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A NOTE ON THE BEST CONSTANT FOR UNCENTERED MAXIMAL FUNCTIONS WITH MEASURE ON $\mathbb{R}$

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ABSTRACT. Let M_μ be the uncentered Hardy–Littlewood maximal operator with a Borel measure μ on $\mathbb{R}$. In this note, we verity that the norm of M_μ on $L^p(\mathbb{R},\mu)$ with $p \in (1,\infty)$ is just the upper bound θ_p obtained by Grafakos and Kinnunen and reobtain the norm of M_μ from $L^1(\mathbb{R},\mu)$ to $L^{1,\infty}(\mathbb{R},\mu)$. Moreover, the norm of the "strong" maximal operator $N^n_{\vec{\mu}}$ on $L^p(\mathbb{R}^n,\vec{\mu})$ is also given.

1. Introduction

Let μ be a Borel measure on $\mathbb{R}^n$; that is, every Borel set is μ-measurable and $\mu(K) < \infty$ for any compact set $K \subset \mathbb{R}^n$. For a μ-locally integrable function $f : \mathbb{R}^n \to [0,\infty]$, the uncentered maximal function of f with respect to μ is defined by

$$M_\mu f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B f d\mu, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where the supremum is taken over all open balls B containing x. The integral average in (1.1) is equal to $f(x)$ if $\mu(B) = 0$.

As we know, when $n = 1$, the operator M_μ always maps $L^1(\mathbb{R},\mu)$ into the Lorenz space $L^{1,\infty}(\mathbb{R},\mu)$ and maps $L^p(\mathbb{R},\mu)$ into itself for $1 < p \leq \infty$; see [1, 10]. It is therefore natural to investigate the best constant C_1^μ such that for any $f \in L^1(\mathbb{R},\mu)$ and $\lambda > 0$,

$$\mu(\{x \in \mathbb{R} : M_\mu f(x) > \lambda\}) \leq \frac{C_1^\mu}{\lambda} \|f\|_{L^1(\mathbb{R},\mu)}, \tag{1.2}$$

as well as the best constant C_p^μ such that for any $f \in L^p(\mathbb{R},\mu)$,

$$\|M_\mu f\|_{L^p(\mathbb{R},\mu)} \leq C_p^\mu \|f\|_{L^p(\mathbb{R},\mu)}. \tag{1.3}$$

In their remarkable work [7], Grafakos and Montgomery-Smith showed that when μ is the n-dimensional Lebesgue measure, the norm of the "strong" uncentered maximal operator on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ tends to ∞ as $n \to \infty$. Moreover, when $n = 1$, they also showed that the best constant C_p^μ for $p \in (1,\infty)$ is the unique positive solution to the equation

$$(p-1)x^p - px^{p-1} - 1 = 0. \tag{1.4}$$

When μ is a non-zero Borel measure satisfying the following assumption:

Assumption (A): for any point $a \in \mathbb{R}$, $\mu(\{a\}) = 0$,

Bernal [1] proved that the best constant C_1^μ in (1.2) is 2. On the other hand, Grafakos and Kinnunen in [6] showed that for a general Borel measure μ on $\mathbb{R}$, the best constants in (1.2) and (1.3) respectively have upper bounds 2 and θ_p with $1 < p < \infty$, where θ_p is the unique positive solution of the equation (1.4); see Lemma 2.1 below. Recently, there are still scholars actively engaged in this field. Jia Wu et al. [11] obtained the best constant of truncated Hardy-Littlewood maximal function on $L^1(\mathbb{R})$. Moyan Qin et al.

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[9] established the limiting weak type behaviors of the strong maximal operators. There are numerous papers on the best constants of maximal functions; see [4, 8, 2].

The purpose of this note is twofold. For $p = 1$, by an alternative method, we reobtain the result of Bernal in [1]. That is, we have the conclusion as follows.

Theorem 1.1. *Let μ be a non-zero Borel measure on $\mathbb{R}$ that satisfies Assumption (A). Then $C_1^\mu = 2$.*

The second purpose of this note is to study the best constant C_p^μ of M_μ with a Borel measure μ for $p \in (1, \infty)$. Under some additional assumptions, we verity that the upper bound θ_p obtained in [6] is just the norm of M_μ. More precisely, we have the following result.

Theorem 1.2. *Let μ be a Borel measure on $\mathbb{R}$ that satisfies Assumption (A) and*

Assumption (B): *there exists a non-negative locally integrable function w on $\mathbb{R}$, such that for any subset $A \subset \mathbb{R}$, $\mu(A) = \int_A w(t)dt$, and*

$$\int_{-\infty}^{0} w(t)dt = \int_{0}^{\infty} w(t)dt = \infty.$$

Then $C_p^\mu = \theta_p$, where θ_p is the unique positive solution of the equation (1.4) with $1 < p < \infty$.

Remark 1.3. (i) It is not clear whether Assumption (B) is necessary.

(ii) Let $w \in A_p$, the class of Muckenhoupt weights; see, for example, [5]. It is not difficult to see the Borel measure $\mu = wdx$ satisfies Assumption (A) and (B). This implies that for a given $w \in A_p$, the norm of M_μ on $L^p(\mathbb{R}, \mu)$ or from $L^1(\mathbb{R}, \mu)$ to $L^{1,\infty}(\mathbb{R}, \mu)$ is independent of w.

We mention that when $p = 1$, by the result in [6], the best constant $C_1^\mu \leq 2$ (see Lemma 2.1 below). Thus to show Theorem 1.1, it suffices to prove the reverse inequality. A novelty of this note is that we take a new method to achieve this aim by finding some suitable "test functions". This method is also used to prove Theorem 1.2.

At the end of this paper, we also briefly discuss the "strong" maximal operator $N_{\vec{\mu}}^n$ with the measure $\vec{\mu}$ on $\mathbb{R}^n$. By an argument similar to Theorem 1.2, we show that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is θ_p^n, $1 < p < \infty$. Moreover, we point out that θ_p^n grows exponentially with n, as $n \to \infty$.

2. Proofs of Theorems 1.1 and 1.2

In this section, we firstly provide the proofs of Theorems 1.1 and 1.2, and then we consider the n-dimensional maximal operator $N_{\vec{\mu}}^n$ in the end. Let $M_\mu f$, C_1^μ and C_p^μ be as in (1.1), (1.2) and (1.3), respectively, and θ_p be the unique positive solution of (1.4). For convenience, we write $Mf := M_\mu f$, $C_1 := C_1^\mu$ and $C_p := C_p^\mu$. We begin with recalling the following result by Grafakos and Kinnunen in [6].

Lemma 2.1. *Let μ be a Borel measure on $\mathbb{R}$ and $p \in (1, \infty)$. Then $C_1 \leq 2$ and $C_p \leq \theta_p$.*

Theorem 1.1 is a consequence of Lemma 2.1 and the following proposition.

Proposition 2.2. *Let μ be a non-zero Borel measure on $\mathbb{R}$ satisfying Assumption (A). Then there exists a collection $\{(a_i, b_i)\}_{i=1}^\infty$ of open intervals such that*

$$a_i < a_{i+1} < b_{i+1} < b_i, \quad \int_{a_i}^{b_i} d\mu > 0, \quad \lim_{i\to\infty} \int_{a_i}^{b_i} d\mu = 0, \tag{2.1}$$

and

$$\lim_{i\to\infty} \frac{\sup_{t>0} t\mu(\{x\in\mathbb{R} : M\chi_{(a_i,b_i)}(x) > t\})}{\mu((a_i,b_i))} = 2.$$

Proof: We claim that there exists $x_0 \in \mathbb{R}$, such that $\mu(I) > 0$ for any open interval $I \ni x_0$. Otherwise, for any $x \in \mathbb{R}$, there exists an open interval $I_x \ni x$ such that $\mu(I_x) = 0$. By the Lindelöf covering theorem, we deduce that there exists a countable collection $\{I_j\}_{j=1}^{\infty}$ of open intervals, such that $\mu(I_j) = 0$ for any $j \in \mathbb{N}$, and

$$\mathbb{R} = \bigcup_{j\in\mathbb{N}} I_j.$$

Therefore,

$$0 < \mu(\mathbb{R}) = \mu\Big(\bigcup_{j\in\mathbb{N}} I_j\Big) \le \sum_{j=1}^{\infty} \mu(I_j) = 0,$$

which yields a contradiction. From this claim, we see that there exists a collection $\{(a_i,b_i)\}_{i=1}^{\infty}$ of open intervals satisfying (2.1).

Let $f_i(x) := \chi_{(a_i,b_i)}(x)$, where $i \in \mathbb{N}$ and $x \in \mathbb{R}$. We then see that for $x < a_i$,

$$Mf_i(x) = \sup_{a_i\le y\le b_i} \frac{\int_{a_i}^{y} d\mu}{\int_{x}^{y} d\mu} = \frac{\int_{a_i}^{b_i} d\mu}{\int_{x}^{b_i} d\mu}.$$

We can treat similarly when $x > b_i$. Therefore we conclude that for any i,

$$Mf_i(x) = \begin{cases} \frac{\int_{a_i}^{b_i} d\mu}{\int_{x}^{b_i} d\mu}, & x < a_i; \\ 1, & a_i \le x \le b_i; \\ \frac{\int_{a_i}^{b_i} d\mu}{\int_{a_i}^{x} d\mu}, & x > b_i. \end{cases}$$

By (2.1), we see that $\mu((a_i,b_i)) \in (0,\infty)$. Without lost of generality, we may further assume that

$$\gamma_i := \max\{\frac{\int_{a_i}^{b_i} d\mu}{\int_{-\infty}^{b_i} d\mu}, \frac{\int_{a_i}^{b_i} d\mu}{\int_{a_i}^{\infty} d\mu}\} < 1.$$

Moreover, since Mf_i is continuous, increasing on $(-\infty, a_i)$, and decreasing on (b_i,∞), we have that

$$E_i := \{x\in\mathbb{R} : Mf_i(x) = \lambda\} \ne \varnothing \text{ for } \gamma_i < \lambda < 1.$$

Let

$$\alpha_i := \inf\{y : y \in E_i\},$$
$$\beta_i := \sup\{y : y \in E_i\}.$$

Then by the continuity of Mf_i, we see that

$$\alpha_i < a_i, \quad \beta_i > b_i,$$

$$\alpha_i, \beta_i \in E_i,$$

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$$\int_{a_i}^{b_i} d\mu = \lambda \int_{\alpha_i}^{b_i} d\mu, \quad \int_{a_i}^{b_i} d\mu = \lambda \int_{a_i}^{\beta_i} d\mu,$$

and

$$\{x \in \mathbb{R} : Mf_i(x) > \lambda\} = (\alpha_i, \beta_i).$$

These facts imply that

$$\begin{aligned}
\lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) &= \lambda \int_{\alpha_i}^{\beta_i} d\mu \\
&= \lambda \int_{\alpha_i}^{b_i} d\mu + \lambda \int_{a_i}^{\beta_i} d\mu - \lambda \int_{a_i}^{b_i} d\mu \\
&= (2 - \lambda) \int_{a_i}^{b_i} d\mu.
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) &\geq \sup_{\gamma_i<\lambda<1} \lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) \\
&= \sup_{\gamma_i<\lambda<1} (2 - \lambda) \int_{a_i}^{b_i} d\mu \\
&= (2 - \gamma_i)\mu((a_i, b_i)).
\end{aligned}$$

By letting $i \to \infty$, we deduce that

$$\underline{\lim}_{i\to\infty} \frac{\sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R} : M\chi_{(a_i,b_i)}(x) > \lambda\})}{\mu((a_i, b_i))} \geq \underline{\lim}_{i\to\infty} (2 - \gamma_i) = 2. \tag{2.2}$$

On the other hand, from Lemma 2.1, we also have

$$\overline{\lim}_{i\to\infty} \frac{\sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R} : M\chi_{(a_i,b_i)}(x) > \lambda\})}{\mu((a_i, b_i))} \leq 2.$$

This together with (2.2) yields that Proposition 2.2 holds and the proof is finished. □

To present the proof of Theorem 1.2, we also need the following proposition.

Proposition 2.3. *Let* $T(x) := \frac{x^{1-\frac{1}{p}}+1}{x+1}$, $x > 0$ *and* $1 < p < \infty$*. Then the maximum value of the function* T *is* $\frac{p-1}{p}\theta_p$*.*

Proof: We firstly see that

$$T'(x) = \frac{-\frac{1}{p}x^{1-\frac{1}{p}} + (1 - \frac{1}{p})x^{-\frac{1}{p}} - 1}{(x + 1)^2}, \quad x > 0.$$

It is not difficult to prove that there exists a unique point $x_0 > 0$ such that $T'(x_0) = 0$ and $\max_{x>0} T(x) = T(x_0)$. Therefore

$$\max_{x>0} T(x) = \frac{x_0^{1-\frac{1}{p}} + 1}{x_0 + 1} = \frac{x_0^{1-\frac{1}{p}} + \frac{\frac{1}{p}x_0^{1-\frac{1}{p}}}{(1-\frac{1}{p})x_0^{-\frac{1}{p}}-1}}{x_0 + \frac{\frac{1}{p}x_0^{1-\frac{1}{p}}}{(1-\frac{1}{p})x_0^{-\frac{1}{p}}-1}} = \frac{p-1}{p} x_0^{-\frac{1}{p}}.$$

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Since $x_0^{-\frac{1}{p}} > 0$ satisfies (1.4), we set $\theta_p := x_0^{-\frac{1}{p}}$, which implies

$$\max_{x>0} T(x) = \frac{p-1}{p}\theta_p.$$

□

Next, we give the proof of Theorem 1.2.

Proof of Theorem 1.2: Let $1 < p < \infty$ and

$$f(x) := \int_0^x w(t)dt, \quad F(x) := |f(x)|^{-\frac{1}{p}}, \quad x \in \mathbb{R}.$$

From assumptions on μ, we see that f is well-defined, continuous and increasing on $\mathbb{R}$ and absolutely continuous in any closed interval on $\mathbb{R}$. Also let

$$\mathscr{A} := \{H : H \text{ is a maximal closed interval such that } \textstyle\int_H w(t)dt = 0\},$$

$E := \cup_{H\in\mathscr{A}} H$, and $E^c := \mathbb{R} \setminus E$. Obviously, $\mathscr{A}$ is a countable set, and $\mu(E) = 0$.

By Proposition 2.3 and Assumption (B), we see that for any $x \in E^c \cap (0,\infty)$, $f(x) > 0$ and

$$\begin{aligned} MF(x) &= \sup_{a<x<b} \frac{\int_a^b F(t)d\mu(t)}{\int_a^b w(t)dt} \\ &\geq \sup_{a<0} \frac{\int_a^x F(t)d\mu(t)}{\int_a^x w(t)dt} \\ &= \sup_{a<0} \frac{p}{p-1} \frac{|f(x)|^{1-\frac{1}{p}} + |f(a)|^{1-\frac{1}{p}}}{|f(x)| + |f(a)|} \\ &= \frac{p}{p-1} F(x) \sup_{a<0} \frac{|\frac{f(a)}{f(x)}|^{1-\frac{1}{p}} + 1}{|\frac{f(a)}{f(x)}| + 1} \\ &= \frac{p}{p-1} F(x) \frac{p-1}{p}\theta_p \\ &= \theta_p F(x). \end{aligned}$$

The above inequality also holds for $x \in E^c \cap (-\infty, 0)$. Hence

$$MF(x) \geq \theta_p F(x), \quad \mu - a.e. \quad x \in \mathbb{R}.$$

Observe that $F \notin L^p(\mathbb{R},\mu)$. So instead, we consider the following function

$$F_\varepsilon(x) := F(x) \min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\} \in L^p(\mathbb{R},\mu), \quad \varepsilon > 0. \tag{2.3}$$

We claim that

$$MF_\varepsilon(x) \geq \frac{1}{\varepsilon + \frac{1}{p'}} F_\varepsilon(x) \frac{1+\theta_p^{1-p-\varepsilon p}}{1+\theta_p^{-p}}, \quad \mu - a.e. \quad x \in \mathbb{R}. \tag{2.4}$$

In fact, assume that $x \in E^c \cap (0,\infty)$, and set

$$\varphi(u) := \sup\{t : f(t) \leq u f(x)\}, \quad u \in \mathbb{R}. \tag{2.5}$$

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We denote by m the Lebesgue measure. From the property of f, $\varphi(u)$ is well-defined, strictly increasing, right continuous, and $m-a.e.$ differentiable on $\mathbb{R}$. We can further verify the following fact,

$$\varphi(1) \geq x,$$

$$f(\varphi(u)) = uf(x), \quad u \in \mathbb{R},$$

$$w(t) = 0, \quad m-a.e. \quad t \in [x, \varphi(1)],$$

and

$$w(\varphi(u))\varphi'(u) = f(x), \quad m-a.e. \quad u \in \mathbb{R}.$$

From these facts, we deduce that

$$\begin{aligned}
MF_\varepsilon(x) &\geq \sup_{y\leq 0} \frac{\int_y^x F_\varepsilon(t)d\mu(t)}{\int_y^x w(t)dt} \\
&\geq \sup_{y\leq 0} \frac{\int_y^x |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\}w(t)dt}{|f(x)|+|f(y)|} \\
&= \sup_{y\leq 0} \frac{\int_y^{\varphi(1)} |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\}w(t)dt}{|f(x)|+|f(y)|} \\
&\geq \frac{\int_{\varphi(-\theta_p^{-p}f(x))}^{\varphi(1)} |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\}w(t)dt}{|f(x)|+|\theta_p^{-p}f(x)|} \\
&= \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}}|f(x)|^{-\frac{1}{p}} \min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\}f(x)du}{|f(x)|+|\theta_p^{-p}f(x)|} \\
&= |f(x)|^{-\frac{1}{p}} \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}} \min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\}du}{1+\theta_p^{-p}} \\
&= |f(x)|^{-\frac{1}{p}} \min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\} \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}} \frac{\min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\}}{\min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\}}du}{1+\theta_p^{-p}} \\
&\geq F_\varepsilon(x) \frac{\int_{-\theta_p^{-p}}^1 |u|^{\varepsilon-\frac{1}{p}}du}{1+\theta_p^{-p}} \\
&= \frac{1}{\varepsilon+\frac{1}{p'}} F_\varepsilon(x) \frac{1+\theta_p^{1-p-\varepsilon p}}{1+\theta_p^{-p}},
\end{aligned}$$

where in the fifth equation, we use integration by substitution with $t = \varphi(u)$; see [3]. When $x \in E^c \cap (-\infty, 0)$, we can treat similarly. Therefore the claim holds.

Then from (2.4), it follows that

$$\lim_{\varepsilon\to 0} \frac{\|MF_\varepsilon\|_{L^p(\mathbb{R},\mu)}}{\|F_\varepsilon\|_{L^p(\mathbb{R},\mu)}} \geq p' \frac{1+\theta_p^{1-p}}{1+\theta_p^{-p}} = \theta_p,$$

which means $C_p \geq \theta_p$. Hence we complete the proof of Theorem 1.2. □

Finally, we briefly discuss the n-dimensional case. Following the method of Grafakos and Montgomery-Smith [7], we present a complete adaptation of their argument here for completeness. Denote by $\mathbf{x} := (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. For a non-negative locally integrable

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function f on $\mathbb{R}^n$ and $w_j(t)$ on $\mathbb{R}$, $j = 1, \dots, n$, satisfying the conditions in Theorem 1.2, we define the "strong" maximal function on $\mathbb{R}^n$, by

$$N_{\vec{\mu}}^n(f)(\mathbf{x}) := \sup_{a_1 < x_1 < b_1} \cdots \sup_{a_n < x_n < b_n} \frac{\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(\mathbf{y}) d\vec{\mu}(\mathbf{y})}{\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} d\vec{\mu}(\mathbf{y})},$$

where $d\vec{\mu}(\mathbf{y}) := w_1(y_1) \cdots w_n(y_n) dy_1 \cdots dy_n$. Clearly $N_{\vec{\mu}}^1 = M_\mu$. Moreover, observe that

$$N_{\vec{\mu}}^n \leq M_\mu^{(1)} \circ \cdots \circ M_\mu^{(n)},$$

where $M_\mu^{(j)}$ denotes the maximal operator M_μ on $\mathbb{R}$ applied to the x_j coordinate. This implies that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is less than or equal to θ_p^n, $1 < p < \infty$. By considering the function

$$G_\varepsilon(\mathbf{x}) := \prod_{j=1}^n F_\varepsilon(x_j), \quad \mathbf{x} \in \mathbb{R}^n$$

where F_ε is as in (2.3), and using an argument similar to Theorem 1.2, we can obtain the reverse inequality. We state this result as follows.

Corollary 2.4. *For $1 < p < \infty$, the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is θ_p^n.*

Furthermore, it is easy to verify $\frac{p}{p-1} < \theta_p < \frac{2p}{p-1}$, which implies that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ grows exponentially with n, as $n \to \infty$.

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