IDEMPOTENT-EQUIVALENT CONGRUENCES ON ORTHODOX SEMIGROUPS

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Introduction

Any congruence on a semigroup S with a nonempty set E_S of idempotents induces a partition of the set E_S. Two congruences ρ and σ on the semigroup S are defined to be *idempotent-equivalent congruences* on S if ρ and σ induce the same partition of E_S. In this paper we investigate idempotent-equivalent congruences on orthodox semigroups (regular semigroups in which the set of idempotents forms a subsemigroup).

If S is an orthodox semigroup and ρ is a congruence on S, then the partition of E_S induced by ρ satisfies certain normality conditions. We determine those partitions of E_S which are induced by congruences on S and we characterize the largest and smallest congruences on S corresponding to such a partition of E_S.

The set of all congruences which are idempotent-equivalent to a given congruence forms a sublattice of the lattice $\Lambda(S)$ of all congruences on S. We investigate some of the properties of this sublattice of $\Lambda(S)$. Specifically, we determine the regular kernels of the meet and join of two idempotent-equivalent congruences ρ and σ on S in terms of the regular kernels of ρ and σ. Finally, we show how this may be simplified in the special case when the lattice of idempotent-equivalent congruences considered coincides with the lattice of idempotent-separating congruences on S.

The corresponding results concerning idempotent-equivalent congruences on inverse semigroups have been obtained by N. R. Reilly and H. E. Scheiblich [3], and the methods which we adopt provide an extension of the methods adopted in [3]. The essential difficulties which arise are due to the necessity to consider the transitive closures of several relations.

1. Preliminary Results and Notation

We shall adhere throughout to the notation and terminology of A. H. Clifford and G. B. Preston [1]. In addition, we shall denote the set of inverses of an element x of a regular semigroup by $V(x)$.

We make frequent use of the following lemma ([3], lemma 1.3 and lemma 1.4).

LEMMA 1.1. *Let S be an orthodox semigroup. Then*

(i) *if a and b are arbitrary elements of S, and if a' and b' are arbitrary inverses of a and b respectively, it follows that $b'a' \in V(ab)$;*

(ii) *if a is an arbitrary element of S and if a' is an arbitrary inverse of a, then $a'E_S a \subseteq E_S$;*

(iii) *if e is an arbitrary idempotent of S, then $V(e) \subseteq E_S$.*

We now give a brief outline of some of the results of the author [2] concerning congruences on orthodox semigroups. The *regular kernel* $\mathscr{B} = \{B_i : i \in I\}$ of a congruence ρ on an orthodox semigroup is defined to be the set of maximal regular subsemigroups of the elements of the kernel $\mathscr{A} = \{A_i : i \in I\}$ of ρ. Indeed, for all $i \in I$, we have the characterization

$$B_i = \{x \in A_i : V(x) \cap A_i \neq \square\}. \tag{1}$$

A *regular kernel normal system* of the orthodox semigroup S is defined to be a set $\mathscr{B} = \{B_i : i \in I\}$ of subsets of S which satisfy the conditions:

(K1) each B_i is a regular subsemigroup of S;

(K2) $B_i \cap B_j = \square$ if $i \neq j$;

(K3) each idempotent of S is contained in some B_i;

(K4) for each $a \in S$, $a' \in V(a)$, and $i \in I$, there is some $j = (a, a', i) \in I$ such that $a'B_i a \subseteq B_j$;

(K5) for each $i, j \in I$, there is some $k \in I$ such that $B_i B_j B_i \subseteq B_k$;

(K6) if $a, ab, bb', b'b \in B_i$ for some $b' \in V(b)$, then $b \in B_i$;

(K7) for each $i \in I$ and for each $j \in I$, there is some $k \in I$ such that $E_i E_j \subseteq E_k$, where E_i denotes the set of idempotents of B_i.

Then we have the following theorem ([2], theorem 3.6).

THEOREM 1.2. *If ρ is a congruence on an orthodox semigroup S then the regular kernel $\mathscr{B}$ of ρ is a regular kernel normal system of S, and $\rho = \rho^t_{\mathscr{B}}$, the transitive closure of the relation $\rho_{\mathscr{B}}$ defined by:*

$$\rho_{\mathscr{B}} = \{(a, b) \in S \times S : \text{there exists } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aa', bb', ab' \in B_i,\ a'a, b'b, a'b \in B_j \text{ for some } i, j \in I\}. \tag{2}$$

Conversely, if $\mathscr{B}$ is a regular kernel normal system of S, then there is precisely one congruence ρ on S such that $\mathscr{B}$ is the regular kernel of ρ, and then $\rho = \rho^t_{\mathscr{B}}$.

Now let S be an orthodox semigroup and let ρ be a congruence on S. Then as mentioned earlier, ρ induces a partition

$$\mathscr{E} = \{E_i : i \in I\}$$

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of the set E_S of idempotents of S. By virtue of lemma 1.1, we easily see that $\mathscr{E}$ satisfies the conditions:

(N1) for all $i, j \in I$, there exists $k \in I$ such that $E_iE_j \subseteq E_k$;

(N2) for all $i \in I$, $a \in S$, and $a' \in V(a)$, there exists $j \in I$ such that $aE_ia' \subseteq E_j$.

We define a partition $\mathscr{E} = \{E_i : i \in I\}$ of the set E_S of idempotents of the orthodox semigroup S to be a *normal partition* of E_S if $\mathscr{E}$ satisfies conditions N1 and N2. We denote by $\pi_{\mathscr{E}}$ the equivalence relation on E_S induced by such a partition $\mathscr{E}$ and show that there exists a congruence ρ on S such that $\rho|_{E_S} = \pi_{\mathscr{E}}$: indeed we determine the maximal and minimal such congruences on S.

Before proceeding to the determination of these congruences, we introduce the following useful notation: *if e and f are two idempotents of the orthodox semigroup S, then we define $e \approx f$ if and only if e and f are in the same class E_i of the normal partition $\mathscr{E} = \{E_i : i \in I\}$ of E_S.*

Using this notation, we have the following lemma.

LEMMA 1.3. *Let $s_1, s_2, \cdots, s_{n-1}$ be elements of the orthodox semigroup S, and let s_i', s_i'' be inverses of s_i for $i = 1, \cdots n-1$ such that relative to some normal partition $\mathscr{E}$ of E_S we have $s_rs_r'' \approx s_{r+1}s_{r+1}'$, $s_r''s_r \approx s_{r+1}'s_{r+1}$,* for $r = 1, \cdots n-2$.

Then the following formulae hold:

$$s_1 s_1' \approx (s_{n-1} s_{n-1}')(s_{n-2} s_{n-2}') \cdots (s_1 s_1'); \tag{3}$$

$$s_1' s_1 \approx (s_1' s_1) \cdots (s_{n-2}' s_{n-2})(s_{n-1}' s_{n-1}); \tag{3'}$$

$$s_{n-1}'' s_{n-1} \approx (s_{n-1}'' s_{n-1}) \cdots (s_2'' s_2)(s_1'' s_1); \tag{4}$$

$$s_{n-1} s_{n-1}'' \approx (s_1 s_1'')(s_2 s_2'') \cdots (s_{n-1} s_{n-1}''). \tag{4'}$$

This may be proved by induction along precisely the same lines as the proof of lemma 3.3 in [2], using the condition (N1) in the appropriate places.

2. The Congruence ζ^t

Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and consider the relation

$$\zeta = \{(a, b) \in S \times S : \text{there exist inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that } i \in I \text{ implies } aE_ia', bE_ib' \subseteq E_j \text{ and } a'E_ia, b'E_ib \subseteq E_k, \text{ for some } j, k \in I\}. \tag{5}$$

We prove that the transitive closure ζ^t of the relation ζ is the largest congruence on S whose restriction to E_S is $\pi_{\mathscr{E}}$.

LEMMA 2.1. *Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and let ζ be defined by* (5). *Then the transitive closure ζ^t of the relation ζ is a congruence on S.*

Proof. ζ is clearly reflexive by normality condition N2, and ζ is also clearly symmetric, so to prove that ζ^t is a congruence on S it suffices to show that ζ is a compatible relation.

Suppose that $(a, b) \in \zeta$, and let c be an arbitrary element of S. Let c' be an arbitrary inverse of c, and let a' and b' be the inverses of a and b respectively which appear in the definition of ζ. Since $a'c' \in V(ca)$ and $b'c' \in V(cb)$, it suffices, in order to establish the left compatibility of ζ, to show that given $i \in I$, there exists $l \in I$ and $m \in I$ such that

$$(ca)E_i(a'c'),\ (cb)E_i(b'c') \subseteq E_l,$$

and

$$(a'c')E_i(ca),\ (b'c')E_i(cb) \subseteq E_m.$$

Now, given $i \in I$, there exists $j \in I$ such that $aE_ia', bE_ib' \subseteq E_j$. Then

$$(ca)E_i(a'c') = c(aE_ia')c' \subseteq cE_jc' \subseteq E_l,$$

some $l \in I$, by condition N2, while

$$(cb)E_i(b'c') = c(bE_ib')c' \subseteq cE_jc' \subseteq E_l,$$

by N2.

Also, $c'E_ic \subseteq E_n$, some $n \in I$ (by N2), and given $n \in I$, there exists $m \in I$ such that $a'E_na, b'E_nb \subseteq E_m$. Hence $(a'c')E_i(ca) \subseteq a'E_na \subseteq E_m$ and

$$(b'c')E_i(cb) \subseteq b'E_nb \subseteq E_m.$$

Hence the left compatibility of ζ is established, and the proof that ζ is right compatible follows in a similar fashion. Thus ζ is compatible, and it follows that ζ^t is a congruence on S.

Lemma 2.2. *Under the conditions of lemma* 2.1 *the restriction* $\zeta^t|_{E_S}$ *of the congruence* ζ^t *to the set* E_S *of idempotents of* S *coincides with* $\pi_{\mathscr{E}}$, *the equivalence relation on* E_S *induced by* $\mathscr{E}$.

Proof. Suppose first that e and f are idempotents of S in the same class E_i of the partition $\mathscr{E}$. Let j be an arbitrary element of I. Then $eE_je \subseteq E_iE_jE_i \subseteq E_k$, some $k \in I$, by N1, and $fE_jf \subseteq E_iE_jE_i \subseteq E_k$. Since $e \in V(e)$ and $f \in V(f)$, it follows that $(e,f) \in \zeta \subseteq \zeta^t$, and hence that $\pi_{\mathscr{E}} \subseteq \zeta^t|_{E_S}$.

Conversely, suppose that e and f are idempotents of S for which $(e,f) \in \zeta^t$. We aim to prove that e and f are in the same class of the partition $\mathscr{E}$, i.e. that $e \approx f$. Now since $\zeta^t = \bigcup_{n=1}^{\infty} \zeta^n$, where ζ^n is the n-fold composition of ζ with itself, it follows that $(e,f) \in \zeta^n$ for some $n \geqq 1$. We consider the cases $n = 1$ and $n > 1$ separately.

Suppose first that $(e,f) \in \zeta$. Then there are inverses e' of e and f' of f such that $i \in I$ implies $eE_ie', fE_if' \subseteq E_j$ and $e'E_ie, f'E_if \subseteq E_k$, some $j, k \in I$. Then $ee' = e(e)e' \approx fef'$, and similarly it follows that $e'e \approx f'e'f$ and that $ff' \approx ef'e'$.

Hence
$$\begin{aligned} e &= (ee')(e'e) \\ &\approx (fef')(f'e'f) \qquad \text{(by N1)} \\ &= f(ef'e')f \\ &\approx f(ff')f = f \qquad \text{(by N1)}. \end{aligned}$$

Hence $e \approx f$, and the proof for the case $n = 1$ is complete. Now suppose that $(e, f) \in \zeta^n$, some $n > 1$. Then there are elements $s_1, s_2, \cdots, s_{n-1}$ of S such that $(e, s_1) \in \zeta, (s_1, s_2) \in \zeta, \cdots (s_{n-1}, f) \in \zeta$, and it follows that there are elements $e' \in V(e)$, $s_l', s_l'' \in V(s_l)$, for $l = 1, \cdots n-1$, and $f' \in V(f)$ such that $i \in I$ implies the existence of $j_1, j_2, \cdots j_n, k_1, k_2, \cdots k_n \in I$ such that

$$\begin{aligned} &eE_i e', s_1 E_i s_1' \subseteq E_{j_1};\ e'E_i e, s_1' E_i s_1 \subseteq E_{k_1}; \\ &s_l E_i s_l'', s_{l+1} E_i s_{l+1}' \subseteq E_{j_{l+1}}, \qquad \text{for } l = 1, \cdots n-2; \\ &s_l'' E_i s_l, s_{l+1}' E_i s_{l+1} \subseteq E_{k_{l+1}}, \qquad \text{for } l = 1, \cdots n-2; \\ &s_{n-1} E_i s_{n-1}'', f E_i f' \subseteq E_{j_n};\ s_{n-1}'' E_i s_{n-1}, f' E_i f \subseteq E_{k_n}. \end{aligned} \tag{6}$$

As the first step in the proof that $e \approx f$, we prove the following formulae:

$$\begin{aligned} &ee' \approx s_1 s_1',\ e'e \approx s_1' s_1, \\ &s_i s_i'' \approx s_{i+1} s_{i+1}',\ s_i'' s_i \approx s_{i+1}' s_{i+1}, \qquad \text{for } i = 1, \cdots n-2, \\ &s_{n-1} s_{n-1}'' \approx ff',\ s_{n-1}'' s_{n-1} \approx f'f. \end{aligned} \tag{7}$$

Now
$$\begin{aligned} s_1 s_1' &= s_1(s_1' s_1) s_1' \approx e(s_1' s_1) e' \qquad \text{(by (6))} \\ &= e(s_1' s_1 e') e' \approx s_1 (s_1' s_1 e') s_1' \qquad \text{(by (6))} \\ &= s_1 e' s_1' \approx ee'e' \qquad \text{(by (6))}, \end{aligned}$$

and hence $s_1 s_1' \approx ee'$. Also, for $i = 1, \cdots n-2$ we have

$$\begin{aligned} s_i s_i'' &= s_i(s_i'' s_i) s_i'' \approx s_{i+1}(s_i'' s_i) s_{i+1}' \qquad \text{(by (6))} \\ &= s_{i+1}[(s_i'' s_i)(s_{i+1}' s_{i+1})] s_{i+1}' \approx s_i (s_i'' s_i)(s_{i+1}' s_{i+1}) s_i'' \qquad \text{(by (6))} \end{aligned}$$

and hence

$$s_i s_i'' \approx s_i (s_{i+1}' s_{i+1}) s_i'' \approx s_{i+1}(s_{i+1}' s_{i+1}) s_{i+1}' = s_{i+1} s_{i+1}'.$$

Finally,

$$\begin{aligned} s_{n-1} s_{n-1}'' &= s_{n-1}(s_{n-1}'' s_{n-1}) s_{n-1}'' \approx f(s_{n-1}'' s_{n-1}) f' \\ &= f[(s_{n-1}'' s_{n-1}) f'] f' \approx s_{n-1}(s_{n-1}'' s_{n-1}) f' s_{n-1}'' \\ &= s_{n-1} f' s_{n-1}'' \approx ff'f' = ff'. \end{aligned}$$

By the dual arguments, we obtain

$$e'e \approx s_1' s_1, \; s_i'' s_i \approx s_{i+1}' s_{i+1}, \qquad \text{for } i = 1, \cdots n-2,$$

and

$$f'f \approx s_{n-1}'' s_{n-1},$$

and this completes the verification of (7). By virtue of the formulae (7) we are now in a position to use lemma 1.3, of course. Now

$$\begin{aligned} ee' &= eee' \approx s_1 e s_1' \text{ (by 6))} \\ &= s_1[e(s_1' s_1)]s_1''(s_1 s_1'). \end{aligned}$$

We use this as a basis for an inductive proof of the formula:

$$ee' \approx s_r e(s_1' s_1)(s_2' s_2) \cdots (s_r' s_r)s_r''(s_r s_r') \cdots (s_2 s_2')(s_1 s_1'), \tag{8}$$

for $r = 1, \cdots n-1$. We have just seen that this holds true for $r = 1$, so suppose that (8) holds for $r = k$. Then we have

$$\begin{aligned} ee' &\approx s_{k+1} e(s_1' s_1)(s_2' s_2) \cdots (s_k' s_k)s_{k+1}'(s_k s_k') \cdots (s_2 s_2')(s_1 s_1') \qquad \text{(by (6))} \\ &= s_{k+1} e(s_1' s_1) \cdots (s_k' s_k)(s_{k+1}' s_{k+1})s_{k+1}''(s_{k+1} s_{k+1}')(s_k s_k') \cdots (s_1 s_1'), \end{aligned}$$

and this completes the inductive proof of (8). From (8), we deduce that

$$\begin{aligned} ee' &\approx s_{n-1} e(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})s_{n-1}''(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1') \\ &\approx fe(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})f'(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1') \qquad \text{(by (6)).} \end{aligned}$$

By the dual of the argument used to prove this, we also have

$$e'e \approx (s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})f'(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')e'f.$$

Hence

$$\begin{aligned} e &= (ee')(e'e) \\ &\approx fe(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})f'(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')e'f \qquad \text{(by N1)} \\ &= u_n v_n, \end{aligned}$$

where

$$u_n = fe(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})f',$$

and

$$v_n = f'(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')e'f.$$

Now

$$\begin{aligned} u_n &\approx s_{n-1} e(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})s_{n-1}'' \\ &= s_{n-1}[e(s_1' s_1)(s_2' s_2) \cdots (s_{n-2}' s_{n-2})]s_{n-1}'(s_{n-1} s_{n-1}''), \end{aligned}$$

and we use this result as a basis for an inductive proof of the formula:

$$u_n \approx s_{n-r}[e(s_1' s_1)(s_2' s_2) \cdots (s_{n-r-1}' s_{n-r-1})]s_{n-r}'(s_{n-r} s_{n-r}'') \cdots (s_{n-1} s_{n-1}''), \tag{9}$$

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for $r = 1, \cdots n-1$. Suppose inductively that (9) holds for $r = k$. Then

$$\begin{aligned} u_n &\approx s_{n-k-1}[e(s'_1 s_1)\cdots(s'_{n-k-1} s_{n-k-1})]s''_{n-k-1}(s_{n-k} s''_{n-k})\cdots s_{n-1} s''_{n-k-1}) \\ &= s_{n-k-1}[e(s'_1 s_1)\cdots(s'_{n-k-2} s_{n-k-2})]s'_{n-k-1}(s_{n-k-1} s''_{n-k-1})\cdots(s_{n-1} s''_{n-1}), \end{aligned}$$

and this completes the inductive proof of (9). From (9) we immediately deduce that

$$\begin{aligned} u_n &\approx (s_1 e s'_1)(s_1 s''_1)(s_2 s''_2)\cdots(s_{n-1} s''_{n-1}) \\ &\approx (eee')(s_1 s''_1)(s_2 s''_2)\cdots(s_{n-1} s''_{n-1}) \qquad \text{(by (6) and N1)} \\ &= (ee')(s_1 s''_1)(s_2 s''_2)\cdots(s_{n-1} s''_{n-1}) \\ &\approx (s_1 s'_1)(s_1 s''_1)(s_2 s''_2)\cdots(s_{n-1} s''_{n-1}) \qquad \text{(by (7) and N1)} \\ &= (s_1 s''_1)(s_2 s''_2)\cdots(s_{n-1} s''_{n-1}) \\ &\approx s_{n-1} s''_{n-1} \qquad \text{by (4') of lemma 1.3} \\ &\approx ff'. \end{aligned}$$

By the dual argument, we deduce that

$$v_n \approx f'f,$$

and hence that

$$e \approx u_n v_n \approx (ff')(f'f) = f,$$

as required.

Hence $\zeta^t|_{E_S} \subseteq \pi_{\mathscr{E}}$, and since the converse inclusion has already been established, we deduce that $\zeta^t|_{E_S} = \pi_{\mathscr{E}}$, as required. This completes the proof of the lemma.

THEOREM 2.3. *Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and let ζ be defined by equation* (5). *Then ζ^t, the transitive closure of ζ, is the largest congruence ρ on S which satisfies $\rho|_{E_S} = \pi_{\mathscr{E}}$.*

PROOF. We have already seen that ζ^t is a congruence on S and that $\zeta^t|_{E_S} = \pi_{\mathscr{E}}$. It remains to verify that if ρ is a congruence on S satisfying $\rho|_{E_S} = \pi_{\mathscr{E}}$, then $\rho \subseteq \zeta^t$.

So let ρ be a congruence on S which satisfies $\rho|_{E_S} = \pi_{\mathscr{E}}$, and let $\mathscr{B} = \{B_i : i \in I\}$ be the regular kernel of ρ. Then since $\rho|_{E_S} = \pi_{\mathscr{E}}$, we have that $E_i \subseteq B_i$, for all $i \in I$, for a suitable indexing of the elements of $\mathscr{B}$. By theorem 1.2, $\rho = \rho^t_{\mathscr{B}}$, the transitive closure of the relation $\rho_{\mathscr{B}}$ defined by (2). Before proceeding to the proof that $\rho = \rho^t_{\mathscr{B}} \subseteq \zeta^t$ *we remark that if $(a, b) \in \rho_{\mathscr{B}}$, then there exist inverses a' of a and b' of b such that $(a', b') \in \rho_{\mathscr{B}}$.*

To prove this, suppose that we have $(a, b) \in \rho_{\mathscr{B}}$ for some $a, b \in S$. Then there are inverses a' of a and b' of b such that $aa', bb', ab' \in B_i$, $a'a, b'b, a'b \in B_j$, for some $i, j \in I$. Since $a \in V(a')$ and $b \in V(b')$, we clearly have that $(a', b') \in \rho_{\mathscr{B}}$.

We now prove that $\rho = \rho^t_{\mathscr{B}} \subseteq \zeta^t$. Let $(a, b) \in \rho^t_{\mathscr{B}} = \bigcup_{n=1}^{\infty} \rho^n_{\mathscr{B}}$. Then $(a, b) \in \rho^n_{\mathscr{B}}$, for some $n \geqq 1$. Hence there exist elements $a = s_0, s_1, s_2, \cdots s_{n-1}, s_n = b \in S$ such that for all $i = 0, \cdots n-1$, we have $(s_i, s_{i+1}) \in \rho_{\mathscr{B}}$, and consequently there

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are elements $s_0'' \in V(s_0), s_i', s_i'' \in V(s_i)$ for $i = 1, \cdots, n-1$, and $s_n' \in V(s_n)$ such that for $i = 0, \cdots, n-1$,

$$(s_i'', s_{i+1}') \in \rho_{\mathscr{B}}.$$

Hence $a\rho = s_0\rho = s_1\rho = \cdots = s_{n-1}\rho = s_n\rho = b\rho$, and

$$a''\rho = s_0''\rho = s_1'\rho, s_1''\rho = s_2'\rho, \cdots s_{n-1}''\rho = s_n'\rho = b'\rho.$$

Now choose E_j, an arbitrary element of the partition $\mathscr{E}$, and let e be an arbitrary element of E_j. Then for $i = 0, \cdots n-1$,

$$\begin{aligned}(s_i e s_i'')\rho = (s_i\rho)(e\rho)(s_i''\rho) &= (s_{i+1}\rho)(e\rho)(s_{i+1}'\rho)\\ &= (s_{i+1}es_{i+1}')\rho,\end{aligned}$$

and

$$\begin{aligned}(s_i''es_i)\rho = (s_i''\rho)(e\rho)(s_i\rho) &= (s_{i+1}'\rho)(e\rho)(s_{i+1}\rho)\\ &= (s_{i+1}'es_{i+1})\rho.\end{aligned}$$

Since s_ies_i'' and $s_{i+1}es_{i+1}'$ are idempotents of S which are equivalent under ρ, and since $\rho|_{E_S} = \pi_{\mathscr{E}}$, it follows that s_ies_i'' and $s_{i+1}es_{i+1}'$ are in the same element E_k of the partition $\mathscr{E}$, and hence by N2,

$$s_iE_js_i'', s_{i+1}E_js_{i+1}' \subseteq E_k.$$

Since $(s_i''es_i)\rho = (s_{i+1}'es_{i+1})\rho$, we also deduce that

$$s_i''E_js_i, s_{i+1}'E_js_{i+1} \subseteq E_l, \qquad \text{some } l \in I.$$

Hence, for $i = 0, \cdots n-1$, we have $(s_i, s_{i+1}) \in \zeta$, and it follows that $(s_0, s_n) \in \zeta^n$, i.e. $(a, b) \in \zeta^n \subseteq \zeta^t$. Hence $\rho = \rho_{\mathscr{B}}^t \subseteq \zeta^t$, and the theorem is proved.

We devote the remainder of this section to the calculation of the regular kernel of the congruence ζ^t. By virtue of theorem 1.2, this provides an alternative characterization of the congruence ζ^t.

We make use of the following theorem, due to N. R. Reilly and H. E. Scheiblich ([3], theorem 1.5).

THEOREM 2.4. *Let E be an idempotent subsemigroup of a semigroup S. Then S has a unique subsemigroup T with the property that T is the largest regular subsemigroup of S with E as its set of idempotents.*

Now let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S. Then for each $i \in I$, E_i is a subsemigroup of S and hence there exists a unique subsemigroup T_i of S with the property that T_i is the largest regular subsemigroup of S with E_i as its set of idempotents. It is obvious that T_i is an orthodox subsemigroup of S. Using this definition of T_i, we now prove the following theorem.

THEOREM 2.5. *Let S be an orthodox semigroup and let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of E_S. For each $i \in I$, define*

$$Z_i = \{x \in T_i : \text{there exists } x' \in V(x) \cap T_i \text{ such that } E_iE_jE_i \subseteq E_k \text{ implies } xE_jx',\ x'E_jx \subseteq E_k\}.$$

Then $\mathscr{Z} = \{Z_i : i \in I\}$ *is the regular kernel of the congruence* ζ^t.

PROOF. Let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of ζ^t and let $\mathscr{B} = \{B_i : i \in I\}$ be the regular kernel of ζ^t. We aim to show that for all $i \in I$, $B_i = Z_i$.

Suppose first that x is an arbitrary element of Z_i for some $i \in I$. Then there exists $x' \in V(x) \cap T_i$ such that $E_iE_jE_i \subseteq E_k$ implies $xE_jx' \subseteq E_k$ and $x'E_jx \subseteq E_k$. Now $xx' \in E_i$, and for all $j \in I$ we have,

$$xx'E_jxx' \subseteq E_iE_jE_i \subseteq E_k, \qquad \text{some } k \in I.$$

It follows that $xE_jx' \subseteq E_k$ and $x'E_jx \subseteq E_k$. Hence

$$(x, xx') \in \zeta \subseteq \zeta^t, \text{ and } (x', xx') \in \zeta \subseteq \zeta^t.$$

Thus $x \in A_i$ and $x' \in A_i$, and so $x \in B_i$ by virtue of the characterization (1) of the B_i. Hence $Z_i \subseteq B_i$, for all $i \in I$.

Conversely, let x be an arbitrary element of B_i for some $i \in I$, and choose $x' \in V(x) \cap B_i$. Then since E_i is the set of idempotents of B_i, it follows that $B_i \subseteq T_i$ and hence that $x, x' \in T_i$. Suppose now that given $j \in I$, we have $E_iE_jE_i \subseteq E_k$ (such a k exists by N1). Choose $e \in E_i$, $f \in E_j$, $g \in E_k$, so that $(e\zeta^t)(f\zeta^t)(e\zeta^t) = g\zeta^t$. Now $x, x' \in A_i$, and so $x\zeta^t = x'\zeta^t = e\zeta^t$, and it follows that

$$(x\zeta^t)(f\zeta^t)(x'\zeta^t) = (x'\zeta^t)(f\zeta^t)(x\zeta^t) = g\zeta^t,$$

i.e.

$$(xfx')\zeta^t = (x'fx)\zeta^t = g\zeta^t.$$

Hence xfx', $x'fx \in E_k$, and so xE_jx', $x'E_jx \subseteq E_k$. Thus $x \in Z_i$, and so $B_i \subseteq Z_i$, for all $i \in I$.

Hence we have $B_i = Z_i$ for all $i \in I$, and the theorem is proved.

3. The Congruence ξ^t

Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and consider the relation

$$\xi = \{(a, b) \in S \times S : \text{there exists } a' \in V(a), b' \in V(b) \text{ and } i, j \in I \text{ such that } aa', bb' \in E_i,\ a'a, b'b \in E_j, \text{ and for some } e \in E_i \text{ and } f \in E_j,\ eaf = ebf \text{ and } fa'e = fb'e\}. \tag{10}$$

In this section we prove that ξ^t, the transitive closure of the relation ξ defined by (10) is the smallest congruence ρ on S which satisfies the condition $\rho|_{E_S} = \pi_{\mathscr{E}}$. We also determine the regular kernel of the congruence ξ^t, thus providing an alternative characterization of this congruence.

LEMMA 3.1. *Let S be an orthodox semigroup and let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of E_S. Then the transitive closure ξ^t of the relation ξ defined by* (10) *is a congruence on S.*

PROOF. It is trivial to verify that ξ is reflexive and symmetric, so to prove that ξ^t is a congruence on S it suffices to prove that ξ is left and right compatible. Let $(a, b) \in \xi$ and let c be an arbitrary element of S. Then there exists $a' \in V(a)$, $b' \in V(b)$, and $i, j \in I$ such that $aa', bb' \in E_i$, $a'a$, $b'b \in E_j$, and for some $e \in E_i$ and $f \in E_j$, $eaf = ebf$ and $fa'e = fb'e$. Let c' be an arbitrary inverse of c. Then since $a'c' \in V(ca)$ and $b'c' \in V(cb)$, in order to prove the left compatibility of ξ it suffices to show that there exist $k, l \in I$ such that $(ca)(a'c')$, $(cb)(b'c') \in E_k$ and $(a'c')(ca)$, $(b'c')(cb) \in E_l$ and that for some $e_1 \in E_k$ and $f_1 \in E_l$, we have

$$e_1(ca)f_1 = e_1(cb)f_1 \text{ and } f_1(a'c')e_1 = f_1(b'c')e_1.$$

Let $(ca)(a'c') \in E_k$. Then since $c(aa')c' \approx c(bb')c'$ (by N2), we also have that $(cb)(b'c') \in E_k$.

Let $(a'c')(ca) \in E_l$ and let $(b'c')(cb) \in E_m$. We prove now that $E_l = E_m$. Now,

$$\begin{aligned}(a'c')(ca) &= (a'a)a'[(aa')(c'c)(aa')]a(a'a)\\ &\approx fa'[e(c'c)e]af \qquad \text{(by N1 and N2)}\\ &= (fa'e)(c'c)(eaf) = (fb'e)(c'c)(ebf)\\ &= fb'[e(c'c)e]bf\\ &\approx (b'b)b'[(bb')(c'c)(bb')]b(b'b) \quad \text{(by N1 and N2)}\\ &= (b'c')(cb), \text{ and it follows that } E_l = E_m.\end{aligned}$$

Now choose $e_1 = ceafa'ec' = cebfb'ec'$, and choose $f_1 = fa'ec'ceaf = fb'ec'cebf$. Note that

$$\begin{aligned} e_1 &\approx c(aa')a(a'a)a'(aa')c' \quad \text{(by N1 and N2)}\\ &= (ca)(a'c'), \text{ so } e_1 \in E_k.\end{aligned}$$

Also,

$$\begin{aligned} f_1 &\approx (a'a)a'(aa')(c'c)(aa')a(a'a) \quad \text{(by N1 by N2)}\\ &= (a'c')(ca), \qquad \text{so } f_1 \in E_l (= E_m).\end{aligned}$$

Now,

$$\begin{aligned} e_1(ca)f_1 &= (ceafa'ec')(ca)(fa'ec'ceaf)\\ &= ce(afa'ec'c)(afa'ec'c)eaf\\ &= ce(afa'ec'c)eaf\\ &= c(eaf)(fa'e)(c'c)(eaf)\\ &= c(ebf)(fb'e)(c'c)(ebf)\\ &= (ce)(bfb'ec'c)ebf\\ &= (ce)(bfb'ec'c)(bfb'ec'c)ebf\\ &= (cebfb'ec')(cb)(fb'ec'cebf)\\ &= e_1(cb)f_1.\end{aligned}$$

Also,

$$
\begin{aligned}
f_1(a'c')e_1 &= (fa'ec'ceaf)(a'c')(ceafa'ec') \\
&= (fa'e)(c'ceafa')(c'ceafa')(ec') \\
&= (fa'e)(c'ceafa')(ec') \\
&= (fb'e)(c'c)(ebf)(fb'e)c' \\
&= (fb'e)(c'cebfb')ec' \\
&= (fb'e)(c'cebfb')(c'cebfb')ec' \\
&= (fb'ec'cebf)(b'c')(cebfb'ec') \\
&= f_1(b'c')e_1 .
\end{aligned}
$$

This completes the proof that ξ is left compatible. The proof that ξ is right compatible follows similarly and is omitted. Thus ξ^t is a congruence on S, and the lemma is proved.

LEMMA 3.2. *Under the conditions of lemma* 3.1, *the restriction* $\xi^t|_{E_S}$ *of the congruence* ξ^t *to the set* E_S *of idempotents of* S *coincides with* $\pi_{\mathscr{E}}$, *the equivalence relation on* E_S *induced by* $\mathscr{E}$.

PROOF. Suppose first that e and f are idempotents of S in the same class E_i of the partition $\mathscr{E}$. Then $ef \in E_i$ and it is easily verified that $(ef)e(ef) = (ef)$ and that $(ef)f(ef) = ef$. Since $e \in V(e)$ and $f \in V(f)$, it follows that $(e,f) \in \xi \subseteq \xi^t$, and consequently that $\pi_{\mathscr{E}} \subseteq \xi^t|_{E_S}$.

Conversely, suppose that e and f are idempotents of S for which $(e,f) \in \xi^t$. Then $(e,f) \in \xi^n$ for some $n \geqq 1$. We consider the cases $n = 1$ and $n > 1$ separately. Suppose first that $(e,f) \in \xi$. Then in particular, there exists inverses e' of e and f' of f such that $ee', ff' \in E_i$ and $e'e, f'f \in E_j$ for some $i, j \in I$. In fact these conditions are sufficient to ensure that $e \approx f$, since $e = (ee')(e'e) \approx (ff')(f'f) = f$, by N1. This completes the proof for the case $n = 1$.

Suppose now that $(e,f) \in \xi^n$ for some $n > 1$. Then there exist elements $s_1, s_2, \cdots s_{n-1} \in S$ such that

$$(e, s_1) \in \xi, (s_1, s_2) \in \xi, \cdots (s_{n-1}, f) \in \xi.$$

Then in particular, there exist elements $e' \in V(e), s_i', s_i'' \in V(s_i)$, for $i = 1, \cdots n-1$, and $f' \in V(f)$ such that

$$
\begin{aligned}
&ee' \approx s_1 s_1', \; e'e \approx s_1' s_1, \\
&s_i s_i'' \approx s_{i+1} s_{i+1}', \; s_i'' s_i \approx s_{i+1}' s_{i+1}, \qquad \text{for } i = 1, \cdots n-2, \\
&s_{n-1} s_{n-1}'' \approx ff', \; s_{n-1}'' s_{n-1} \approx f'f.
\end{aligned}
\tag{11}
$$

Now $e = (ee')(e'e) \approx (s_1 s_1')(s_1' s_1)$, by N1, and so by (3) and (3′) of lemma 1.3, and by N1, we have,

$$\begin{aligned} e &\approx (s_{n-1}s'_{n-1})\cdots(s_2s'_2)(s_1s'_1)(s'_1s_1)(s'_2s_2)\cdots(s'_{n-1}s_{n-1}) \\ &= (s_{n-1}s''_{n-1})(s_{n-1}s'_{n-1})\cdots(s_2s'_2)(s_1s'_1)(s'_1s_1)(s'_2s_2)\cdots(s'_{n-1}s_{n-1})(s''_{n-1}s_{n-1}) \\ &\approx (ff')(s_{n-1}s'_{n-1})\cdots(s_2s'_2)(s_1s'_1)(s'_1s_1)(s'_2s_2)\cdots(s'_{n-1}s_{n-1})(f'f) \\ &= fsf, \end{aligned}$$

where

$$s = f'(s_{n-1}s'_{n-1})\cdots(s_2s'_2)(s_1s'_1)(s'_1s_1)(s'_2s_2)\cdots(s'_{n-1}s_{n-1})f'.$$

Now $f' = (f'f)(ff') \approx (s''_{n-1}s_{n-1})(s_{n-1}s''_{n-1})$, by N1, so

$$\begin{aligned} s &\approx (s''_{n-1}s_{n-1})(s_{n-1}s''_{n-1})(s_{n-1}s'_{n-1})\cdots(s_2s'_2)(s_1s'_1)(s'_1s_1)(s'_2s_2) \\ &\qquad\cdots(s'_{n-1}s_{n-1})(s''_{n-1}s_{n-1})(s_{n-1}s''_{n-1}) \\ &= (s''_{n-1}s_{n-1})(s_{n-1}s'_{n-1})\cdots(s_1s'_1)(s'_1s_1)\cdots(s'_{n-1}s_{n-1})(s_{n-1}s''_{n-1}). \end{aligned}$$

Thus by (3), (3′), (4), and (4′) of lemma 1.3, and by N1, we have

$$\begin{aligned} s &\approx (s''_{n-1}s_{n-1})\cdots(s''_2s_2)(s''_1s_1)(s_1s'_1)(s'_1s_1)(s_1s''_1)(s_2s''_2)\cdots(s_{n-1}s''_{n-1}) \\ &= (s''_{n-1}s_{n-1})\cdots(s''_2s_2)s''_1[s_1s_1s'_1s'_1s_1s_1]s''_1(s_2s''_2)\cdots(s_{n-1}s''_{n-1}) \\ &= (s''_{n-1}s_{n-1})\cdots(s''_2s_2)(s''_1s_1)(s_1s''_1)(s_2s''_2)\cdots(s_{n-1}s''_{n-1}), \end{aligned}$$

since $s'_1s'_1 \in V(s_1s_1)$. Hence by (4) and (4′) of lemma 1.3, and by N1, we finally obtain

$$s \approx (s''_{n-1}s_{n-1})(s_{n-1}s''_{n-1}) \approx (f'f)(ff') = f',$$

and so by N1,

$$e \approx fsf \approx ff'f = f,$$

as required. Hence $\xi^t|_{E_S} \subseteq \pi_{\mathscr{E}}$, and since we have already proved the reverse inclusion, it follows that $\xi^t|_{E_S} = \pi_{\mathscr{E}}$, and the proof of the lemma is completed.

THEOREM 3.3. *Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and let ξ be defined by equation* (10). *Then ξ^t, the transitive closure of ξ, is the smallest congruence ρ on S which satisfies $\rho|_{E_S} = \pi_{\mathscr{E}}$.*

PROOF. We have already seen that ξ^t is a congruence on S which satisfies $\xi^t|_{E_S} = \pi_{\mathscr{E}}$. Thus it remains to verify that if ρ is a congruence on S such that $\rho|_{E_S} = \pi_{\mathscr{E}}$, then $\xi^t \subseteq \rho$. Let ρ be a congruence on S for which $\rho|_{E_S} = \pi_{\mathscr{E}}$. It suffices to prove that $\xi \subseteq \rho$, for then it follows that $\xi^t \subseteq \rho$, since ξ^t is the smallest transitive relation containing ξ, and ρ is a transitive relation on S.

So let a and b be elements of S for which $(a, b) \in \xi$. Then there exist elements $a' \in V(a)$ and $b' \in V(b)$, and $i, j \in I$ such that $aa', bb' \in E_i$, $a'a, b'b \in E_j$, and for some $e \in E_i, f \in E_j$, $eaf = ebf$ and $fa'e = fb'e$. Then

$$\begin{aligned} a\rho &= ((aa')a(a'a))\rho \\ &= (aa')\rho(a\rho)(a'a)\rho \\ &= e\rho a\rho f\rho \qquad\qquad (\text{since } \rho|_{E_S} = \pi_{\mathscr{E}}) \\ &= (eaf)\rho = (ebf)\rho = e\rho b\rho f\rho \\ &= (bb')\rho b\rho(b'b)\rho \qquad (\text{since } \rho|_{E_S} = \pi_{\mathscr{E}}) \\ &= (bb'bb'b)\rho = b\rho, \qquad \text{and so } (a, b) \in \rho. \end{aligned}$$

Hence $\xi \subseteq \rho$, and the proof of the theorem is completed.

We now proceed to the determination of the regular kernel of the congruence ξ^t. If $\mathscr{E} = \{E_i : i \in I\}$ is a normal partition of the set E_S of idempotents of the orthodox semigroup S, then as in § 2, we denote by T_i the unique subsemigroup of S having the property that T_i is the largest orthodox subsemigroup of S with E_i as its set of idempotents. We now prove the following theorem.

THEOREM 3.4. *Let S be an orthodox semigroup and let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of E_S. For each $i \in I$, define*

$$X_i = \{x \in T_i : \text{there exists } x' \in V(x) \cap T_i \text{ such that } \\ exf = ef, fx'e = fe, \text{ for some } e, f \in E_i\}.$$

Then $\mathscr{X} = \{X_i : i \in I\}$ is the regular kernel of the congruence ξ^t.

PROOF. Let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of ξ^t and let $\mathscr{B} = \{B_i : i \in I\}$ be the regular kernel of ξ^t. We aim to prove that for all $i \in I$, $B_i = X_i$.

Suppose first that x is an arbitrary element of X_i. Then there exists $x' \in V(x) \cap T_i$ such that $exf = ef$ and $fx'e = fe$, for some $e, f \in E_i$. Then $xx' \in E_i$ and $x'x \in E_i$, and $exf = ef = eef$, $fx'e = fe = fee$, so it follows that $(x, e) \in \xi \subseteq \xi^t$ and that $(x', e) \in \xi \subseteq \xi^t$. Hence $x, x' \in A_i$, and so $x \in B_i$, by virtue of the characterization (1) of the B_i. It follows that $X_i \subseteq B_i$, for all $i \in I$.

Conversely, let x be an arbitrary element of B_i, for some $i \in I$, and choose $x^* \in V(x) \cap B_i$. Then since E_i is the set of idempotents of B_i, we have $B_i \subseteq T_i$, and it follows that $x, x^* \in T_i$. To prove that $x \in X_i$ it suffices to prove the existence of elements $e, f \in E_i$ such that

$$exf = ef \text{ and } fx^*e = fe.$$

To prove this, it suffices to show that there exist elements $g_1, h_1 \in E_i$ such that $g_1 x h_1 = g_1 h_1$: for if this is true, then since x^* is also an element of B_i, we also have that there exist elements $g_2, h_2 \in E_i$ such that $g_2 x^* h_2 = g_2 h_2$, and taking $e = h_2 g_1 \in E_i$ and $f = h_1 g_2 \in E_i$, we see that

$$exf = (h_2 g_1)x(h_1 g_2) = h_2(g_1 x h_1)g_2 = (h_2 g_1)(h_1 g_2) = ef,$$

and

$$fx^*e = (h_1 g_2)x^*(h_2 g_1) = h_1(g_2 x^* h_2)g_1 = (h_1 g_2)(h_2 g_1) = fe.$$

Now $x, x^* \in A_i$, so $xx^*x^*x \in E_i$ and $(x, xx^*x^*x) \in \xi^t$. Hence $(x, xx^*x^*x) \in \xi^n$ for some $n \geqq 1$. As usual, we consider the cases $n = 1$ and $n > 1$ separately. Suppose first that $(x, xx^*x^*x) \in \xi$. Then there exist elements $x' \in V(x)$, $z \in V(xx^*x^*x)$, and $j, k \in I$ such that $xx', xx^*x^*xz \in E_j$ and $x'x, zxx^*x^*x \in E_k$, and for some $e_1 \in E_j, f_1 \in E_k$, we have

$$e_1 x f_1 = e_1 xx^*x^*x f_1, f_1 x' e_1 = f_1 z e_1.$$

Let $g_1 = e_1 xx^* \approx xx'xx^* = xx^* \in E_i$, and let

$$h_1 = x^*xf_1 \approx x^*xx'x = x^*x \in E_i.$$

Then

$$g_1 x h_1 = e_1 xx^*xx^*x f_1 = e_1 x f_1 = (e_1 xx^*)(x^*x f_1) = g_1 h_1.$$

Now suppose that $(x, xx^*x^*x) \in \xi^n$ for some $n > 1$. Then there exist elements $s_1, s_2, \cdots s_{n-1} \in S$ such that $(x, s_1) \in \xi$, $(s_1, s_2) \in \xi, \cdots (s_{n-1}, xx^*x^*x) \in \xi$, and hence there exist $x' \in V(x)$, $s_i', s_i'' \in V(s_i)$ for $i = 1, \cdots n-1$, and $z \in V(xx^*x^*x)$ such that

$$\begin{aligned} & xx' \approx s_1 s_1', \; x'x \approx s_1' s_1, \\ & s_i s_i'' \approx s_{i+1} s_{i+1}', \; s_i'' s_i \approx s_{i+1}' s_{i+1}, \qquad \text{for } i = 1, \cdots, n-2, \\ & s_{n-1} s_{n-1}'' \approx xx^*x^*xz, \; s_{n-1}'' s_{n-1} \approx zxx^*x^*x, \end{aligned} \tag{12}$$

and for some elements $e_i, f_i \in E_S$, for $i = 1, \cdots, n$ which satisfy

$$e_i \approx s_i s_i', f_i \approx s_i' s_i, \text{ for } i = 1, \cdots, n-1,$$

and $e_n \approx s_{n-1} s_{n-1}''$, $f_n \approx s_{n-1}'' s_{n-1}$, we have

$$\begin{aligned} & e_1 x f_1 = e_1 s_1 f_1, \; f_1 x' e_1 = f_1 s_1' e_1; \\ & e_i s_{i-1} f_i = e_i s_i f_i, \; f_i s_{i-1}'' e_i = f_i s_i' e_i, \qquad \text{for } i = 2, \cdots n-1; \\ & e_n s_{n-1} f_n = e_n xx^*x^*x f_n, \; f_n s_{n-1}'' e_n = f_n z e_n. \end{aligned} \tag{13}$$

Now take

$$g_1 = xf_n(f_1 x' e_1) u_n (e_n s_{n-1} f_n) e_n xx^*,$$

and

$$h_1 = x^*xf_n(f_1 x' e_1) u_n (e_n s_{n-1} f_n e_n) x,$$

where

$$u_n = (e_2 s_1 f_2 s_1'' e_2)(e_3 s_2 f_3 s_2'' e_3) \cdots (e_{n-1} s_{n-2} f_{n-1} s_{n-2}'' e_{n-1}).$$

Then,

$$g_1 x h_1 = x(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x)(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x),$$

but since $e_n s_{n-1} f_n = e_n xx^*x f_n \in E_S$, and since $u_n \in E_S$, it follows that

$$f_n f_1 x'(e_1 u_n e_n s_{n-1} f_n e_n) x \in E_S,$$

and hence that

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$$g_1 x h_1 = x(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x)$$
$$= x(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n) e_n x.$$

Now

$$\begin{aligned} f_n f_1 x' e_1 u_n e_n s_{n-1} f_n &= f_n(f_1 x' e_1)(e_2 s_1 f_2 s_1'' e_2) \cdots (e_{n-1} s_{n-2} f_{n-1} s_{n-2}'' e_{n-1})(e_n s_{n-1} f_n) \\ &= f_n(f_1 s_1' e_1)(e_2 s_1 f_2 s_2' e_2) \cdots (e_{n-1} s_{n-2} f_{n-1} s_{n-1}' e_{n-1})(e_n s_{n-1} f_n) \\ &= f_n f_1(s_1' e_1 e_2 s_1) f_2(s_2' e_2 e_3 s_2) f_3 \cdots f_{n-1}(s_{n-1}' e_{n-1} e_n s_{n-1}) f_n, \end{aligned}$$

and hence $f_n f_1 x' e_1 u_n e_n s_{n-1} f_n \in E_S$. It follows that

$$\begin{aligned} g_1 x h_1 &= x(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n)(f_n f_1 x' e_1 u_n e_n s_{n-1} f_n) e_n x \\ &= x f_n f_1 x' e_1 u_n (e_n s_{n-1} f_n) f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x \\ &= x f_n f_1 x' e_1 u_n (e_n s_{n-1} f_n)(e_n x x^* x^* x f_n) f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x \\ &= (x f_n f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x x^*)(x^* x f_n f_1 x' e_1 u_n e_n s_{n-1} f_n e_n x) \\ &= g_1 h_1. \end{aligned}$$

It remains to verify that $g_1 \in E_i$ and $h_1 \in E_i$. We first remark that, for $i = 2, \cdots n-1$,

$$e_i(s_{i-1} f_i s_{i-1}'') e_i \approx (s_{i-1} s_{i-1}'')(s_{i-1} s_{i-1}'' s_{i-1} s_{i-1}'')(s_{i-1} s_{i-1}'') = s_{i-1} s_{i-1}'',$$

and it follows that

$$u_n \approx (s_1 s_1'')(s_2 s_2'') \cdots (s_{n-2} s_{n-2}'').$$

Also, $e_1 \approx s_1 s_1'$, $e_n \approx s_{n-1} s_{n-1}''$, and

$$e_n s_{n-1} f_n = e_n x x^* x^* x f_n \approx (s_{n-1} s_{n-1}'')(x x^* x^* x)(s_{n-1}'' s_{n-1}),$$

while

$$\begin{aligned} x f_n f_1 x' &\approx x(s_{n-1}'' s_{n-1})(x' x) x' = x(s_{n-1}'' s_{n-1}) x' \\ &= x(x' x)(s_{n-1}'' s_{n-1}) x' \approx x(s_1' s_1)(s_{n-1}'' s_{n-1}) x' \\ &\approx x(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})(s_{n-1}'' s_{n-1}) x' \qquad \text{(by (3'))} \\ &= x(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1}) x' \\ &\approx x(s_1' s_1) x' \approx x(x' x) x' = x x'. \end{aligned}$$

Hence

$$\begin{aligned} g_1 &\approx (x x')(x x')(s_1 s_1'')(s_2 s_2'') \cdots (s_{n-2} s_{n-1}'')(s_{n-1} s_{n-1}'') \\ &\qquad (x x^* x^* x)(s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'') x x^* \\ &\approx (x x')(s_{n-1} s_{n-1}'')(x x^* x^* x)(s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'') x x^* \\ &\approx (s_1 s_1')(s_{n-1} s_{n-1}'')(x x^* x^* x)(x' x)(s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'')(x x')(x x^*) \\ &\approx (s_1 s_1')(s_{n-1} s_{n-1}'')(x x^* x^* x)(s_1' s_1)(s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'')(s_1 s_1')(x x^*). \end{aligned}$$

Using (3) and (3′) of lemma 1.3, we obtain

$$\begin{aligned} g_1 &\approx (s_1 s_1')(xx^*x^*x)(s_1' s_1)(s_1 s_1')(xx^*) \\ &\approx (xx')(xx^*x^*x)(x'x)(xx')(xx^*) \\ &= (xx^*)(x^*x)(xx^*) \in E_i, \text{ and it follows that } g_1 \in E_i. \end{aligned}$$

Since $h_1 = x^* g_1 x$, we have also

$$h_1 \approx x^*(xx^*)(x^*x)(xx^*)x = x^*x^*xx \in E_i.$$

Hence $h_1 \in E_i$, and we deduce that $x \in X_i$. Hence $B_i \subseteq X_i$, and since the reverse inclusion has already been proved, we have $B_i = X_i$, and this completes the proof of the theorem.

4. Lattice properties of idempotent-equivalent congruences

Let $\mathscr{E} = \{E_i : i \in I\}$ be a normal partition of the set E_S of idempotents of the orthodox semigroup S and denote by $\Lambda_{\mathscr{E}}(S)$ the set of congruences on S which induce the partition $\mathscr{E}$ of E_S. In [3] (theorem 3.4), Reilly and Scheiblich have proved that $\Lambda_{\mathscr{E}}(S)$ is a complete sublattice of $\Lambda(S)$ of commuting congruences on X. We make use of this theorem to calculate the regular kernels of the meet and join of two idempotent-equivalent congruences ρ and σ on S in terms of the regular kernels of ρ and σ.

We now introduce the following notation. If T is a subsemigroup of the orthodox semigroup S for which $T \cap E_S \neq \square$, then we denote by $R(T)$ the maximal regular subsemigroup of S which is contained in T. (Since E_T is a subsemigroup of T, $R(T)$ always exists, by virtue of lemma 2.4). Thus if $\mathscr{A} = \{A_i : i \in I\}$ is the kernel of a congruence ρ on the orthodox semigroup S, and if $\mathscr{B} = \{B_i : i \in I\}$ is the regular kernel of ρ, then for all $i \in I$, $B_i = R(A_i)$. In general if T is a subsemigroup of S, then we easily see that

$$R(T) = \{x \in T : V(x) \cap T \neq \square\}. \tag{14}$$

We also adopt the following notation: if B is a subset of the semigroup T, and if ρ is a congruence on T, then we define

$$B\rho = \{x \in T : (b, x) \in \rho \text{ for some } b \in B\}.$$

We now prove the following lemma.

LEMMA 4.1. *Let ρ and σ be idempotent-equivalent congruences on the orthodox semigroup S and let $\mathscr{N} = \{N_i : i \in I\}$ be the kernel of ρ and let $\mathscr{M} = \{M_i : i \in I\}$ be the kernel of σ. Then for all $i \in I$,*

$$N_i\sigma = M_i\rho = E_i(\rho \vee \sigma)$$

where

$$E_i = N_i \cap E_S = M_i \cap E_S.$$

PROOF. We first remark that since ρ and σ are idempotent-equivalent congruences on S, it follows that $\rho \circ \sigma = \sigma \circ \rho$, by the result of Reilly and Scheiblich. Hence $\rho \circ \sigma$ is the smallest transitive relation containing ρ and σ ([1], lemma 1.4), and as the proof that $\rho \circ \sigma$ is compatible is trivial, it follows that $\rho \circ \sigma = \sigma \circ \rho = \rho \vee \sigma$.

Now let e be an arbitrary element of E_i and let x be an arbitrary element of $N_i\sigma$. Then there exists $n \in N_i$ such that $(n, x) \in \sigma$, and hence we have $(e, n) \in \rho$ and $(n, x) \in \sigma$, and it follows that $(e, x) \in \rho \circ \sigma = \rho \vee \sigma$; that is, $x \in e(\rho \vee \sigma) = E_i(\rho \vee \sigma)$. Conversely, if $x \in E_i(\rho \vee \sigma)$, then $(x, e) \in \rho \vee \sigma = \sigma \circ \rho$ for some $e \in E_i$. Hence there exists $n \in S$ such that $(x, n) \in \sigma$ and $(n, e) \in \rho$, and it follows that $n \in N_i$ and that $x \in N_i\sigma$. Hence $N_i\sigma = E_i(\rho \vee \sigma)$ and the proof that $M_i\rho = E_i(\rho \vee \sigma)$ is similar.

COROLLARY 4.2. *Under the conditions of Lemma* 4.1, *we have*

$$R(N_i\sigma) = R(M_i\rho) = R(E_i(\rho \vee \sigma)),$$

for all $i \in I$.

PROOF. Since $E_i(\rho \vee \sigma)$ is an element of the kernel of the congruence $\rho \vee \sigma$, $E_i(\rho \vee \sigma)$ is a subsemigroup of S, and the result follows immediately.

Suppose now that ρ and σ are idempotent-equivalent congruences on the orthodox semigroup S and let $\mathcal{N}' = \{N_i' : i \in I\}$ and $\mathcal{M}' = \{M_i' : i \in I\}$ be the regular kernels of ρ and σ respectively. We prove that $\{(N' \vee M')_i : i \in I\}$ is the regular kernel of the congruence $\rho \vee \sigma$, where for each $i \in I$, $(N' \vee M')_i$ is defined by

$$(N' \vee M')_i = \{k \in S : \text{there exists } k' \in V(k) \text{ such that } kk', k'k \in E_i, \text{ and } kn,\ k'n' \in M_i', \text{ some } n \in N_i',\ n' \in V(n) \cap N_i'\}. \tag{15}$$

LEMMA 4.3. *Let ρ and σ be idempotent-equivalent congruences on the orthodox semigroup S. Let $\mathcal{N} = \{N_i : i \in I\}$ and $\mathcal{M} = \{M_i : i \in I\}$ be the kernels of ρ and σ respectively, and let $\mathcal{N}' = \{N' : i \in I\}$ and $\mathcal{M}' = \{M_i' : i \in I\}$ be the regular kernels of ρ and σ respectively. Then for all $i \in I$, we have*

$$(N' \vee M')_i = R(M_i\rho) = R(N_i\sigma),$$

where $(N' \vee M')_i$ is defined by (15).

PROOF. It clearly suffices to prove that $(N' \vee M')_i = R(N_i\sigma)$, since we have already proved that $R(N_i\sigma) = R(M_i\rho)$. Let $E_i = N_i \cap E_S = M_i \cap E_S$, for each $i \in I$. Suppose first that k is an arbitrary element of $(N' \vee M')_i$. Then there exists $k' \in V(k)$ such that $kk', k'k \in E_i$, and for some $n \in N_i'$ and $n' \in V(n) \cap N_i'$, we have $kn, k'n' \in M_i'$. Then $kk', n'n \in E_i$, $kn \in M_i'$, and $k'k, nn' \in E_i$, $k'n' \in M_i'$, and so $(k, n') \in \rho_{\mathcal{M}'} \subseteq \rho^t_{\mathcal{M}'} = \sigma$. Hence $k \in N_i'\sigma \subseteq N_i\sigma$, and it follows that $(N' \vee M')_i \subseteq N_i\sigma$. But if $k \in (N' \vee M')_i$, then it follows from the definition of $(N' \vee M')_i$

that there exists an element $k' \in V(k) \cap (N' \vee M')_i \subseteq N_i\sigma$, and hence $(N' \vee M')_i \subseteq R(N_i\sigma)$.

Conversely, let k be an arbitrary element of $R(N_i\sigma)$ and choose

$$k^* \in V(k) \cap R(N_i\sigma).$$

Since $R(N_i\sigma) = R(E_i(\rho \vee \sigma))$ is the maximal regular subsemigroup of $E_i(\rho \vee \sigma)$, we have $k^l k^{*l}, k^{*l} k^l \in E_i$ for any positive integer l. Also, since $k \in R(N_i\sigma) \subseteq N_i\sigma$, and $k \in R(N_i\sigma) = R(M_i\rho) \subseteq M_i\rho$, there exist $n_1 \in N_i$ and $m \in M_i$ such that $(k, n_1) \in \sigma$ and $(k, m) \in \rho$. We choose $n_1^* \in V(n_1)$ and $m^* \in V(m)$ arbitrarily.

Here we remark that $(k^2m^{*2}k^2, k^2) \in \rho$, $(k^2m^{*2}k^2, k^4) \in \sigma$, $(kn_1^*k, k^2) \in \rho$, and $(kn_1^*k, k) \in \sigma$. In fact, since $(k, m) \in \rho$, we have

$$(k^2m^{*2}k^2, m^2) = (k^2m^{*2}k^2, m^2m^{*2}m^2) \in \rho, \text{ and } (m^2, k^2) \in \rho.$$

Hence $(k^2m^{*2}k^2, k^2) \in \rho$. Also, since $(m^2, k^2k^{*2}) \in \sigma$ and $(m^2, k^{*2}k^2) \in \sigma$, we have

$$(k^2m^{*2}k^2, k^2m^2k^2) = (k^2k^{*2}k^2m^{*2}k^2k^{*2}k^2, k^2m^2m^{*2}m^2k^2) \in \sigma$$

and $(k^2m^2k^2, k^4) = (k^2m^2k^2, k^2(k^2k^{*2})k^2) \in \sigma$. Hence $(k^2m^{*2}k^2, k^4) \in \sigma$. That $(kn_1^*k, k^2) \in \rho$ and $(kn_1^*k, k) \in \sigma$ can be proved similarly.

Now we set

$$n = k^{*2}m^2k^{*2}k^4k^*n_1k^* \text{ and } n' = kn_1^*kk^{*4}k^2m^{*2}k^2.$$

Then clearly $n' \in V(n)$. Moreover,

$$(n, k^{*2}k^4k^{*2}) = (k^{*2}m^2k^{*2}k^4k^*n_1k^*, k^{*2}k^2k^{*2}k^4k^*kk^*k^*) \in \rho$$

and

$$(n', k^2k^{*4}k^2) = (kn_1^*kk^{*4}k^2m^{*2}k^2, k^2k^{*4}k^2) \in \rho$$

and so $n, n' \in N_i$. Hence $n, n' \in R(N_i) = N_i'$.

Furthermore,

$$(kn, kk^{*4}k^4k^*) = (kk^{*2}m^2k^{*2}k^4k^*n_1k^*, kk^{*2}(k^2k^{*2})k^{*2}k^4k^*kk^*) \in \sigma,$$
$$(n'k^*, kk^{*4}k^4k^*) = (kn_1^*kk^{*4}k^2m^{*2}k^2k^*, kk^{*4}k^4k^*) \in \sigma,$$
$$(k^*n', k^{*4}k^4) = (k^*kn_1^*kk^{*4}k^2m^{*2}k^2, k^*kk^{*4}k^4) \in \sigma,$$
$$(nk, k^{*4}k^4) = (k^{*2}m^2k^{*2}k^4k^*n_1k^*k, k^{*2}k^2k^{*4}k^4k^*kk^*k) \in \sigma$$

and so kn, $n'k^*$, k^*n', $nk \in M_i$. Hence kn, $k^*n' \in R(M_i) = M_i'$. Therefore, by definition, $k \in (M' \vee N')_i$. Hence $R(N_i\sigma) \subseteq (M' \vee N')_i$ and this completes the proof of the lemma.

We are now in a position to prove the following theorem.

THEOREM 4.4. *Let ρ and σ be idempotent-equivalent congruences on the orthodox semigroup S with regular kernels $\mathcal{N}' = \{N_i' : i \in I\}$ and $\mathcal{M}' = \{M_i' : i \in I\}$ respectively. For each $i \in I$, define $(N' \wedge M')_i = N_i' \cap M_i'$, and define $(N \vee M')_i$*

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by (15). *Then* $\{(N' \wedge M')_i : i \in I\}$ *is the regular kernel of* $\rho \cap \sigma$ *and* $\{(N' \vee M')_i : i \in I\}$ *is the regular kernel of* $\rho \vee \sigma$.

PROOF. Let $\mathcal{N} = \{N_i : i \in I\}$ be the kernel of ρ and let $\mathcal{M} = \{M_i : i \in I\}$ be the kernel of σ. Then we first remark that $\{N_i \cap M_i : i \in I\}$ is the kernel of $\rho \cap \sigma$. This follows easily since for each $e \in E_i$, $e(\rho \cap \sigma) = e\rho \cap e\sigma = N_i \cap M_i$. We now verify that for each $i \in I$, $R(N_i \cap M_i) = R(N_i) \cap R(M_i)$. It is trivial to verify that $R(N_i \cap M_i) \subseteq R(N_i) \cap R(M_i)$, so suppose that x is an arbitrary element of $R(N_i) \cap R(M_i)$. Then $x \in R(N_i)$ and $x \in R(M_i)$, so there exist inverses x' and x^* of x such that $x, x' \in N_i$ and $x, x^* \in M_i$. But then $x'xx^* \in V(x)$, and $x'xx^* \in N_iE_i \subseteq N_i$, and $x'xx^* \in E_iM_i$. Hence $x, x'xx^* \in N_i \cap M_i$, and it follows that $x \in R(N_i \cap M_i)$. Thus $R(N_i \cap M_i) = R(N_i) \cap R(M_i) = N_i' \cap M_i'$, and we see that $\{(N' \wedge M')_i : i \in I\}$ is the regular kernel of $\rho \cap \sigma$.

To prove that $\{(N' \vee M')_i : i \in I\}$ is the regular kernel of $\rho \vee \sigma$ it suffices to note that $\{R(E_i(\rho \vee \sigma)) : i \in I\}$ is the regular kernel of $\rho \vee \sigma$, and that for each $i \in I$, $R(E_i(\rho \vee \sigma)) = R(N_i\sigma) = (N' \vee M')_i$, by lemma 4.3 and corollary 4.2.

5. The lattice of idempotent-separating congruences

We now show how the result of theorem 4.4 may be simplified in the case when the partition $\mathcal{E}$ of E_S considered is the maximum partition of E_S. In this case, $\Lambda_{\mathcal{E}}(S) = \Sigma(\mathcal{H})$, the lattice of idempotent-separating congruences on S.

A set $\mathcal{N} = \{N_e : e \in E_S\}$ of normal subgroups of the maximal subgroups $\{H_e : e \in E_S\}$ of the orthodox semigroup S is defined to be a *group kernel normal system* of S if the N_e satisfy the conditions:

(i) $a'N_e a \subseteq N_{a'ea}$ for all $a \in S$, $a' \in V(a)$, and $e \in E_S$;

(ii) $N_e N_f \subseteq N_{ef}$ for all $e, f \in E_S$.

Then we have the following theorem ([2], theorem 4.2).

THEOREM 5.1. *If* ρ *is an idempotent-separating congruence on an orthodox semigroup* S *then the kernel* $\mathcal{N}$ *of* ρ *is a group kernel normal system of* S, *and* $\rho = \rho_{\mathcal{N}}$, *where* $\rho_{\mathcal{N}}$ *is defined by*

$$\begin{aligned}\rho_{\mathcal{N}} = \{(a, b) \in S \times S &: \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that} \\ aa' = bb' = e,\ ab' \in N_e,\ &a'a = b'b = f,\ a'b \in N_f, \text{ for some } e, f \in E_S\}.\end{aligned} \tag{17}$$

Conversely, if $\mathcal{N}$ *is a group kernel normal system of* S, *then there is precisely one congruence* ρ *on* S *such that* $\mathcal{N}$ *is the kernel of* ρ. *This congruence* ρ *is an idempotent-separating congruence on* S *and* $\rho = \rho_{\mathcal{N}}$.

We now determine the kernels of the meet and join of two idempotent-separating congruences ρ and σ on S in terms of the kernels of ρ and σ.

The following theorem has a precise analogue for inverse semigroups ([1], theorem 7.56).

THEOREM 5.2. *Let ρ and σ be idempotent-separating congruences on the orthodox semigroup S with kernels $\mathscr{N} = \{N_e : e \in E_S\}$ and $\mathscr{M} = \{M_e : e \in E_S\}$ respectively. Define $\mathscr{M}\mathscr{N} = \mathscr{M} \vee \mathscr{N} = \{M_e N_e : e \in E_S\}$, and $\mathscr{M} \wedge \mathscr{N} = \{M_e \cap N_e : e \in E_S\}$. Then $\mathscr{M} \vee \mathscr{N}$ and $\mathscr{M} \wedge \mathscr{N}$ are group kernel normal systems of S, and $\mathscr{M} \wedge \mathscr{N}$ is the kernel of $\rho \cap \sigma$ and $\mathscr{M} \vee \mathscr{N}$ is the kernel of $\rho \vee \sigma$.*

PROOF. That $\mathscr{M} \wedge \mathscr{N}$ is a group kernel normal system and is the kernel of $\rho \cap \sigma$ follows immediately from theorem 4.4 and theorem 5.1. Furthermore, by theorem 4.4, it follows that the kernel of the congruence $\rho \vee \sigma$ is $\{(N \vee M)_e : e \in E_S\}$, where for each $e \in E_S$,

$$\begin{aligned}(N \vee M)_e = \{k \in S : &\text{ there exists } k' \in V(k) \text{ such that} \\ &kk' = k'k = e, \text{ and } kn, k'n' \in M_e, \text{ some} \\ &n \in N_e, n' \in V(n) \cap N_e\}.\end{aligned}$$

Thus to complete the proof of the theorem it suffices to show that for each $e \in E_S$ we have $M_e N_e = (N \vee M)_e$.

Let k be an arbitrary element of $M_e N_e$. Then $k = mn$, for some $m \in M_e$, $n \in N_e$. Let m' be the inverse of m which is in M_e and let n' be the inverse of n which is in N_e, and let $k' = n'm'$. Then $kk' = m(nn')m' = mem' = mm' = e$, and similarly $k'k = e$. Moreover, $kn' = mnn' = me = m \in M_e$, and $k'n = n'm'n$: but $n'm'n$ and $e = n'en$ are in the same element M_f of the group kernel normal system $\mathscr{M}$ by condition (i) of the definition of group kernel normal systems, and hence $k'n \in M_e$. Thus $k \in (N \vee M)_e$ and it follows that $M_e N_e \subseteq (M \vee N)_e$ for each $e \in E_S$.

Conversely, choose $k \in (N \vee M)_e$. Then there exists $k' \in V(k)$ such that $kk' = k'k = e$ and $kn, k'n' \in M_e$ for some $n \in N_e$ and $n' \in V(n) \cap N_e$. Now $ke = kk'k = k$, and hence $k = ke = k(nn') = (kn)n' \in M_e N_e$, and it follows that for each $e \in E_S$, $(N \vee M)_e \subseteq M_e N_e$. Hence $M_e N_e = (N \vee M)_e$ for each $e \in E_S$, and the theorem is proved.

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