HOMOLOGICAL INVARIANTS OF LOCAL RINGS

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Introduction

In this paper R is a commutative noetherian local ring with unit element 1 and M is its maximal ideal. Let K be the residue field R/M and let $\{t_1, t_2, \ldots, t_n\}$ be a minimal system of generators for M. By a complex $R<T_1, \ldots, T_\rho>$ we mean an R-algebra* obtained by the adjunction of the variables $T_1, \ldots, T_\rho$ of degree 1 which kill $t_1, \ldots, t_\rho$. The main purpose of this paper is, among other things, to construct an R-algebra resolution of the field K, so that we can investigate the relationship between the homology algebra $H(R<T_1, \ldots, T_n>)$ and the homological invariants of R such as the algebra $\mathrm{Tor}^R(K, K)$ and the Betti numbers $B_\rho = \dim_K \mathrm{Tor}^R_\rho(K, K)$ of the local ring R. The relationship was initially studied by Serre [5]. Then Tate [6] gave the correct lower bound for the Betti numbers of a nonregular local ring. In his M. I. T. lecture (See a footnote of [6]) Eilenberg proves that

$$B_2 = \binom{n}{2} + \binom{n}{0} b_1 \text{ and } B_3 \geq \binom{n}{3} + \binom{n}{1} b_1,$$

where $b_1 = \dim_K H_1(R<T_1, \ldots, T_n>)$. In this paper these results of Eilenberg are generalized as follows:

$$B_3 = \binom{n}{3} + \binom{n}{1} b_1 + \varepsilon_2,$$

$$B_4 = \binom{n}{4} + \binom{n}{2} b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2} + \varepsilon_2 \binom{n}{1} + \varepsilon_3 \binom{n}{0},$$

and so forth, where $\varepsilon_2 = \dim_K H_2(\Lambda)/H_1(\Lambda)^2$, $\varepsilon_3 = \dim_K H_3(\Lambda)/H_1(\Lambda) \cdot H_2(\Lambda)$, and $\Lambda = R<T_1, \ldots, T_n>$. As corollaries of the above computation we obtain part of the results by Tate [6],

$$B_\rho \geq \binom{n}{\rho} + \binom{n}{\rho-2} + \binom{n}{\rho-4} + \cdots, \text{ for } \rho \leq 4,$$

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* For definition, see a paper of Tate [6]. Throughout the paper the numbers in square brackets refer to the papers of the bibliography at the end of the paper.

if R is not regular.

If R is a complete intersection, we have

$$B_3 = \binom{n}{3} + \binom{n}{1} b_1,$$

$$B_4 = \binom{n}{4} + \binom{n}{2} b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2}.$$

§ 1. The complex $R < T_1, \cdots, T_p >$

Let us consider a filtered complex $\Lambda = R < T_1, \ldots, T_n >$ with an increasing sequence of subcomplexes $R \subset R < T_1 > \subset R < T_1, T_2 > \subset \cdots \subset R < T_1, \ldots, T_p > \subset \cdots \subset \Lambda$.

Then the graded differential algebra Λ over R (in the sequel we shall call it simply "R-algebra" in the sense of Tate) has the increasing filtration $\{R < T_1, \ldots, T_p >\}$ such that $R < T_1, \ldots, T_p >$ is an R-subalgebra. Defining R-modules

$$D_{p,q} = H_{p+q}(R < T_1, \ldots, T_p >)$$
$$E_{p,q} = H_{p+q}(R < T_1, \ldots, T_p > / R < T_1, \ldots, T_{p-1} >),$$

we have the usual exact sequence

$$\cdots \xrightarrow{k} D_{p-1,q+1} \xrightarrow{i} D_{p,q} \xrightarrow{j} E_{p,q} \xrightarrow{k} D_{p-1,q} \xrightarrow{i} \cdots$$

for each pair $(R < T_1, \ldots, T_p >, R < T_1, \ldots, T_{p-1} >)$.

Thus the exact couple $C(\Lambda) = < D, E; i, j, k >$ is associated with R-algebra Λ, where

$$D = \sum_{p,q} D_{p,q} \quad \text{and} \quad E = \sum_{p,q} E_{p,q}.$$

LEMMA 1.1.

$$E_{p,q} \simeq D_{p-1,q}$$

Proof. It is sufficient to show chain equivalences λ and μ

$$R < T_1, \ldots, T_p > / R < T_1, \ldots, T_{p-1} > \underset{\mu}{\overset{\lambda}{\rightleftarrows}} R < T_1, \ldots, T_{p-1} >$$

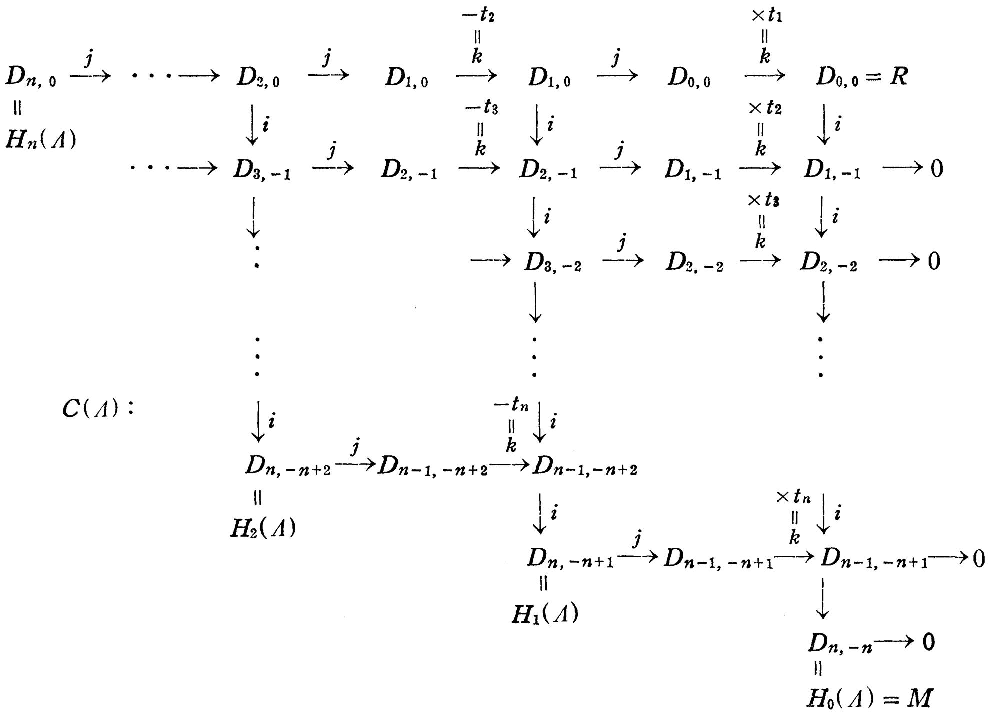
such that $\lambda\mu = 1$ and $\mu\lambda = 1$. Let x be a homogeneous element of degree $p+q$ in $R < T_1, \ldots, T_p >$. Then $x = x_1 + x_2 \cdot T_p$, where x_1 and x_2 are homogeneous elements of $R < T_1, \ldots, T_{p-1} >$ with degrees $p+q$ and $p+q-1$ respectively.

Obviously the residue class $\bar{x}$ is represented by $x_2 \cdot T_p$. Define $\lambda(\bar{x}) = x_2$. It is immediate to verify that λ is well defined and is a chain mapping. Defining μ by

$$\mu(y) = \overline{y \cdot T_p},$$

we see by straightforword computation that λ and μ are chain equivalences. This completes the proof.

By replacing the E-terms by the corresponding isomorphic D-terms, the exact couple $C(\Lambda)$ can be developed into a "lattice-like" diagram



The steps from upper left to lower right are exact sequences. It is easy to see that $k_{p,q}$; $D_{p,q} \longrightarrow D_{p,q}$ is the multiplication by $(-1)^{p+q} t_{p+1}$. This diagram provides us with the whole story about the following known results which have been proved by several authors [2], [6].

PROPOSITION 1.2. *The following statements are equivalent.*

i) $H_1(\Lambda) = 0$

ii) $H_\rho(R < T_1, \ldots, T_p >) = 0$ *for any* $\rho \geq 1$ *and for any* $p (n \geq p \geq 0)$.

iii) $\{t_1, t_2, \dots, t_n\}$ *is an R-sequence.*

iv) *R is regular.*

Proofs. i) → ii) Since $H_1(A)=0$, $k_{n-1,-n+2}$ which is the multiplication by $-t_n$, is onto. It follows that any element $x \in D_{n-1,-n+2}$ belongs to $\bigcap_{\rho=0}^{\infty} M^{\rho} \cdot D_{n-1,-n+2}$. By virtue of Krull (for example see [7]) x vanishes, because $D_{n-1,-n+2} = H_1(R<T_1, \dots, T_{n-1}>)$ is a noetherian module over R. By the repeated use of the same argument, we can prove that $H_1(R<T_1, \dots, T_p>) = D_{p,-p+1}$ vanishes for all $p(n \geq p \geq 1)$. Then $i_{p,-p+2}: D_{p,-p+2} \to D_{p+1,-p+1}$ are all onto, because of the exactenss of the diagram $C(A)$. Since $D_{2,0}$ vanishes*, all $H_2(R<T_1, \dots, T_p>)$ vanish. By repeating this process the proof of i) → ii) is established. ii) → iii) It is immediate by definition that $D_{p,-p} = H_0(R<T_1, \dots, T_p>) = R/(t_1, \dots, t_p)$. Since $k_{p,-p}$ is isomorphic, t_{p+1} is a non zero divisor for $R/(t_1, \dots, t_p)$. This completes the proof.

iii) → iv) It is immediate by definition.

iv) → i) Without loss of generality we may assume that $\{t_1, \dots, t_n\}$ is an R-sequence. Then all $k_{p,-p}$ are isomorphic so that all $i_{p,-p+1}$ are onto. Since $D_{1,0}=0$ in this case, we have $H_1(A)=0$.

§ 2. Construction of a minimal algebra resolution

Let us denote by b_ρ $\dim_K H_\rho(A)$ and let 1-cycles $\mathfrak{Z}_1^1, \dots, \mathfrak{Z}_{b_1}^1$ represent the homology classes $Z_1^1, \dots, Z_{b_1}^1 \in H_1(A)$ respectively. Then by adjoining $S_1, \dots, S_{b_1}$ of degree 2 which kill the cycles $\mathfrak{Z}_1^1, \dots, \mathfrak{Z}_{b_1}^1$ we obtain an R-algebra

$$A^{(2)} = A<S_1, \dots, S_{b_1}> ; \ \partial_2^{(2)} S_i = \mathfrak{Z}_i^1,$$

satisfying the following conditions:

a) $A^{(2)} \supset A = A^{(1)}$, and $A_\lambda^{(2)} = A_\lambda$ for $\lambda < 2$,

b) $H_1(A^{(2)}) = 0$.

Let

$$V_\rho = H_\rho(A)/(H_{\rho-1}(A) \cdot H_1(A) + H_{\rho-2}(A) \cdot H_2(A) + \cdots + H_{\rho-\lambda}(A) \cdot H_\lambda(A))$$

for $\rho \geq 2$, where $\lambda = \frac{\rho}{2}$ if ρ is even and $\lambda = \frac{\rho-1}{2}$ if ρ is odd, and let $\varepsilon_\rho = \dim_K V_\rho$. Selecting ρ-cycles $\mathfrak{Z}_1^\rho, \dots, \mathfrak{Z}_{\varepsilon_\rho}^\rho$ representing the homology classes $Z_1^\rho, \dots, Z_{\varepsilon_\rho}^\rho \in V_\rho$ and adjoining $U_1^{\rho+1}, \dots, U_{\varepsilon_\rho}^{\rho+1}$ of degree $\rho+1$, we have an R-algebra

* For t_1 is a non-zero divisor for R.

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$$A^{(\rho+1)} = A^{(\rho)} < U_1^{\rho+1}, \dots, U_{\varepsilon_\rho}^{\rho+1} > \; ; \; \partial_{\rho+1}^{(\rho+1)} U_i^{\rho+1} = \mathfrak{Z}_i^\rho$$

satisfying

$$\begin{aligned} &\text{a) } A^{(\rho+1)} \supset A^{(\rho)}, \; A_\lambda^{(\rho+1)} = A_\lambda^{(\rho)} \text{ for } \lambda < \rho+1 \\ &\text{and } A_{\rho+1}^{(\rho+1)} = A_{\rho+1}^{(\rho)} \oplus RU_1^{\rho+1} \oplus \cdots \oplus RU_{\varepsilon_\rho}^{\rho+1} \\ &\text{b) } H_\rho(A^{(\rho+1)}) = H_\rho(A^{(\rho)})/RZ_1^\rho + \cdots + RZ_{\varepsilon_\rho}^\rho \\ &\qquad = H_\rho(A^{(\rho)})/V_\rho \end{aligned}$$

Letting $X_\rho = A_\rho^{(\rho)}$ and defining $\partial_{\rho+1} : X_{\rho+1} \to X_\rho$ by $\partial_{\rho+1} = \partial_{\rho+1}^{(\rho+1)}$, we obtain an R-algebra $X = \bigcup_\rho X_\rho$

$$X : \longrightarrow X_{\rho+1} \xrightarrow{\partial_{\rho+1}} X_\rho \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} K \longrightarrow 0$$

where $X_0 = R$ and the mapping ε is the augmentation homomorphism.

Defining vector spaces over K, $D_{p,q} = H_{p+q}(A^{(p)})$ and $E_{p,q} = H_{p+q}(A^{(p)}/A^{(p-1)})$, we obtain a spectral sequence

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & D_{1,3} = H_4(A) & & & & & & & & & & & \\
& \downarrow i_{13} & & & & & & & & & & & \\
\cdots \longrightarrow & D_{2,2} & \longrightarrow & E_{2,2} & \xrightarrow{k_{22}} & D_{1,2} = H_3(A) & & & & & & & \\
& \downarrow i_{22} & & & & \downarrow i_{12} & & & & & & & \\
\cdots \longrightarrow & D_{3,1} & \longrightarrow & E_{3,1} & \xrightarrow{k_{3,1}} & D_{2,1} & \longrightarrow & E_{2,1} & \xrightarrow{k_{21}} & D_{1,1} = H_2(A) & & & \\
& \downarrow i_{31} & & & & \downarrow i_{21} & & & & \downarrow i_{11} & & & \\
\cdots \longrightarrow & D_{4,0} & \longrightarrow & E_{4,0} & \longrightarrow & D_{3,0} & \longrightarrow & E_{3,0} & \longrightarrow & D_{2,0} & \longrightarrow & E_{2,0} & \longrightarrow D_{1,0} = H_1(A) \\
& \downarrow i_{40} & & & & \downarrow i_{30} & & & & \downarrow i_{20} & & & \downarrow \\
\cdots \longrightarrow & D_{5,-1} & \longrightarrow & 0 & \longrightarrow & D_{4,-1} & \longrightarrow & 0 & \longrightarrow & D_{3,-1} & \longrightarrow & 0 & \longrightarrow D_{2,-1} \longrightarrow 0 \\
& \parallel & & & & \parallel & & & & \parallel & & & \\
& H_4(X) & & & & H_3(X) & & & & H_2(X) & & & H_1(X)
\end{array}$$

By virtue of the construction of X it is seen that $D_{\rho+1,-1} = H_\rho(A^{(\rho+1)}) = H_\rho(X)$ for $\rho \geq 1$, $H_1(X) = H_1(A^{(2)}) = 0$, and $H_\rho(X) = D_{\rho,0}/V_\rho$. If we can prove $D_{\rho,0} \simeq V_\rho$, X is aspherical so that we have a desired R-algebra minimal resolution of K. In this paper we contend

PROPOSITION 2.1.

i) $D_{2,0} \simeq H_2(A)/H_1(A)^2 = V_2$,

ii) $D_{3,0} \simeq H_3(A)/H_2(A) \cdot H_1(A) = V_3$.

For the proposition we need the following two lemmas.

LEMMA 2.2.

i_{11}, i_{21} and i_{12} are onto.

LEMMA 2.3.

a) $k_{21}(E_{2,1}) \simeq H_1(\Lambda)^2$,

b) $k_{22}(E_{2,2}) + i_{12}^{-1}k_{31}(E_{3,1}) \simeq H_2(\Lambda) \cdot H_1(\Lambda)$.

Proof of Proposition 2.1.

It is immediate from the exactness of the spectral sequence and the above two lemmas.

Proof of Lemma 2.2.

Let $Z \in D_{2,0}$, then Z is represented by a cycle

$$\mathfrak{Z} = c + \sum_{i=1}^{b_1} \lambda^i S_i,$$

where $c \in \Lambda_2$ and $\lambda^i \in R$. Since $0 = \partial_2 \mathfrak{Z} = \partial_2 c + \sum_{i=1}^{b_1} \lambda^i \mathfrak{Z}_i^1$, we have $\sum_{i=1}^{b_1} \bar{\lambda}^i Z_i^1 = 0$ where $\bar{\lambda}^i \in K$. Therefore $\lambda^i \in M$ for all i. Let $\lambda^i = \sum_{j=1}^{n} r^{ij} \cdot t_j$, then

$$\begin{aligned} \mathfrak{Z} &= c + \sum_{i,j} r^{ij} t_j S_i \\ &= (c + \sum_{i,j} r^{ij} T_j \mathfrak{Z}_i^1) + \partial_3(\sum_{i,j} r^{ij} T_j S_i). \end{aligned}$$

The cycle $\mathfrak{Z}' = (c + \sum_{i,j} r^{ij} T_j \mathfrak{Z}_i^1)$ represents an element $Z' \in D_{1,1}$ whose image under i_{11} is Z. Therefore i_{11} is onto.

Secondly we wish to show that i_{21} and i_{12} are onto. Let $y \in X_3$ represent an element $Y \in D_{3,0}$. Then

$$y = d + \sum_{j=1}^{b_1} \sum_{i=1}^{n} \mu^{ij}(T_i \cdot S_j) + \sum_{k=1}^{\varepsilon_2} \nu^k U_k^3,$$

where $d \in \Lambda_3$.

$$0 = \partial_3 y = \sum_{j=1}^{b_1} (\sum_{i=1}^{n} \mu^{ij} t_i) S_j + (\partial_3 d - \sum_{i,j} \mu^{ij} T_i \mathfrak{Z}_j^1 + \sum_k \nu^k \mathfrak{Z}_k^2).$$

Thus we have

$$\sum_{i=1}^{n} \mu^{ij} t_i = 0 \qquad \text{for all } j,$$

so that $\sum_i \mu^{ij} T_i$ is 1-cycle of Λ and $\sum_j (\sum_i \mu^{ij} T_i) \mathfrak{Z}_j^1$ represents an element $Z'' \in$

$H_1(\Lambda)^2$. From this $Z'' = \sum_{k=1}^{\varepsilon_2} \bar{\nu}^k Z_k^2$, and hence $\nu^k \in M$. Letting $\nu^k = \sum_{l=1}^{n} \nu^{kl} t_l$ and considering $\partial_4(\sum_{k,l} \nu^{kl} T_l U_k^3) = \sum_k \nu^k U_k^3 - \sum_{k,l} \nu^{kl} T_l \mathfrak{Z}_k^2$, we find 2-cycle of $\Lambda^{(2)}$,

$$d + \sum_{j,i} \mu^{ij} (T_i \cdot S_j) + \sum_{k,l} \nu^{kl} T_l \mathfrak{Z}_k^2,$$

whose homology class Y' is mapped onto Y under i_{21}. From the analogous argument it is easy to see that i_{12} is onto. Thus the proof is omitted. This completes the proof of the Lemma.

Proof of Lemma 2.3.

Select 3-relative cycle $\mathfrak{Z}$ of $\Lambda^{(2)}/\Lambda$ representing an element $Z \in E_{2,1}$. Then $\mathfrak{Z} = x + \sum_{i,j} \lambda^{ij} T_i \cdot S_j$, where $x \in \Lambda_2$ and $\sum_{i=1}^{n} \lambda^{ij} T_i$ is 1-cycle of Λ. Since $k_{21}(Z)$ is represented by 2-cycle of $\sum_{j=1}^{b_1} (\sum_{i=1}^{n} \lambda^{ij} T_i) \mathfrak{Z}_j^1$, we have $k_{21}(Z) \in H_1(\Lambda)^2$. Conversely it is obvious that $H_1(\Lambda)^2 \subset k_{21}(E_{2,1})$, beause $\mathfrak{Z}_i^1 \mathfrak{Z}_j^1 = \partial_3(-\mathfrak{Z}_i^1 S_j)$ for any pair (i, j). This completes the proof of Lemma 2.3. a).

Let $Y \in E_{3,1}$ and y be 4-relative cycle of $\Lambda^{(3)}/\Lambda^{(2)}$ representing Y. Then we have

$$y = c + \sum_{i,j} \lambda^{ij} T_i U_j^3,$$

where $c \in \Lambda_3^{(2)}$ and $\sum_i \lambda^{ij} T_i$ is 1-cycle of Λ. By considering k_{31} and i_{12}, $i_{12}^{-1} k_{31}(Y)$ is represented by 3-cycle of Λ, $\sum_{j=1}^{\varepsilon_2} (\sum_{i=1}^{n} \lambda^{ij} T_i) \mathfrak{Z}_j^2$, whose homology class is in $H_1(\Lambda) \cdot V_2 \subset H_1(\Lambda) \cdot H_2(\Lambda)$.

Let $\mathfrak{Z}$ be a relative 4-cycle representing an element $Z \in E_{2,2}$, and let

$$\mathfrak{Z} = a + \sum_{b_1 \geq k > i \geq 1} \lambda^{ik} S_i \cdot S_k + \sum_{b_1 \geq k \geq 1} \lambda^{kk} S_k^{(2)} + \sum_{\substack{n \geq j > i \geq 1 \\ b_1 \geq k \geq 1}} \mu^{ijk} \; (T_i T_j S_k),$$

where $a \in \Lambda_4$ and $1 \cdot S_k^{(2)}$ is a generator of $\Lambda_4^{(2)}$, whose boundary is defined by $\mathfrak{Z}_k^1 S_k$ (refer to [6]). Considering the boundary of $\mathfrak{Z}$, we have

$$\Lambda_3 \ni \partial_4 \mathfrak{Z} = (\partial_4 a + \sum_{i,j,k} \mu^{ijk} (T_i \cdot T_j) \mathfrak{Z}_k^1) + \sum_{k=1}^{b_1} \{ \sum_{i=1}^{k} \lambda^{ik} \mathfrak{Z}_i^1 + \sum_{i=k+1}^{b_1} \lambda^{ki} \mathfrak{Z}_i^1 + \partial_2 (\sum_{i,j} \mu^{ijk} T_i \cdot T_j) \} S_k,$$

so that

$$\sum_{j=1}^{k} \lambda^{ik} \mathfrak{Z}_i^1 + \sum_{i=k+1}^{b_1} \lambda^{ki} \mathfrak{Z}_i^1 + \partial_2 (\sum \mu^{ijk} T_i \cdot T_j) = 0 \text{ for each } k.$$

Therefore all $\lambda^{ik} \in M$ for any pair (i, k) satisfing $b_1 \geq k \geq i \geq 1$. Letting $\lambda^{ik} = \sum_{j=1}^{n} \lambda^{ijk} t_j$, considering $\xi_k = \sum_{i=1}^{k} \sum_{j=1}^{n} \lambda^{ijk} T_j \mathfrak{Z}_i^1 + \sum_{i=k+1}^{b_1} \sum_{j=1}^{n} \lambda^{kji} T_j \mathfrak{Z}_i^1$, we obtain a 2-cycle η_k

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of Λ by

$$\eta_k - \xi_k = \sum_{i,j} \mu^{ijk}(T_i \cdot T_j),$$

because $\partial_2(\xi_k) = \sum_{i=1}^{k} \lambda^{ik} \mathfrak{Z}_i^1 + \sum_{i=k+1}^{b_1} \lambda^{ki} \mathfrak{Z}_i^1$. The straightforward computation shows $\sum_{k=1}^{b_1} \xi_k \mathfrak{Z}_k^1 = 0$, so that we have

$$\sum_{i,j,k} \mu^{ijk}(T_i \cdot T_j)\mathfrak{Z}_k^1 = \sum_{k=1}^{b_1} \eta_k \mathfrak{Z}_k^1.$$

Since $k_{22}(Z)$ is represented by $\sum_{k=1}^{b_1} \eta_k \mathfrak{Z}_k^1$, $k_{22}(E_{2,2}) \subset H_2(\Lambda) \cdot H_1(\Lambda)$. It is immediate to show that $H_2(\Lambda) \cdot H_1(\Lambda) \subset k_{22}(E_{2,2})$, because $\partial_4(\eta \cdot S_k) = \eta \cdot \mathfrak{Z}_k^1$ for any 2-cycle η of Λ. This completes the proof of Lemma 2.3.

§ 3. Computation of B_ρ ($\rho \leq 4$)

Proposition 3.1.

i) $B_1 = \binom{n}{1}$. $B_2 = \binom{n}{2} + b_1$,

ii) $B_3 = \binom{n}{3} + \binom{n}{1} \cdot b_1 + \varepsilon_2$

iii) $B_4 = \binom{n}{4} + \binom{n}{2} \cdot b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2} + \binom{n}{1}\varepsilon_2 + \binom{n}{0}\varepsilon_3$.

Proof.

In the previous section we have proved that the sequence

$$X_4 \xrightarrow{\partial_4} X_3 \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} K \longrightarrow 0$$

is exact. By definition $\mathrm{Tor}_\rho^R(K, K)$ is computed by $X_\rho \otimes_R K$ for all $\rho \leq 3$. Therefore we get i) and ii). From a general theory (for example, see [5] or [4]) we know that there exists $\tilde{X}_5$ such that $\tilde{X}_5 \xrightarrow{\tilde{\partial}_5} X_4 \xrightarrow{\partial_4} X_3$ is exact and $\tilde{\partial}_5(\tilde{X}_5) \subset MX_4$. Therefore B_4 can be computed as stated in 3.1. iii) without knowing explicitely a system of generators for $\tilde{X}_5$.

Note that $\tilde{X}_5$ may be considered as X_5 which we constructed in § 2.

§ 4. Corollaries and a conjecture

Corollary 4.1.

If R is a complete intersection, we have

$$B_3 = \binom{n}{3} + \binom{n}{1} b_1$$

$$B_4 = \binom{n}{4} + \binom{n}{2} b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2}$$

COROLLARY 4.2.

$$B_\rho \geq \binom{n}{\rho} + \binom{n}{\rho-2} + \binom{n}{\rho-4} + \cdots$$

for $\rho \leq 4$, *if R is not regular.*

Proofs

By a Theorem of Assmus [1] R is a local complete intersection if and only if $H(A)$ is the exterior algebra on $H_1(A)$. Therefore we have $\varepsilon_2 = \varepsilon_3 = 0$ in this case. The corollary 4.1. coincides with a result of Tate [6]. The special case when $b_1 = 1$. $b_2 = b_3 = 0$, provides us with the proof of Corollary 4.2., which is the estimation of Tate [6].

Tate said in [6] that it is doubtful whether minimal R-algebra resolutions exist in all cases. It seems to the author that such resolution may be probable in view of the construction we consider in this paper.

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