value downstream at infinity, p_∞. While this is the hypothesis adopted in this paper, it should be noted that the general theory (see equation (17) below) is by no means restricted to this case, so if a better hypothesis is found the theory given below could easily be extended.

Let the wake displacement thickness equal a downstream at infinity, then this will be the width of our stationary air model of the wake. Therefore the equation of continuity of mass applied to the "channel" $A_\infty C'_\infty A_\infty$ (see figure 1) yields

$$HV\cos\alpha = (H - a)U\cos\beta, \tag{2}$$

assuming the flow to be two-dimensional and incompressible.

Let D, L be the Ox, Oy components of the force acting on an aerofoil of the cascade, then it readily follows from the momentum equation, Bernoulli's theorem and (2) that

$$D = \tfrac{1}{2}H\rho V^2\cos^2\alpha\left\{\left(\frac{H}{H-a}\right)^2\tan^2\beta - \tan^2\alpha + \left(\frac{a}{H-a}\right)^2\right\}, \tag{3}$$

and

$$L = H\rho V^2 \cos^2 \alpha \left\{ \tan \alpha - \left(\frac{H}{H - a} \right) \tan \beta \right\}, \tag{4}$$

where ρ is the fluid density. When $a = 0$ these equations reduce to the classical results for non-separating cascade flow (see [5]).

The problem solved in this paper is that of calculating (U, β) given (V, α), the blade geometry and the point of flow separation D_0. After this calculation equations (2), (3) and (4) enable us to deduce a, D and L.

3. The Conformal Transformations

Let $w = \varphi + i\psi$, where φ is the velocity potential, and ψ is the stream function, then the w-plane will appear as shown in figure 2. We have taken the origin of the w-plane to be at the front stagnation point S of one of the aerofoils; the other aerofoils then lie on $\psi = \pm nh \cos \alpha$, $n = 1, 2, \ldots$, where $h = HV$, and these aerofoils have their front stagnations

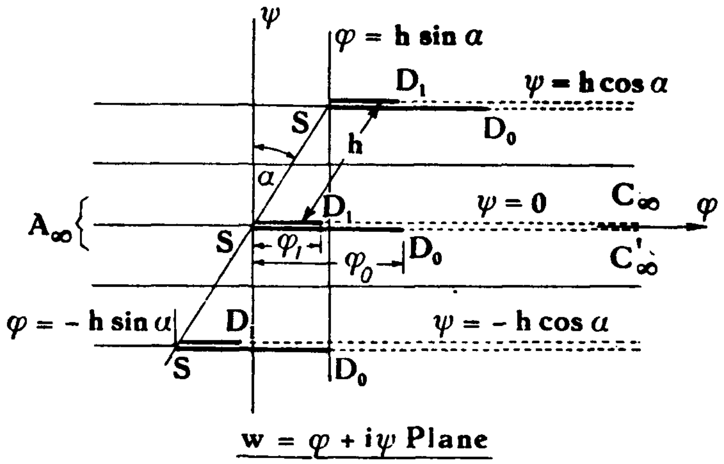


Figure 2

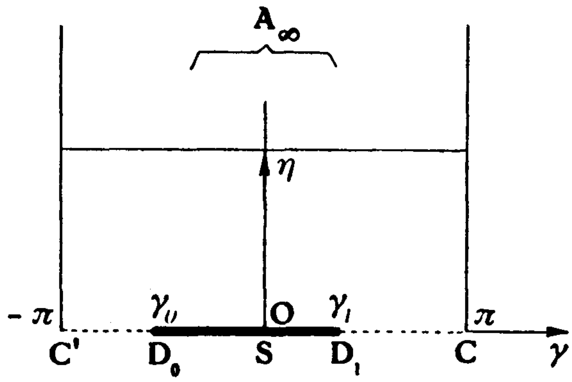
points at $\varphi = \mp nh \sin \alpha$. The flow is assumed to separate at $\varphi = \varphi_1$, on the upper surface and at $\varphi = \varphi_0$ (the trailing edge) on the lower surface of the aerofoil on $\psi = 0$. (We shall refer only to this particular aerofoil in the sequel.)

If $\tau(w)$ is a function characteristic of the flow it is apparent that τ must satisfy the periodic relation

$$\tau(w) = \tau(w + nih\,e^{-i\alpha}). \tag{5}$$

Now consider the conformal transformation

$$w = \frac{h}{2\pi} \{ \zeta \sin \alpha - 2 \cos \alpha \ln (\cos \tfrac{1}{2}\zeta) \}. \tag{6}$$



$\zeta = \gamma + i\eta$ Plane

Figure 3

It is readily verified that this maps the aerofoil and wake on to $-\pi < \gamma < \pi$, $\eta = 0$, as shown in figure 3. Furthermore as $w(\pi, \eta) = w(-\pi, n) + ih\,e^{-i\alpha}$, it follows from (5) that

$$\tau(\pi, \eta) = \tau(-\pi, \eta). \tag{7}$$

The separation points D_1, D_0 map on to $\eta = 0$, $\gamma = \gamma_1, \gamma_0$, where

$$\begin{aligned} \varphi_0 &= \frac{h}{2\pi}\{\gamma_0 \sin\alpha - 2\cos\alpha \ln(\cos\tfrac{1}{2}\gamma_0)\}, \\ \varphi_1 &= \frac{h}{2\pi}\{\gamma_1 \sin\alpha - 2\cos\alpha \ln(\cos\tfrac{1}{2}\gamma_1)\}, \end{aligned} \tag{8}$$

and the upper and lower surfaces of the wake are mapped on to $\eta = 0$, $-\pi < \nu < \gamma_0$, and $\eta = 0$, $\gamma_1 < \gamma < \pi$ respectively. What may be described as the "wetted" surface of the aerofoil — the surface over which θ_s, the surface slope, is known — is mapped on to $\eta = 0$, $\gamma_0 < \gamma < \gamma_1$. The point downstream at infinity maps on to $\gamma = \pm\pi$, $\eta = 0$, while the point upstream at infinity ($\varphi = -\infty$) maps on to the "point" $\eta = \infty$.

We shall use the variable ζ as our independent variable in the stalled cascade problem.

4. The Boundary-Value Problem of Stalled Cascade Flow and its Solution

The most convenient dependent variable for our purpose is that defined in

$$\tau \equiv \ln\left(\frac{U dz}{dw}\right) = \ln\frac{U}{q} + i\theta \equiv \Omega + i\theta, \tag{9}$$

which is obviously an analytic function of z, w and by (6) of ζ. The variable Ω is clearly depedent on the fluid pressure p, in fact

$$p = p_\infty + \tfrac{1}{2}\rho U^2(1 - e^{-2\Omega}) \tag{10}$$

where p_∞ is the pressure at $\varphi = \infty$.

It now follows from equations (1) and (7) that in the ζ-plane τ satisfies the mixed boundary conditions:—

(11) 1. $$\tau(\pi, \eta) = \tau(-\pi, \eta),$$

(12) 2. On $\eta = 0$: $$\left.\begin{array}{l} \Omega = \Omega_s \ (-\pi < \gamma < \gamma_0,\ \gamma_1 < \gamma < \pi) \\ \theta = \theta_s \ (\gamma_0 < \gamma < \gamma_1) \end{array}\right\},$$

3.

(13) $$\lim_{\eta\to\infty} \tau(\gamma, \eta) = \ln \frac{U}{V} + i\alpha,$$

(14) $$\lim_{\gamma\to\pm\pi} \tau(\gamma, 0) = i\beta,$$

where Ω_s, θ_s are functions of γ, which are known or can be deduced.

The solution to this mixed and periodic boundary value problem can be written down directly from an equation due to Woods [9] for the flow past a porous aerofoil. The equation in question is

(15) $$\tau = \frac{1}{2\pi} \exp\left\{\frac{1}{2}\int_{-\pi}^{\pi} \varepsilon \cot \tfrac{1}{2}(\gamma - \zeta) d\gamma\right\} \Big[A + \int_{-\pi}^{\pi} (\theta_s \cos \pi\varepsilon - \Omega_s \sin \pi\varepsilon) \exp\left\{-\tfrac{1}{2}\,\mathscr{Re}\int_{-\pi}^{\pi} \varepsilon \cot \tfrac{1}{2}(\gamma^* - \gamma) d\gamma^*\right\} \cot \tfrac{1}{2}(\gamma - \zeta) d\gamma\Big],$$

where A is a real constant, and ε is a porosity factor of which it is sufficient for our present purposes to know that it enters into the aerofoil boundary condition as follows:

(16) $$\theta \cos \pi\varepsilon - \Omega \sin \pi\varepsilon = \theta_s \cos \pi\varepsilon - \Omega_s \sin \pi\varepsilon,$$

and that it is positive on the lower surface of the aerofoil and negative on the upper.

Comparison of (12) and (16) shows that in our application of (15), $\varepsilon = \frac{1}{2}$ in $-\pi < \gamma < \gamma_0$, $\varepsilon = 0$ in $\gamma_0 < \gamma < \gamma_1$, $\varepsilon = -\frac{1}{2}$ in $\gamma_1 < \gamma < \pi$, and consequently (15) yields

(17) $$\tau(\zeta) = \frac{\{\sin \frac{1}{2}(\zeta - \gamma_0) \sin \frac{1}{2}(\gamma_1 - \zeta)\}^{\frac{1}{2}}}{2\pi \cos \frac{1}{2}\zeta} \times \Big[A + \int_{\gamma_0}^{\gamma_1} \theta_s(\gamma) \frac{\cos \frac{1}{2}\gamma \cot \frac{1}{2}(\gamma - \zeta) d\gamma}{\{\sin \frac{1}{2}(\gamma - \gamma_0) \sin \frac{1}{2}(\gamma_1 - \gamma)\}^{\frac{1}{2}}} + \left(\int_{\gamma_1}^{\pi} - \int_{-\pi}^{\gamma_0}\right) \Omega_s(\gamma) \frac{\cos \frac{1}{2}\gamma \cot \frac{1}{2}(\gamma - \zeta) d\gamma}{\{\sin \frac{1}{2}(\gamma - \gamma_0) \sin \frac{1}{2}(\gamma - \gamma_1)\}^{\frac{1}{2}}}\Big].$$

If we now adopt the Helmholtz hypothesis for the wake pressure then $q = U$ on the free streamlines, and so from (9), $\Omega_s = 0$, which eliminates the last term of (17).

The limit in (14) and the denominator term $\cos \frac{1}{2}\zeta$ in (17) require that the term in the square brackets in (17) vanishes at $\zeta = \pm\pi$. This enables

https://doi.org/10.1017/S1446788700025568 Published online by Cambridge University Press

us to calculate the constant A, and reduce (17) to

$$(18)\qquad \begin{aligned}\tau(\zeta) = \frac{1}{2\pi}\{\sin\tfrac{1}{2}(\zeta-\gamma_0)\sin\tfrac{1}{2}(\gamma_1-\zeta)\}^{\frac{1}{2}}\\ \int_{\gamma_0}^{\gamma_1}\theta_s(\gamma)\frac{\operatorname{cosec}\tfrac{1}{2}(\gamma-\zeta)d\gamma}{\{\sin\tfrac{1}{2}(\gamma-\gamma_0)\sin\tfrac{1}{2}(\gamma_1-\gamma)\}^{\frac{1}{2}}}.\end{aligned}$$

for the case of Helmholtz flow.

Equation (18) is the required solution of our boundary value problem. If $\theta_s(\gamma)$ is given or can be deduced, (18) gives $\tau(\zeta)$, then it follows from (6) and (9) that $z(\zeta)$ is given by

$$(19)\qquad z(\zeta) = \frac{h}{2\pi U}\int^{\zeta} e^{\tau(\zeta)}\frac{\sin(\alpha+\tfrac{1}{2}\zeta)}{\sin\tfrac{1}{2}\zeta}d\zeta.$$

The flow pattern is completely determined by (18) and (19).

More generally $\theta_s(\gamma)$ is initially unknown, and (18) and (19) are in effect a complicated integro-differential equation for $\theta_s(\gamma)$. We start with the linear approximation, which consists in putting $\tau(\zeta) = 0$ in (19), and deducing $\theta_s^{(1)}(\gamma)$ from the resulting $z^{(1)}(\gamma)$ relation on $\eta = 0$. In (18) this value of $\theta_s^{(1)}(\gamma)$ yields the "linear perturbation" solution, $\tau^{(1)}(\zeta)$. The process is now repeated with $\tau^{(1)}(\zeta)$ in (19); this yields $\tau^{(2)}(\gamma)$, and so on. The calculation continues in this way until $\tau^{(n)} - \tau^{(n-1)}$ is negligible. The details of this iterative method are exactly parallel to those given in [7] for separating flow past an isolated body, and we shall not consider it further. The linear perturbation solution is often sufficient in practical problems.

5. The Outlet Conditions

From equations (12), (13) and (18) we deduce that

$$(20)\qquad \ln\left(\frac{U}{V}\right) = \frac{1}{2\pi}\int_{\gamma}^{\gamma_1}\theta_s(\gamma)\frac{\sin\tfrac{1}{2}(\gamma-\lambda_1)d\gamma}{\{\sin\tfrac{1}{2}(\gamma-\gamma_0)\sin\tfrac{1}{2}(\gamma_1-\gamma)\}^{\frac{1}{2}}},$$

$$(21)\qquad \alpha = \frac{1}{2\pi}\int_{\gamma}^{\gamma_1}\theta_s(\gamma)\frac{\cos\tfrac{1}{2}(\gamma-\lambda_1)d\gamma}{\{\sin\tfrac{1}{2}(\gamma-\gamma_0)\sin\tfrac{1}{2}(\gamma_1-\gamma)\}^{\frac{1}{2}}}$$

and

$$(22)\qquad \beta = \frac{1}{2\pi}\{\cos\tfrac{1}{2}\gamma_0\cos\tfrac{1}{2}\gamma_1\}^{\frac{1}{2}}\int_{\gamma}^{\gamma_1}\theta_s(\gamma)\frac{\sec\tfrac{1}{2}\gamma\, d\gamma}{\{\sin\tfrac{1}{2}(\gamma-\gamma_0)\sin\tfrac{1}{2}(\gamma_1-\gamma)\}^{\frac{1}{2}}}$$

where

$$(23)\qquad \lambda_1 \equiv \tfrac{1}{2}(\gamma_1+\gamma_0).$$

As the inlet conditions (V, α) and the separation points (γ_0, γ_1) are supposed given, equation (21) is a restriction on the distribution $\theta_s(\gamma)$, which we must be careful to satisfy throughout the iterative process described above. It is in effect a "closure" condition for the blade, which, while automatically satisfied with a given blade profile, may not necessarily be satisfied with an approximating profile, such as will occur in an iterative solution of equations (18) and (19).

Equations (20) and (22) determine the outlet conditions immediately $\theta_s(\gamma)$ has been calculated. The forces acting on the blading can next be deduced from equations (2) to (4).

6. An Example: A Cascade of Flat Plates

The simplest example of the theory of any interest occurs with a cascade of flat plates. Suppose the plates are at an angle of attack of $\alpha + \alpha_0$, so

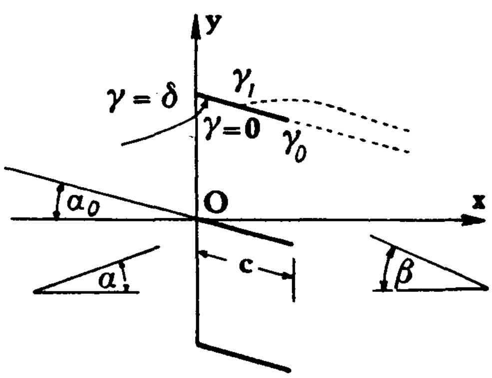


Figure 4

that $-\alpha_0$ is the angle between the plates and Ox (see figure 4), then

$$\theta_s = -\alpha_0 + \pi\{(U(0) - U(\delta)\},$$

where $U(\gamma)$ is the unit function and δ is the value of γ at the sharp leading edge of the typical plate. Substituting this value of θ_s in (18) we find that

$$\tau(\zeta) = -i\alpha_0 - 2\,\{\coth^{-1} F(\delta,\ \zeta) - \coth^{-1} F(0,\ \zeta)\}, \tag{24}$$

where

$$F(\gamma,\ \zeta) = \left\{\frac{\sin\frac{1}{2}(\gamma-\gamma_0)\sin\frac{1}{2}(\gamma_1-\zeta)}{\sin\frac{1}{2}(\gamma_1-\gamma)\sin\frac{1}{2}(\zeta-\gamma_0)}\right\}^{\frac{1}{2}}. \tag{25}$$

The limits (13) and (14) give

$$\begin{aligned}\ln\frac{U}{V} + i\alpha = -i\alpha_0 &- 2i\cot^{-1}\left\{e^{i\lambda_0}\frac{\sin\frac{1}{2}(\delta-\gamma_0)}{\sin\frac{1}{2}(\gamma_1-\delta)}\right\}^{\frac{1}{2}}\\ &+ 2i\cot^{-1}\left\{e^{i\lambda_0}\frac{\sin(-\frac{1}{2}\gamma_0)}{\sin\frac{1}{2}\gamma_1}\right\}^{\frac{1}{2}},\end{aligned} \tag{26}$$

and

$$\beta = -\alpha_0 - 2\cot^{-1}\left\{\frac{\sin\frac{1}{2}(\delta-\gamma_0)\cos\frac{1}{2}\gamma_1}{\sin\frac{1}{2}(\gamma_1-\delta)\cos\frac{1}{2}\gamma_0}\right\}^{\frac{1}{2}} + 2\cot^{-1}\left\{\frac{\tan(-\frac{1}{2}\gamma_0)}{\tan\frac{1}{2}\gamma_1}\right\}^{\frac{1}{2}}. \tag{27}$$

When (V, β) and α_0 are given these equations enable (U, β) and δ to be calculated.

With flat plates separation is very likely to occur at the sharp leading edge, $\gamma = \delta$. In this case $\gamma_1 = \delta$, and (24), (26) and (27) become

$$\tau(\zeta) = i\alpha_0 + 2\coth^{-1}\left\{\frac{\sin(-\frac{1}{2}\gamma_0)}{\sin\frac{1}{2}\gamma_1}\,\frac{\sin\frac{1}{2}(\gamma_1-\zeta)}{\sin\frac{1}{2}(\zeta-\gamma_0)}\right\}^{\frac{1}{2}} \tag{28}$$

$$\ln\frac{U}{V} = -\coth^{-1}\left[\frac{\cos\frac{1}{2}(\gamma_1+\gamma_0)}{\{\sin(-\frac{1}{2}\gamma_0)\sin\frac{1}{2}\gamma_1\}^{\frac{1}{2}}}\right], \tag{29}$$

$$\alpha = -\alpha_0 - \cot^{-1}\left[\frac{\sin\frac{1}{2}(\gamma_1+\gamma_0)}{\{\sin(-\frac{1}{2}\gamma_0)\sin\frac{1}{2}\gamma_1\}^{\frac{1}{2}}}\right], \tag{30}$$

and

$$\beta = -\alpha_0 + 2\cot^{-1}\left\{\frac{\tan(-\frac{1}{2}\gamma_0)}{\tan\frac{1}{2}\gamma_1}\right\}^{\frac{1}{2}}. \tag{31}$$

Several flow problems of practical interest can be calculated from these equations. The three examples shown in figure 5 are the special cases $\alpha = -\frac{1}{2}\pi + \varepsilon$, $\alpha_0 = \frac{1}{2}\pi$, ε small and positive (figure 5a) $\alpha = 0$, $\alpha_0 = \frac{1}{2}\pi$, $\gamma_1 = -\gamma_0$ (figure 5b) and $\alpha_0 = 0$, $\gamma_0 = -\pi$ (figure 5c). Thus our theory is applicable to (a) the flow past a slotted wall, (b) the flow through a family of apertures and (c) the flow through a bank of tubes.

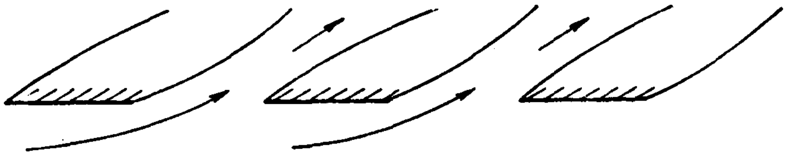


Figure 5a

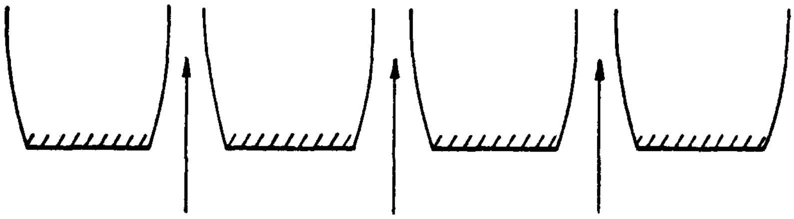


Figure 5b

https://doi.org/10.1017/S1446788700025568 Published online by Cambridge University Press

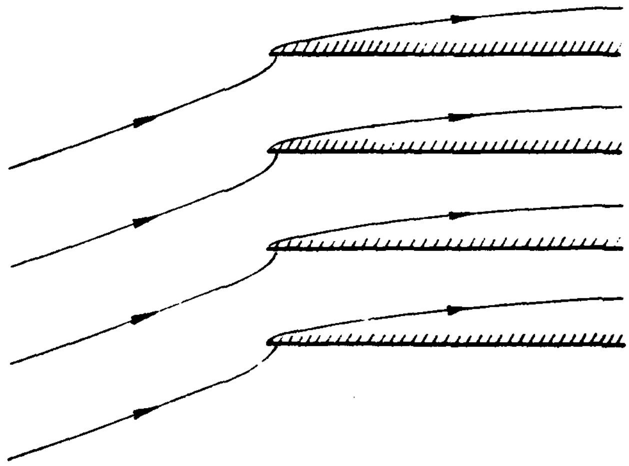


Figure 5c

References

[1] Birkhoff, G. and Zarantonello, E. H., Jets, Wakes and Cavities. Academic Press, New York. (1957).
[2] Chang, C. C. and Chu, V. C., J. Appl. Mech. (1956).
[3] Kemp, H. N. and Sears, W. R., J. Aero. Sci. (9), 20, (1953). 585.
[4] Pearson, H., Proc. 4th Anglo-American Aero. Conference (R. Ae. S.), (1953). 127–162.
[5] Robinson, A. and Laurmann, J. A., Wing Theory, Camb. Univ. Press. (1956).
[6] Rosenblat, S. and Woods, L. C., Aust. Aero. Res. Comm. Report ACA-58. (1956).
[7] Woods, L. C., Proc. Roy. Soc. A, 227, (1955a). 367–386.
[8] Woods, L. C., Proc. Roy. Soc. A, 228, (1955b). 50–65.
[9] Woods, L. C., Proc. Roy. Soc. A, 238, (1956). 358–388.
[10] Woods, L. C., Proc. Roy. Soc. A, 239, (1957). 328–337.

The University of New South Wales, Sydney.

https://doi.org/10.1017/S1446788700025568 Published online by Cambridge University Press