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RELATIVE STABILITY, CHARACTERISTIC FUNCTIONS AND STOCHASTIC COMPACTNESS

R. A. MALLER

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Abstract

A recent result of Rogozin on the relative stability of a distribution function is extended, by giving equivalences for relative stability in terms of truncated moments of the distribution and in terms of the real and imaginary parts of the characteristic function. As an application, the known results on centering distributions in the domain of attraction of a stable law are extended to the case of stochastically compact distributions.

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1. Introduction and results

Let X, X_i, $i \geqslant 1$, be independent random variables with distribution F not degenerate at 0, and let $S_n = X_1 + X_2 + \ldots + X_n$. We say that F is *relatively stable* if there are constants $B_n > 0$, $B_n \to +\infty$, for which either $S_n/B_n \xrightarrow{P} 1$, or $S_n/B_n \xrightarrow{P} -1$ (we abbreviate this to $S_n/B_n \xrightarrow{P} \pm 1$). Rogozin (1976) and Maller (1978) showed that, in the case when $P(|X| > x) > 0$ for $x > 0$, F is relatively stable if and only if $xP(|X| > x)/v(x) \to 0$ as $x \to +\infty$, where $v(x) = \int_{-x}^{x} u\, dF(u)$ for $x > 0$. Rogozin also showed that then B_n may be taken to be nondecreasing, is regularly varying with index 1, and satisfies $B_n \sim nv(B_n)$. These results extend earlier ones of Khintchine (1936) who restricted himself to the case of positive X_i with a continuous distribution.

In this paper, we supplement Rogozin's theorem by finding two further equivalences for relative stability, one involving the distribution F through $v(x)$ and the truncated second moment, $V(x) = \int_{-x}^{x} u^2\, dF(u)$, and the second involving the real and imaginary parts of the characteristic function F, which we define as $\varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$. We go on to show that these equivalences have counterparts for the convergence of a subsequence of S_n/B_n to ± 1, and indicate how those results are related to some problems in the theory of stochastically compact distributions.

THEOREM 1. *The following are equivalent:*
(1.1) *F is relatively stable;*
(1.2) $V(x)/xv(x)\to 0$ *as* $x\to+\infty$;
(1.3) $[1-\operatorname{Re}\varphi(t)]/\operatorname{Im}\varphi(t)\to 0$ as $t\to 0$, *and* $|\operatorname{Im}\varphi(t)|$ *is regularly varying with index* 1 *as* $t\to 0$.
If (1.1) *holds*, $\operatorname{Im}\varphi(t)\sim tv(|t|^{-1})$ *as* $t\to 0$.

According to Feller (1965–66) (with his restriction of the symmetry of F removed), F is *stochastically compact* if there are sequences A_n, B_n, $B_n>0$, $B_n\to+\infty$, for which, given any sequence of integers $n''\to+\infty$, there is a subsequence $n'\to+\infty$ for which $(S_{n'}/B_{n'})-A_{n'}$ converges in distribution to a nondegenerate random variable. Feller gives necessary and sufficient conditions for this as: $V(x\lambda)/V(x)\leqslant C\lambda^{\rho}$ for some $\rho<2$ and $C>0$, if $x>1$ and $\lambda>\tau$; equivalently, $\limsup_{x\to+\infty} x^2P(|X|>x)/V(x)<+\infty$. For our purpose it is useful to note that these conditions are equivalent to $\limsup_{x\to+\infty} V(x\lambda)/V(x)\leqslant c\lambda^{2-\alpha}$ for some $\alpha>0$ and $c\geqslant 1$, when $\lambda\geqslant 1$. This may be proved either by a minor modification of the method of Letac (1970) or by an argument similar to the criterion on page 110 of Feller (1969). When F is in a domain of attraction (that is, when the whole sequence $(S_n/B_n)-A_n$ converges to a nondegenerate random variable), V is regularly varying with index in $[0,2)$; thus $\lim_{x\to+\infty} V(x\lambda)/V(x)=\lambda^{2-\alpha}$ for some $\alpha>0$ when $\lambda>0$. The condition $\limsup_{x\to+\infty} V(x\lambda)/V(x)\leqslant c\lambda^{2-\alpha}$ is a natural generalization of this, also embodying the idea of an 'index of variation'.

We now show that relatively stable and stochastically compact distributions are connected in a couple of interesting ways. We first state the following subsequential version of Theorem 1:

THEOREM 2. *If*
(1.4) *there are sequences* n_i, m_i, B_i, C_i, *for which either* $S_{n_i}/B_i\xrightarrow{P}1$, $S_{m_i}/C_i\xrightarrow{P}-1$, *or both, then*
(1.5) $\liminf_{x\to+\infty} V(x)/x|v(x)|=0$, *and*
(1.6) $\liminf_{t\to 0}[1-\operatorname{Re}\varphi(t)]/|\operatorname{Im}\varphi(t)|=0$.
Conversely, if in addition F is stochastically compact, then (1.6) *implies* (1.5) *and* (1.5) *implies* (1.4).

In Maller (1978) it was shown that (1.4) implies

$$\liminf_{x\to+\infty} xP(|X|>x)/|v(x)|=0,$$

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while the reverse implication is true if in addition it is assumed that F is not in the domain of partial attraction of the normal distribution.

Theorem 2 is related to a result of Kesten (1972) who showed that, if $n^{-\frac{1}{2}} S_n$ has a finite almost sure limit point, then it has as almost sure limit points the whole real line, provided $\limsup_{t\to 0} |\operatorname{Im}\varphi(t)|/[1-\operatorname{Re}\varphi(t)] < +\infty$. The latter condition is just the opposite of (1.6).

It was proved by Maller (1974) that (1.6) does not hold when the tail function $P(|X|>x)$ is regularly varying with index in $(-1,0)$. It is not hard to show that this condition is sufficient for F to be stochastically compact, and in fact the results of the paper just quoted hold, and can be extended, when only the stochastic compactness of F is assumed.

We now establish a connection between Theorem 2 and the problem of centering S_n for stochastically compact F. For such distributions, if A_n and B_n are the centering and norming constants, we say that "A_n may be chosen as 0" if $\limsup_{n\to+\infty} |A_n| < +\infty$; because, in this case, any sequence n'' contains a subsequence n' for which $(S_{n'}/B_{n'}) - A_{n'}$ converges, and $A_{n'} \to A$, where A is a finite constant. Thus, $S_{n'}/B_{n'}$ also converges to a nondegenerate random variable and so no centering is necessary. We prove

THEOREM 3. *Suppose F is stochastically compact. The centering constants may be chosen as zero if and only if* (1.4) *does not hold.*

Using Theorem 3, we can generalize the known results on centering distributions in the domain of attraction of a stable law with index $\neq 1$ as follows (we omit the proof of these results): say that $F \in SC(\alpha)$ if $\limsup_{x\to+\infty} V(x\lambda)/V(x) \leqslant c\lambda^{2-\alpha}$ for $\lambda \geqslant 1$ for some $c \geqslant 1$ and $\alpha > 0$. Then if $F \in SC(\alpha)$ and $\alpha > 1$, A_n may be chosen as $nB_n^{-1} EX$ ($E|X|$ being finite in this case). If $F \in SC(\alpha)$ and α is *necessarily less than* 1, in the sense that $\liminf_{x\to+\infty} V(x\lambda_0)/V/(x) > \lambda_0$ for some $\lambda_0 > 1$, then A_n may be chosen as zero. Following Matuszewska (1962), p. 324 define

$$s_v = \lim (\log\lambda)^{-1} \log [\liminf_{x\to+\infty} V(x\lambda)/V(x)],$$

$$\sigma_v = \lim (\log\lambda)^{-1} \log [\limsup_{x\to+\infty} V(x\lambda)/V(x)],$$

called by Goldie (1977) the *lower and upper indices of variation* of V. With these, the stochastic compactness of F can be simply expressed as $\sigma_v < 2$, while by the result mentioned above, we can say that if $\sigma_v < 1$ or if $1 < s_v \leqslant \sigma_v < 2$ then F is stochastically compact and the centering constants may be chosen as $nB_n^{-1} EX$ or 0, respectively.

Rogozin's (1976) first example is of a relatively stable F for which $E|X| < +\infty$ and $EX = 0$. It is easy to check that his F is also in the domain of attraction of a stable law with index 1, yet by Theorem 3, the centering constants cannot be taken as zero.

For a nonnegative X, relative stability is equivalent to the slow variation of v; this is a consequence of Theorem 2 of Rogozin (1971), who gives a comprehensive discussion of the nonnegative case with regard to the fluctuations of S_n. In general, relative stability is not implied by the slow variation of v or the regular variation with index 1 of Im φ alone; a counter-example consisting of an 'almost symmetric' distribution is not difficult to construct.

We mention finally that two sufficient conditions for relative stability which can be deduced from Theorem 2 of Klass and Teicher (1977) can easily be derived as special cases of either Theorem 1 or of Rogozin's condition for relative stability.

2. Proofs

PROOF OF THEOREM 1. From Gnedenko and Kolmogorov (1968), p. 24, $S_n/B_n \xrightarrow{P} \pm 1$ if and only if for every $\lambda > 0$

(2.1) $nP(|X| > \lambda B_n) \to 0$,

(2.2) $nv(\lambda B_n)/B_n \to 1$ (or to -1), and

(2.3) $nB_n^{-2}\left\{V(\lambda B_n) - \left[\int_{-\lambda B_n}^{\lambda B_n} u\, dF(u)\right]^2\right\} \to 0.$

When (2.2) holds, it is clear that (2.3) is equivalent to

(2.4) $nB_n^{-2}\, V(\lambda B_n) \to 0$,

so $S_n/B_n \xrightarrow{P} \pm 1$ implies $V(B_n)/B_n\, v(B_n) \to 0$. Also a simple consequence of $S_n/B_n \xrightarrow{P} \pm 1$ is that $B_{n+1}/B_n \to 1$, and now a standard argument shows that $V(x)/xv(x) \to 0$. Thus (1.1) implies (1.2).

Again if $S_n/B_n \xrightarrow{P} 1$, $\varphi^n(t/B_n) \to e^{it}$ for every real t, so $\varphi^n(t/B_n) = \exp(it + \delta_n(t))$ where $\delta_n(t) \to 0$ as $n \to +\infty$. Thus, introducing the notation $\chi(t) = \operatorname{Re} \varphi(t)$, $\psi(t) = \operatorname{Im} \varphi(t)$, we have

$$\begin{aligned} n[1 - \chi(t/B_n) - i\psi(t/B_n)] &= n[1 - \exp(it + \delta_n(t))/n] \\ &= n[1 - \exp(it/n)] + n \exp(it/n)\,[1 - \exp(\delta_n(t)/n)] \\ &\to -it, \end{aligned}$$

which means that $n[1 - \chi(t/B_n)] \to 0$ and $n\psi(t/B_n) \to t$ as $n \to +\infty$ for each real t.

Now χ and ψ are continuous functions, and since $[1 - \chi(t/B_n)]/\psi(t/B_n) \to 0$ as $n \to +\infty$ for each $t \neq 0$, and $B_{n+1}/B_n \to 1$, we conclude from Theorem 1 of Kingman (1964) that $[1 - \chi(t)]/\psi(t) \to 0$ as $t \to 0+$, and hence as $t \to 0$. Since F is not degenerate at 0, $\chi(t) < 1$ in a neighbourhood $(0, \delta)$ [Lukacs (1970), p. 19]; so $|\psi(t)| > 0$ in

$(0,\delta)$, and since ψ is continuous, ψ is either positive or negative in $(0,\delta)$ (and hence, since it is an odd function, is correspondingly negative or positive in $(-\delta,0)$). Clearly, for the case under consideration $\psi>0$ in $(0,\delta)$, and since $\psi(t/B_n)/\psi(1/B_n)\to t$ as $n\to+\infty$ we can apply Theorem B of Seneta (1971) to conclude that ψ is regularly varying with index 1 as $t\to 0+$. Thus $|\psi|$ is regularly varying with index 1 as $t\to 0$. Also for $t\neq 0$,

$$\begin{aligned}|\psi(t)-tv(|t|^{-1})| &= \left|\int_{-\infty}^{\infty}\sin tx\,dF(x)-t\int_{-|t|^{-1}}^{|t|^{-1}}x\,dF(x)\right|\\ &\leqslant \int_{-|t|^{-1}}^{|t|^{-1}}|tx-\sin tx|\,dF(x)+1-F(|t|^{-1})+F(-|t|^{-1})\\ &\leqslant t^2\int_{-|t|^{-1}}^{|t|^{-1}}x^2\,dF(x)+o|tv(|t|^{-1})|\\ &= t^2V(|t|^{-1})+o|tv(|t|^{-1})|=o|tv(|t|^{-1})|\end{aligned}$$

using the result of Rogozin (1976) and what we proved earlier. Thus we have $\psi(t)\sim tv(|t|^{-1})$ as $t\to 0$.

Suppose now that (1.2) holds, and suppose first that F is a *continuous* function; later, we reduce the general case to this. Under this assumption,

$$v(x)=xF(x)+xF(-x)-\int_0^x[F(u)+F(-u)]\,du$$

is continuous in $x>0$. Given $\varepsilon>0$ we can choose $x_0(\varepsilon)$ so large that $x\geqslant x_0$ implies $\varepsilon^{-1}x^{-1}V(x)\leqslant|v(x)|$, and since $V(x)>0$ for $x>0$ (F not being degenerate at 0) this means $|v(x)|>0$ for $x\geqslant x_0$, and since v is continuous this means either $v(x)>0$ or $v(x)<0$ for $x\geqslant x_0$. We suppose $v(x)>0$; this leads to $S_n/B_n\xrightarrow{P}+1$, and the other case may be proved similarly. Thus for $x\geqslant x_0$ we have

$$\begin{aligned}\varepsilon>\frac{V(x)}{xv(x)} &= \left|\int_0^x u^2\,d[1-F(u)+F(-u)]\right|\Big/xv(x)\\ &\geqslant \left|\int_0^x u^2\,d[1-F(u)-F(-u)]\right|\Big/xv(x)\\ &= \left|\int_0^x u\,dv(u)\right|\Big/xv(x)=\left|1-\int_0^x\frac{v(u)}{xv(x)}du\right|,\end{aligned}$$

which means $\int_0^x v(u)\,du/xv(x)\to 1$. So by Seneta (1976), p. 54, for example, we

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have that $v(x)$ is a slowly varying function. An integration by parts shows that

$$P(|X|>x) = -\int_x^\infty dP(|X|>u) = \int_x^\infty u^{-2}\, dV(u)$$
$$= -x^{-2}V(x) + 2\int_x^\infty u^{-3}V(u)\,du$$

so that, if $x \geqslant x_0$,

$$\frac{xP(|X|>x)}{v(x)} \leqslant 2x\int_x^\infty u^{-2}\frac{V(u)}{uv(u)}\frac{v(u)}{v(x)}du$$
$$\leqslant 2\varepsilon\int_1^\infty u^{-2}\frac{v(ux)}{v(x)}du \to 2\varepsilon\int_1^\infty u^{-2}\,du = 2\varepsilon/3$$

as $x\to+\infty$ by a well-known property of slow variation. Thus $xP(|X|>x)/v(x)\to 0$ as $x\to+\infty$, and by the results quoted in the Introduction, this means F is relatively stable if $P(|X|>x)>0$ for $x>0$. If $P(|X|>x_0)=0$ for some x_0 (or even if X_i have finite variance) note that if $EX_i=0$, $xv(x)=x\int_{|u|\geqslant x}u\,dF(u)\leqslant\int_{|u|\geqslant x}u^2\,dF(u)\to 0$ as $x\to+\infty$; so (1.2) cannot hold in this case, F not being degenerate at 0. But if $EX_i\neq 0$ in this case, then by the weak law of large numbers $S_n/n|EX|\xrightarrow{P}\pm 1$; so F is relatively stable.

Now we remove the assumption of the continuity of F. Let F be any distribution for which $V(x)/xv(x)\to 0$, equivalently, $x|v(x)|/V(x)\to+\infty$. Let U_i be uniform r.v.'s on $[-1,1]$, independent of each other and of the X_i. Since the distribution of U_i is continuous, so is that of X_i+U_i, and we can apply the result just proved if we can show that $V^*(x)/xv^*(x)\to 0$, where V^* and v^* are the truncated second and first moments of X_i+U_i. We have, after some elementary manipulation,

$$2V^*(x) = 2\int_{-x}^x y^2\,dP(X_i+U_i<y) = 2\int_{-x}^x y^2\,d\int_{-1}^1 F(y-u)\,dP(U_i<u)$$
$$= \int_{-1}^1\int_{-x}^x y^2\,dF(y-u)\,du$$
$$= \int_{-1}^1\int_{-x-u}^{x-u} y^2\,dF(y)\,du + 2\int_{-1}^1 u\int_{-x-u}^{x-u} y\,dF(y)\,du + \int_{-1}^1 u^2\int_{-x-u}^{x-u} dF(y)\,du$$
$$\leqslant \int_{-1}^1\int_{-x-1}^{x+1} y^2\,dF(y)\,du + 2\int_{-1}^1 |u|\int_{-x-1}^{x+1}|y|\,dF(y)\,du + \int_{-1}^1 u^2\,du$$
$$\leqslant 2V(x+1) + O(V(x+1)) + \tfrac{2}{3} = [2+O(1)]\,V(x+1),$$

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because

$$\int_{-x-1}^{x+1} |y|\,dF(y) \leqslant \int_{-1}^{1} |y|\,dF(y) + \int_{1\leqslant |y| \leqslant x+1} y^2\,dF(y)$$
$$\leqslant \int_{-1}^{1} |y|\,dF(y) + V(x+1)$$
$$= O(V(x+1))$$

on noting that, since $V(+\infty)>0$ (F not being degenerate at 0), any constant is $O(V(x+1))$. Now consider

$$2x|v^*(x)-v(x+1)| = x\left|\int_{-1}^{1}\int_{-x}^{x} y\,dF(y-u)\,du - 2v(x+1)\right|$$
$$= x\left|\int_{-1}^{1}\int_{-x-u}^{x-u} y\,dF(y)\,du - 2\int_{-x-1}^{x+1} y\,dF(y) + \int_{-1}^{1} u\int_{-x-u}^{x-u} dF(y)\,du\right|$$
$$\leqslant x\left|\int_{-1}^{1}\int_{x-u}^{x+1} y\,dF(y)\,du\right| + x\left|\int_{-1}^{1}\int_{-x-1}^{-x-u} y\,dF(y)\,du\right| + x\left|\int_{-1}^{1} u[F(x-u)-F(-x-u)]\,du\right|$$
$$\leqslant 2x(x+1)[F(x+1)-F(x-1)] + 2x(x+1)[F(-x+1) - F(-x-1)] + x\left|\int_{-1}^{1} u[F(x-u)-F(-x-u)]\,du\right|.$$

The integral in the last expression has modulus

$$\leqslant \left|\int_{-1}^{1} uF(x-u)\,du\right| + \left|\int_{-1}^{1} uF(-x-u)\,du\right|$$
$$= \tfrac{1}{2}[F(x-1)-F(x+1)] - \tfrac{1}{2}\int_{-1}^{1} u^2\,dF(x-u) + \tfrac{1}{2}[F(-x-1)-F(-x+1)] - \tfrac{1}{2}\int_{-1}^{1} u^2\,dF(-x-u)$$
$$\leqslant \tfrac{1}{2}[F(x+1)-F(x-1)] + \tfrac{1}{2}[F(-x+1)-F(-x-1)]$$

so that for large enough x there is a constant $c>0$ for which

$$2x|v^*(x)-v(x+1)| \leqslant c(x-1)^2[F(x+1)-F(x-1)] + c(x-1)^2[F(-x+1)-F(-x-1)].$$

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But we have

$$1 \geqslant \frac{V(x+1)-V(x-1)}{V(x+1)} = \left[\int_{x-1}^{x+1} y^2\,dF(y)\Big/ V(x+1)\right] + \left[\int_{-x-1}^{-x+1} y^2\,dF(y)\Big/ V(x+1)\right]$$

$$\geqslant (x-1)^2\frac{F(x+1)-F(x-1)}{V(x+1)} + (x-1)^2\frac{F(-x+1)-F(-x-1)}{V(x+1)}$$

so that, for x large enough,

$$\frac{2x|\,\nu^*(x)-\nu(x+1)\,|}{V(x+1)} \leqslant 2c.$$

Then since $x|\,\nu(x+1)\,|/V(x+1)\to+\infty$, we must have $x|\,\nu^*(x)\,|/V(x+1)\to+\infty$, equivalently, since $V^*(x)=O(V(x+1))$, $V^*(x)/x\nu^*(x)\to 0$. Thus if

$$T_n = U_1+U_2+\ldots+U_n,$$

we have by the first part of the proof that $(S_n+T_n)/B_n \xrightarrow{P} \pm 1$ where $B_n\to+\infty$ and B_n satisfies $B_n \sim n\nu^*(B_n)$, while $\nu^*(x)$ is ultimately positive (say) and is slowly varying as $x\to+\infty$. A well-known property of such functions is that $\nu^*(x)\geqslant x^{-\varepsilon}$ for x large enough for any $\varepsilon>0$. So $B_n \geqslant nB_n^{-\frac{1}{2}}$ if n is large enough, and thus $B_n^2/n\to+\infty$. But $T_n/n^{\frac{1}{2}}$ converges to normality and so $T_n/B_n \xrightarrow{P} 0$ and $S_n/B_n \xrightarrow{P} \pm 1$. Thus (1.2) implies (1.1).

Suppose (1.3) holds; then, as in an earlier part of the proof, ψ is of constant sign (say $\psi>0$) in a neighbourhood $(0,\delta)$. Defining

$$B_n^{-1} = \inf\{0<t<\delta\colon \psi(t)>n^{-1}\}$$

we have by continuity that $n\psi(1/B_n)=1$, and that B_n is a positive nondecreasing sequence converging to $+\infty$. Since ψ is regularly varying with index 1 as $t\to 0+$, $n\psi(t/B_n)\to t$ for $t>0$ (and hence for $t<0$) and $n[1-\chi(t/B_n)]\to 0$. It is now easy to see that $\varphi^n(t/B_n)\to e^{it}$ as $n\to+\infty$ for each t; so $S_n/B_n \xrightarrow{P} 1$.

Proof of Theorem 2. If $S_{n_i}/B_i \xrightarrow{P} \pm 1$ then (2.2) and (2.4) hold with n replaced by n_i and B_n replaced by B_i, and so $V(B_i)/B_i\,\nu(B_i)\to 0$ and (1.5) holds. Also, just as in Theorem 1 we have $n_i[1-\chi(1/B_i)]\to 0$ and $n_i\psi(1/B_i)\to\pm 1$ and so $[1-\chi(1/B_i)]/\psi(1/B_i)\to 0$ and (1.6) holds.

To show that (1.5) implies (1.4), let $x_i\to+\infty$ be such that $V(x_i)/x_i|\,\nu(x_i)\,|\to 0$; since F is not degenerate at 0, this means $|\,\nu(x_i)\,|>0$ for i large enough, so by taking a further subsequence if necessary, we can assume that either $\nu(x_i)>0$ or $\nu(x_i')<0$ for two possibly different sequences x_i, x_i'. Assume that $\nu(x_i)>0$, since the other case can be dealt with similarly, and let n_i denote the integer nearest to

$x_i/v(x_i)$; since $v(x)/x \to 0$, this means $n_i \to +\infty$. Also then, $n_i x_i^{-2} V(x_i) \to 0$, and if we assume in addition that F is stochastically compact, we have

$$\limsup x^2 P(|X| > x)/V(x) < +\infty;$$

so $n_i P(|X| > x_i) \to 0$. Thus if $\lambda > 1$, from

$$\begin{aligned} n_i x_i^{-2} V(x_i \lambda) &= n_i x_i^{-2} V(x_i) + n_i x_i^{-2} \int_{x_i}^{x_i \lambda} u^2 \,|\, dP(|X| > u) \,| \\ &\leqslant o(1) + \lambda^2 n_i P(|X| > x_i) = o(1) \end{aligned}$$

and the monotonicity of V we have that $n_i x_i^{-2} V(x_i \lambda) \to 0$ for every $\lambda > 0$, and in turn this means, if $0 < \lambda < 1$

$$\begin{aligned} o(1) = n_i x_i^{-2} [V(x_i) - V(x_i \lambda)] &= n_i x_i^{-2} \int_{x_i \lambda}^{x_i} u^2 \,|\, dP(|X| > u) \,| \\ &\geqslant \lambda^2 n_i [P(|X| > x_i \lambda) - P(|X| > x_i)] \\ &= \lambda^2 n_i P(|X| > x_i \lambda) + o(1). \end{aligned}$$

Finally if $\lambda > 0$

$$\begin{aligned} |n_i x_i^{-1} v(x_i \lambda) - n_i x_i^{-1} v(x_i)| &= n_i x_i^{-1} \left| \int_{x_i \leqslant |u| \leqslant \lambda x_i} u \, dF(u) \right| \\ &\leqslant \max(1, \lambda) \, n_i P(|X| > \min(x_i, \lambda x_i)) \\ &\to 0. \end{aligned}$$

So, since $n_i x_i^{-1} v(x_i) \to 1$, $n_i x_i^{-1} v(x_i \lambda) \to 1$. We have shown that (2.1), (2.2) and (2.4) hold with n_i in place of n and x_i in place of B_n, so $S_{n_i}/x_i \xrightarrow{P} +1$.

Finally, let (1.6) hold, together with the stochastic compactness of F. We have for $t > 0$

$$\begin{aligned} 1 - \chi(t) = \int_{-\infty}^{\infty} (1 - \cos tx) \, dF(x) &\geqslant \int_{-t^{-1}}^{t^{-1}} (1 - \cos tx) \, dF(x) \geqslant \tfrac{1}{3} t^2 \int_{-t^{-1}}^{t^{-1}} x^2 \, dF(x) \\ &= \tfrac{1}{3} t^2 V(t^{-1}), \end{aligned}$$

so (1.6) means that for every $\varepsilon > 0$, for some sequence $t_i \to 0+$,

$$\begin{aligned} t_i^2 V(t_i^{-1}) \leqslant \varepsilon |\psi(t_i)| &= \varepsilon \left| \int_{-\infty}^{\infty} \sin t_i x \, dF(x) \right| \\ &\leqslant \varepsilon \left| \int_{-t_i^{-1}}^{t_i^{-1}} (t_i x - \sin t_i x) \, dF(x) \right| + \varepsilon t_i |v(t^{-1})| + \varepsilon [1 - F(t_i^{-1}) \\ &\qquad + F(-t_i^{-1})] \\ &\leqslant \varepsilon t_i^2 \int_{-t_i^{-1}}^{t_i^{-1}} x^2 \, dF(x) + \varepsilon t_i |v(t_i^{-1})| + \varepsilon c t_i^2 V(t_i^{-1}). \end{aligned}$$

Here we used the stochastic compactness to deduce that, for some $c>0$,

$$1-F(t_i^{-1})+F(t_i^{-1})\leqslant ct_i^2V(t_i^{-1})$$

for i large enough. It follows that

$$(1-\varepsilon-c\varepsilon)t_i^2V(t_i^{-1})\leqslant \varepsilon t_i|v(t_i^{-1})|$$

and thus that (1.5) holds. This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Suppose there are no n_i, B_i', for which $S_{n_i}/B_i'\xrightarrow{P}\pm 1$. Suppose there is an n_i for which $A_{n_i}\to\pm\infty$; by taking a subsequence we can make $(S_{n_i}/B_{n_i})-A_{n_i}\xrightarrow{D}X$, where X is a proper r.v. But then $S_{n_i}/B_{n_i}|A_{n_i}|\xrightarrow{P}\pm 1$, a contradiction.

Conversely suppose there are sequences n_i, B_i' for which $S_{n_i}/B_i'\xrightarrow{P}1$, and that A_n may be chosen as zero. By taking a subsequence we can make $S_{n_i}/B_{n_i}\xrightarrow{D}X$, where X is a proper nondegenerate r.v., and $B_{n_i}/B_i'\to a$ where $-\infty\leqslant a\leqslant+\infty$. Thus.

$$\begin{aligned}P(X<x)&=\lim P(S_{n_i}<xB_{n_i})=\lim P(S_{n_i}/B_i'<xB_{n_i}/B_i')\\&=P(1<ax)=0 \text{ or } 1,\end{aligned}$$

which is impossible. Similarly there can be no n_i, B_i' for which $S_{n_i}/B_i'\xrightarrow{P}-1$.

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CSIRO Division of Mathematics and Statistics
Melbourne
Australia

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