SOME REMARKS ON THE CHARACTERS OF THE SYMMETRIC GROUP, II

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Introduction. Let p be a fixed prime number. We denote by $k(n)$ the number of partitions of n. As is well known, the number of ordinary irreducible characters of the symmetric group S_n is $k(n)$. We set $k(0) = 1$ and

$$(1) \qquad l(b) = \sum_{b_0,\ldots,b_{p-1}} k(b_0)\,k(b_1)\ldots k(b_{p-1}) \qquad \left(\sum_{i=0}^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right),$$

$$(2) \qquad l^*(b) = \sum_{b_1,\ldots,b_{p-1}} k(b_1)\,k(b_2)\ldots k(b_{p-1}) \qquad \left(\sum_{i=1}^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

Two ordinary irreducible representations of S_n belong to the same p-block if and only if they have the same p-core **(10; 2; 11)**. The number of ordinary irreducible characters belonging to a p-block of weight b is independent of the p-core and is equal to $l(b)$ **(16; 12**; also **11; 15)**. This may be also easily proved by applying the theory of p-quotients **(6; 4)**. Moreover we have the following theorem **(13**; also **4a; 8; 15; 16)**.

THEOREM 1. *The number of modular irreducible characters belonging to a p-block of weight b is $l^*(b)$.*

In the present paper we shall give a simple proof for this theorem. We shall then derive some new properties of decomposition numbers of S_n.

1. We denote by χ_α the character of the irreducible representation $[\alpha]$ corresponding to a Young diagram $[\alpha]$. We set $r(\alpha, \alpha') = (-1)^s$ if a diagram $[\alpha']$ of S_{n-g} is obtained from $[\alpha]$ by removing a g-hook of leg length s. Otherwise we set $r(\alpha, \alpha') = 0$. Then the Murnaghan-Nakayama recursion formula **(7; 9)** is expressed as follows:

If G is an element of S_n containing a g-cycle P and $\bar{G}$ is the permutation of $n - g$ symbols arising from G by removing this cycle, then

$$(3) \qquad \chi_\alpha(G) = \sum_{\alpha'} r(\alpha, \alpha')\,\chi_{\alpha'}(\bar{G}),$$

where $[\alpha']$ ranges over all diagrams of S_{n-g}.

If $[\alpha]$ is a diagram with p-core $[\alpha_0]$ then the summation in (3) may be limited to those $[\alpha']$ with the same p-core $[\alpha_0]$.

We set $n = n' + tp$ $(0 \leqslant n' < p)$ and consider an element G of S_n such that

$$G = W \,.\, Q_1 \,.\, Q_2 \ldots Q_s,$$

where no two of Q_i have common symbols and each Q_i is a cycle of length

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$a_i p$ $(a_1 \geqslant a_2 \geqslant \ldots \geqslant a_s)$ and where W is any permutation on the fixed symbols of $P = Q_1 . Q_2 \ldots Q_s$. We set

$$a = \sum_i a_i \qquad (0 \leqslant a \leqslant t).$$

Then P is called an element of *type* $(a_1, a_2, \ldots, a_s)$ and of *weight* a. The number of elements of weight a such that they all lie in different conjugate classes of S_n is $k(a)$. If we set

$$\sum_{a=0}^{t} k(a) = r, \tag{4}$$

then we have a system of elements of weight a $(a = 0, 1, 2, \ldots, t)$

$$P_0 = 1, P_1, \ldots, P_{r-1},$$

such that they all lie in different conjugate classes of S_n and every element of weight a $(0 \leqslant a \leqslant t)$ is conjugate to one of them. Every conjugate class contains an element of the form VP_i, where i is uniquely determined by the class and where V is a p-regular element of S_{n-ap}, if P_i is of weight a. Since the number $k^*(n)$ of modular irreducible representations of S_n is equal to the number of p-regular classes of S_n, we have

$$k(n) = \sum_{a=0}^{t} k^*(n - ap)\, k(a). \tag{5}$$

Let P_i be an element of type $(a_1, a_2, \ldots, a_s)$ and of weight a. Let $[\alpha_0]$ be a p-core with m nodes and $n = m + bp$. Then the number of diagrams of S_{m+jp} with p-core $[\alpha_0]$ is $l(j)$. We denote by $\chi_\beta{}^{(a)}$ the character of the irreducible representation $[\beta]$ of S_{n-ap} corresponding to a diagram $[\beta]$. Let us denote by B the block of S_n with p-core $[\alpha_0]$. Applying the Murnaghan-Nakayama recursion formula iterated s times to $[\alpha] \subset B$, we obtain

$$\chi_\alpha(VP_i) = \begin{cases} \sum_\beta h(\alpha, \beta)\ \chi_\beta^{(a)}(V),\ [\beta] \subset B^{(a)} & (\text{for } a \leqslant b), \\ 0 & (\text{for } b < a), \end{cases} \tag{6}$$

where the $h(\alpha, \beta)$ are rational integers and $B^{(a)}$ denotes the block of S_{n-ap} with p-core $[\alpha_0]$. If $a \leqslant b$ then $B^{(a)}$ is of weight $b - a$. Let $\phi_\lambda{}^{(a)}$ be the character of S_{n-ap} in the modular irreducible representation λ. We then have

$$\chi_\beta^{(a)}(V) = \sum_\lambda d_{\beta\lambda}^{(a)} \phi_\lambda^{(a)}(V) \qquad (V \text{ in } S_{n-ap},\ p\text{-regular}), \tag{7}$$

where the $d_{\beta\lambda}{}^{(a)}$ are the decomposition numbers **(1)** of S_{n-ap}. Hence (6), combined with (7), yields

$$\chi_\alpha(VP_i) = \sum_\lambda u_{\alpha\lambda}{}^i \phi_\lambda^{(a)}(V), \tag{8}$$

where the $u_{\alpha\lambda}{}^i$ are rational integers. If $b < a$ then $u_{\alpha\lambda}{}^i = 0$ for every λ, and if $a \leqslant b$ then $u_{\alpha\lambda}{}^i = 0$ for $\lambda \not\subset B^{(a)}$. Let $D = (d_{\alpha\lambda})$ be the decomposition matrix of S_n. Then

$$\chi_\alpha(V) = \sum_\lambda d_{\alpha\lambda} \phi_\lambda(V) \qquad (V \text{ in } S_n,\ p\text{-regular}). \tag{9}$$

Hence, for $P_0 = 1$, we have

(10) $$u_{\alpha\lambda}^{0} = d_{\alpha\lambda}.$$

We arrange these numbers $u_{\alpha\lambda}^{i}$ for a fixed i in the form of a matrix

(11) $$U^{i} = (u_{\alpha\lambda}^{i}),$$

with α as row index and λ as column index, and set

(12) $$U = (U^0, U^1, \ldots, U^{r-1}).$$

Each column of U is given by a pair (i, λ). It follows from (5) that the number of such columns is $k(n)$ (note that the number of elements P_i of weight a is $k(a)$), whence U is a square matrix of the same degree as the matrix $Z = (\chi_\alpha(G))$ of the group characters χ_α of S_n. According to (8) we have the formula

(13) $$Z = UA.$$

Here A is a square matrix such that

(14) $$A = \begin{bmatrix} \Phi^{(0)} & & & & 0 \\ & \Phi^{(1)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \cdot & \\ 0 & & & & \Phi^{(t)} \end{bmatrix},$$

where, for each a, the matrix $\Phi^{(a)} = (\phi_\lambda^{(a)}(V))$ of the modular group characters of S_{n-ap} appears in the main diagonal with multiplicity $k(a)$ if the rows and columns are arranged suitably. Since Z is non-singular, so is U:

(15) $$|U| \neq 0.$$

Proof of Theorem 1. It follows from (8) that, if the rows and columns of U are taken in a suitable order, U breaks up completely into q matrices $U_1, U_2, \ldots, U_q$, each U_k corresponding to a block B_k of S_n. Denote by x_k the number of ordinary irreducible characters in B_k. It follows from $|U| \neq 0$ that each U-matrix U_k of B_k must necessarily be a square matrix of degree x_k and $|U_k| \neq 0$. Let B_k be a block of weight b with p-core $[\alpha_0]$. We then have $x_k = l(b)$. Denote by $f(a)$ the number of modular irreducible characters in a block of weight a with p-core $[\alpha_0]$. Since U_k is a square matrix of degree $l(b)$ we have by (8)

(16) $$l(b) = \sum_{a=0}^{b} f(a)\, k(b-a).$$

Since $l^*(0) = f(0) = 1$ and $l^*(1) = f(1) = p - 1$, we shall assume that $l^*(a) = f(a)$ for $a < b$. We then have by (**12**; Lemma 1)

$$\begin{aligned} f(b) &= l(b) - \sum_{a=0}^{b-1} f(a)\, k(b-a) \\ &= l(b) - \sum_{a=0}^{b-1} l^*(a)\, k(b-a) = l^*(b). \end{aligned}$$

This completes the proof.

2. In what follows we shall be concerned with representations belonging to a fixed block B_k of weight b, so we may drop the subscript k. Applying (8) to the orthogonality relations

$$\sum_\alpha \chi_\alpha(VP_i)\ \chi_\alpha(V'P_j) = 0 \qquad (i \neq j),$$

we obtain

$$\sum_\alpha \chi_\alpha(VP_i)\ \chi_\alpha(V'P_j) = 0 \qquad [\alpha] \subset B, \quad (i \neq j), \tag{17}$$

whence

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(V'P_j) = 0 \qquad [\alpha] \subset B, \quad (i \neq j). \tag{18}$$

We then have

$$\sum_\alpha u_{\alpha\lambda}{}^i u_{\alpha\kappa}{}^j = 0 \qquad [\alpha] \subset B, \quad (i \neq j). \tag{19}$$

For $P_j = P_0 = 1$, it follows from (18) that

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(V) = 0 \qquad [\alpha] \subset B, \quad (i \neq 0), \tag{20}$$

where V is any p-regular element of S_n. Hence

$$\sum_\alpha u_{\alpha\lambda}{}^i d_{\alpha\kappa} = 0 \qquad [\alpha] \subset B, \quad (i \neq 0). \tag{21}$$

Since the U-matrix U_k of B is non-singular the identities (21) are linearly independent. Moreover the number of identities (21) is $l(b) - l^*(b)$ and hence the system of linearly independent identities (21) satisfied by the rows of the decomposition matrix D_k of B is complete.

We shall denote by $n(G)$ the order of the normalizer $N(G)$ of G in S_n. Applying (8) to the orthogonality relations

$$\sum_\alpha \chi_\alpha(VP_i)\ \chi_\alpha(VP_i) = n(VP_i),$$

we have

$$\sum_\lambda \Big(\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(VP_i)\Big)\ \phi_\lambda^{(a)}(V) = n(VP_i).$$

Let $\eta_\lambda{}^{(a)}$ be the character of the indecomposable constituent of the regular representation of S_{n-ap} which corresponds to $\phi_\lambda{}^{(a)}$. Then we have the character relation

$$\sum_\lambda \eta_\lambda^{(a)}(V)\ \phi_\lambda^{(a)}(V) = n^{(a)}(V),$$

where $n^{(a)}(V)$ denotes the order of the normalizer of V in S_{n-ap}. Hence

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(VP_i) = \frac{n(VP_i)}{n^{(a)}(V)}\ \eta_\lambda^{(a)}(V), \qquad [\alpha] \subset B. \tag{22}$$

If P_i is an element of weight a with $n - ap$ 1-cycles, k_1 p-cycles, k_2 $2p$-cycles, $\ldots$, k_m mp-cycles, then (22) yields

$$(23)\qquad \sum_{\alpha} u_{\alpha\lambda}{}^{i} u_{\alpha\kappa}{}^{i} = \frac{n(VP_i)}{n^{(a)}(V)}\, c_{\lambda\kappa}^{(a)} = c_{\lambda\kappa}^{(a)} \prod_i (k_i!(ip)^{k_i}), \qquad [\alpha] \subset B,$$

where the $c_{\lambda\kappa}{}^{(a)}$ denote the Cartan invariants of S_{n-ap}.

3. Let $[\alpha]$ with p-core $[\alpha_0]$ belong to a block B of weight b and let $[\alpha]^*$ be its star diagram (**14**; also **4**; **11**; **17**). We shall write

$$[\alpha]^* = [\nu_0]\cdot[\nu_1]\cdot \ldots \cdot[\nu_{p-1}],$$

where the $[\nu_r]$ are the disjoint right constituents of $[\alpha]^*$. We assume that $[\nu_r]$ contains b_r nodes, where

$$(24)\qquad b = b_0 + b_1 + \ldots + b_{p-1},$$

and r is the leg length of the p-hook represented by its upper left-hand corner node. We denote by χ_{α^*} the character of (reducible) representation $[\alpha]^*$ of S_b corresponding to the star diagram $[\alpha]^*$ and by $f_\alpha{}^*$ its degree. Then

$$(25)\qquad f_\alpha^* = \frac{b!}{b_0!\,b_1!\ldots b_{p-1}!} f_{\nu_0} f_{\nu_1} \ldots f_{\nu_{p-1}},$$

where f_{ν_r} denotes the degree of the ordinary irreducible representation $[\nu_r]$ of S_{b_r} (**14**).

If P_b represents the product of b cycles, each of length p, on the last bp of n symbols, then P_b is of weight b and of type $(1, 1, \ldots, 1)$. Denote by $N(P_b)$ the normalizer of P_b in S_n. We then have $N(P_b) = \mathfrak{G}_1 \times \mathfrak{G}_2$, where $\mathfrak{G}_1$ is the subgroup of S_n which permutes only the first $n - bp$ symbols and which may be identified with S_{n-bp}. On the other hand

$$(26)\qquad \mathfrak{G}_2 = S_b{}^*\mathfrak{Q}, \quad S_b{}^* \cap \mathfrak{Q} = 1,$$

where $\mathfrak{Q}$ is the subgroup generated by the b individual cycles of length p of P_b and is the normal subgroup of $\mathfrak{G}_2$, and $S_b{}^*$ is the subgroup of permutations which permute the cycles of P_b amongst themselves. We see that $S_b{}^*$ is isomorphic to the symmetric group S_b of b symbols. We denote by W the element of S_b which corresponds to W^* of $S_b{}^*$. The transitive subgroup $\mathfrak{G}_2$ of S_n is called the *generalized symmetric group* and is denoted by $S(b, p)$. The order of $S(b, p)$ is $b!p^b$. It may be verified that there are $l(b)$ conjugate classes of $S(b, p)$. For example we shall determine the conjugate classes of $S(2, 3)$. We set

$$Q_1 = (1\ 2\ 3), \qquad Q_2 = (4\ 5\ 6).$$

Then there exist two conjugate classes which are represented by

$$W_0{}^* = 1, \qquad W_1{}^* = (1\ 4)(2\ 5)(3\ 6).$$

A complete system of representatives for the conjugate classes of $S(2, 3)$ is given by

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$$W_0^*,\ W_1^*,\ Q_1,\ Q_1^2,\ Q_1Q_2,\ Q_1Q_2^2,\ Q_1^2Q_2^2,\ W_1^*Q_1,\ W_1^*Q_1^2.$$

Each element is associated with a star diagram with 2 nodes by the following way:

$$\begin{array}{rll}
W_0^* = & 1 & [1^2]\cdot[0]\cdot[0] \\
W_1^* = & (1\ 4)(2\ 5)(3\ 6) & [2]\cdot[0]\cdot[0] \\
Q_1Q_2 = & (1\ 2\ 3)(4\ 5\ 6) & [0]\cdot[1^2]\cdot[0] \\
W_1^*Q_1 = & (1\ 4\ 2\ 5\ 3\ 6) & [0]\cdot[2]\cdot[0] \\
Q_1^2Q_2^2 = & (1\ 3\ 2)(4\ 6\ 5) & [0]\cdot[0]\cdot[1^2] \\
W_1^*Q_1^2 = & (1\ 4\ 3\ 6\ 2\ 5) & [0]\cdot[0]\cdot[2] \\
Q_1 = & (1\ 2\ 3) & [1]\cdot[1]\cdot[0] \\
Q_1^2 = & (1\ 3\ 2) & [1]\cdot[0]\cdot[1] \\
Q_1Q_2^2 = & (1\ 2\ 3)(4\ 6\ 5) & [0]\cdot[1]\cdot[1].
\end{array}$$

By the same way each conjugate class of $S(b, p)$ is uniquely associated with a star diagram with b nodes. Every conjugate class of $S(b, p)$ associated with $[\alpha]^*$ such that $[\nu_0] = [0]$ contains the elements of weight b. But the converse is not valid generally.

THEOREM 2. *The number of ordinary irreducible representations of $S(b, p)$ is $l(b)$ and there is a* (1-1) *correspondence between ordinary irreducible representations of $S(b, p)$ and star diagrams $[\alpha]^*$ containing b nodes.*

This, together with related theorems, will be proved in a forthcoming paper **(13a)**.

We denote by ζ_{α^*} the ordinary irreducible characters of $S(b, p)$ corresponding to a star diagram $[\alpha]^*$. Let VP be an element of S_n such that P is an element of type $(a_1, a_2, \ldots, a_s)$ and of weight b $(b = \sum a_i)$ and V is any permutation on the fixed symbols of P, and let W be an element of S_b with a_1-cycle, a_2-cycle, $\ldots$, a_s-cycle. We have by (6)

$$\chi_\alpha(VP) = h(\alpha, \alpha_0)\, \chi_{\alpha_0}(V). \tag{27}$$

Since $h(\alpha, \alpha_0)$ is determined by $(a_1, a_2, \ldots, a_s)$, we may set $h(\alpha, \alpha_0) = u(W)$. We then have by Thrall and Robinson **(18**; **14**; also cf. **6)**

$$u(W) = \sigma_\alpha\, \chi_{\alpha^*}(W), \tag{28}$$

where $\sigma_\alpha = \pm 1$ is the product of the parities of the b hooks of length p of $[\alpha]$. On the other hand we can prove that

$$\chi_{\alpha^*}(W) = \zeta_{\alpha^*}(W^*), \qquad W^* \in S_b^*. \tag{29}$$

Thus we may denote without confusion by $\chi_{\alpha^*}(G^*)$, $G^* \in S(b, p)$, the character of the ordinary irreducible representation of $S(b, p)$ corresponding to $[\alpha]^*$.

Let W_i ($i = 0, 1, 2, \ldots, k(b) - 1$) be a complete system of representatives for conjugate classes of S_b. If we denote by $n^*(W_i^*)$ the order of the normalizer $N^*(W_i^*)$ of W_i^* in $S(b, p)$ then it follows from (19) and (23) that

$$\sum_{\alpha^*} \chi_{\alpha^*}(W_i^*)\, \chi_{\alpha^*}(W_j^*) = \delta_{ij} n^*(W_i^*).$$

Evidently these relations are the orthogonality relations for the characters of $S(b, p)$.

4. Let V be any p-regular element of S_n and let W^* be any element of S_b^*. We have by (20)

$$\sum_{\alpha} \chi_{\alpha^*}(W^*)\, \chi_\alpha(V) = 0, \qquad [\alpha] \subset B. \tag{30}$$

It was shown in **(2)** that $S(b, p)$ possesses only one p-block. If we denote by

$$D^* = (d^*_{\alpha\lambda})$$

the decomposition matrix of $S(b, p)$, then (30) yields:

$$\sum_{\alpha} \sigma_\alpha\, d_{\alpha\kappa}\, \chi_{\alpha^*}(W^*) = 0, \qquad [\alpha] \subset B, \tag{31}$$

$$\sum_{\alpha} \sigma_\alpha\, d^*_{\alpha\lambda}\, \chi_\alpha(V) = 0, \qquad [\alpha] \subset B, \tag{32}$$

and hence

$$\sum_{\alpha} \sigma_\alpha\, d_{\alpha\kappa}\, d^*_{\alpha\lambda} = 0, \qquad [\alpha] \subset B. \tag{33}$$

Moreover we have the following

THEOREM 3. *Let B be a p-block of weight b and let $G = VP$ be an element of S_n such that P is any element of weight a different from b and V is any p-regular permutation on the fixed symbols of P. Then for any element $W^* \in S_b^*$,*

$$\sum_{\alpha} \sigma_\alpha\, \chi_\alpha(G)\, \chi_{\alpha^*}(W^*) = 0, \qquad [\alpha] \subset B.$$

This follows immediately from (19).

We obtain the generalization of the Murnaghan-Nakayama recursion formula for the character χ_{α^*} of $S(b, p)$ and this yields

THEOREM 4. *Let B be a p-block of weight b and let S be any element of $S(b, p)$ associated with a star diagram $[\beta]^* = [\lambda_0]\cdot[\lambda_1]\cdot \ldots \cdot[\lambda_{p-1}]$ such that $[\lambda_0] \neq [0]$. Then*

$$\sum_{\alpha} \sigma_\alpha\, \chi_\alpha(V)\, \chi_{\alpha^*}(S) = 0 \qquad (V \text{ in } S_n,\ p\text{-regular}).$$

Let R be any element of $S(b, p)$ associated with a star diagram $[\beta]^*$ such that $[\lambda_0] = [0]$. The number of conjugate classes of $S(b, p)$ which contain the element R defined above is $l^*(b)$. We denote by $R_1, R_2, \ldots, R_{l^*(b)}$ the representatives for these classes.

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THEOREM 5. *Let $D = (d_{\alpha\lambda})$ be the decomposition matrix of a p-block B of weight b. Then*

$$d_{\alpha\lambda} = \sigma_\alpha \sum_{\kappa=1}^{l^*(b)} v_{\kappa\lambda}\, \chi_{\alpha^*}(R_\kappa), \qquad \text{for } [\alpha] \subset B,$$

where the $v_{\kappa\lambda}$ are complex numbers and are independent of α.

COROLLARY. *Let $D = (d_{\alpha\lambda})$ and $D' = (d'_{\alpha'\lambda})$* with $[\alpha]^* = [\alpha']^*$ *be the decomposition matrices of p-blocks B and B' of same weight respectively. Then*

$$d'_{\alpha'\nu} = \sigma_\alpha\, \sigma_{\alpha'} \sum_{\lambda=1}^{l^*(b)} w_{\nu\lambda}\, d_{\alpha\lambda}, \qquad \text{for } [\alpha'] \subset B',$$

where the $w_{\nu\lambda}$ are rational integers and $|w_{\nu\lambda}| = \pm 1$.

Consequently we have

THEOREM 6. *Two matrices of Cartan invariants corresponding to the p-blocks of same weight have the same elementary divisors.*

Example. The following is the U-matrix for the 2-block B of S_6 with 2-core [0].

[6]	1	0	0	1	0	1	1	1	1	1
[5, 1]	1	1	0	1	1	1	1	−1	−1	−1
[4, 2]	1	1	1	1	1	1	−1	3	1	0
$[4, 1^2]$	2	1	1	0	1	−2	0	−2	0	1
$[3^2]$	1	0	1	1	0	1	−1	−3	−1	0
$[2^3]$	1	0	1	−1	0	1	1	3	−1	0
$[3, 1^3]$	2	1	1	0	−1	−2	0	2	0	−1
$[2^2, 1^2]$	1	1	1	−1	−1	1	1	−3	1	0
$[2, 1^4]$	1	1	0	−1	−1	1	−1	1	−1	1
$[1^6]$	1	0	0	−1	0	1	−1	−1	1	−1

The matrix occupying the first three columns of this U-matrix is the decomposition matrix of B and the matrix occupying the last three columns is the matrix $(\sigma_\alpha\, \chi_{\alpha^*}\, (W_i^*))$ of $S(3, 2)$. We set

$$Q_1 = (1\ 2),\quad Q_2 = (3\ 4),\quad Q_3 = (5\ 6),\quad P = (1\ 2)(3\ 4)(5\ 6).$$

Then

$$W_0^* = 1,\quad W_1^* = (1\ 3)(2\ 4),\quad W_2^* = (1\ 3\ 5)(2\ 4\ 6),$$

$$Q_1,\quad W_1^*Q_3,\quad Q_1Q_2,\quad W_1^*Q_1,\quad P,\quad W_1^*Q_1Q_3,\quad W_2^*Q_1$$

form a complete system of representatives for conjugate classes of $S(3, 2)$. We then obtain easily Table I, showing the group characters χ_{α^*} of $S(3, 2)$ (cf. **5**, p. 275).

TABLE I

class	$[1^3]\cdot[0]$	$[2,1]\cdot[0]$	$[3]\cdot[0]$	$[1^2]\cdot[1]$	$[2]\cdot[1]$	$[1]\cdot[1^2]$	$[1]\cdot[2]$	$[0]\cdot[1^3]$	$[0]\cdot[2,1]$	$[0]\cdot[3]$
element	1	(13)(24)	(135)(246)	(12)	(13)(24)(56)	(12)(34)	(1324)	(12)(34)(56)	(1324)(56)	(135246)
order	1	6	8	3	6	3	6	1	6	8
$[3]\cdot[0]$	1	1	1	1	1	1	1	1	1	1
$[0]\cdot[3]$	1	1	1	−1	−1	1	−1	−1	1	−1
$[2]\cdot[1]$	3	1	0	1	−1	−1	1	−3	−1	0
$[2,1]\cdot[0]$	2	0	−1	2	0	2	0	2	0	−1
$[1]\cdot[2]$	3	1	0	−1	1	−1	−1	3	−1	0
$[1^2]\cdot[1]$	3	−1	0	1	1	−1	−1	−3	1	0
$[0]\cdot[2,1]$	2	0	−1	−2	0	2	0	−2	0	1
$[1]\cdot[1^2]$	3	−1	0	−1	−1	−1	1	3	1	0
$[1^3]\cdot[0]$	1	−1	1	1	−1	1	−1	1	−1	1
$[0]\cdot[1^3]$	1	−1	1	−1	1	1	1	−1	−1	−1

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The decomposition matrix D^* and the matrix C^* of Cartan invariants of $S(3, 2)$ are given by

$$D^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad C^* = \begin{bmatrix} 8 & 4 \\ 4 & 6 \end{bmatrix}.$$

The following are the D-matrices $(d_{\alpha\lambda})$ and $(d'_{\alpha'\lambda})$ for the 2-block of S_6 with 2-core [0] and the 2-block of S_7 with 2-core [1] respectively:

$$\begin{matrix} [6] \\ [5, 1] \\ [4, 2] \\ [4, 1^2] \\ [3^2] \\ [2^3] \\ [3, 1^3] \\ [2^2, 1^2] \\ [2, 1^4] \\ [1^6] \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} [7] \\ [4, 2, 1] \\ [5, 1^2] \\ [5, 2] \\ [3^2, 1] \\ [3, 2^2] \\ [2^2, 1^3] \\ [3, 1^4] \\ [3, 2, 1^2] \\ [1^7] \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We see from the table of the group characters χ_{α^*} of $S(3, 2)$ that

$$(d_{\alpha\lambda}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -3 & -1 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 0 \\ \frac{4}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

There exists the following relation between $(d'_{\alpha'\lambda})$ and $(\sigma_\alpha \sigma_{\alpha'} d_{\alpha\lambda})$:

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$$(d'_{\alpha'\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \\ -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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