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PRODUCT MAPS AND COUNTABLE PARACOMPACTNESS

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1. Introduction. In all that follows, we let S denote the space $\{0, 1, 1/2, \ldots, 1/n, \ldots\}$ with the relative usual topology and $i : S \to S$ denote the identity map on S. In this note, by a *map* or *mapping* we always mean a continuous surjection. A map $f : X \to Y$ is said to be *hereditarily quotient* if $y \in \operatorname{int} f(V)$ whenever V is open in X and $f^{-1}(y) \subset V$. E. Michael has defined a map $f : X \to Y$ to be *bi-quotient* if whenever $\mathscr{V}$ is a collection of open sets in X which covers $f^{-1}(y)$, there exists finitely many $f(V)$, with $V \in \mathscr{V}$, which cover some neighbourhood of y. In [**10**, Theorem 1.3], Michael has shown that if f is a map onto a Hausdorff space, then f is bi-quotient if and only if $f \times i_Z$ is a quotient map for every space Z, where i_Z denotes the identity map on Z.

In [**13**], P. Zenor showed that the continuous closed image of a countably paracompact Hausdorff space need not be countably paracompact. The problem of the preservation of countable paracompactness under closed maps takes on importance in view of the following result, essentially due to Zenor: Let X be a countably paracompact space and Z denote the discrete space of integers; then, X is normal if and only if every continuous closed image of $X \times Z$ is countably paracompact. This follows from Theorem 3.2 and the proof of Lemma 2.8 of [**13**].

In § 2 we shall study the problem of the preservation of countable paracompactness under a closed map f via the product map $f \times i$. In particular, it is shown that if $f : X \to Y$ is a closed map, X is countably paracompact, and $f \times i$ is hereditarily quotient, then Y is countably paracompact. Numerous corollaries follow from this result.

2. Countable paracompactness. If X is a space, then in $X \times S$ let $X_0 = X \times \{0\}$ and $X_n = X \times \{1/n\}$ for $n = 1, 2, \ldots$. If A is a subset of $X \times S$, we let $A_n = A \cap X_n$ for $n = 0, 1, 2, \ldots$.

In the following lemma, (a) implies (b) is due to C. H. Dowker [**1**]. The equivalence of (a) and (e) is due to T. Ishikawa [**7**], or see [**2**, p. 178].

LEMMA 1. *The following are equivalent for a topological space X.*
(a) *X is countably paracompact.*
(b) *If Y is compact, then $X \times Y$ is countably paracompact.*
(c) *$X \times S$ is countably paracompact.*

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(d) *If A is closed in $X \times S$ and $A_0 = \emptyset$, then there exists an open set H in $X \times S$ with $A \subset H$ and $(\mathrm{cl}(H))_0 = \emptyset$.*

(e) *If $\{F(n)\}$ is a decreasing sequence of closed subsets of X with a void intersection, then there exists a sequence $\{G(n)\}$ of open subsets of X with $F(n) \subset G(n)$ for all n and the closures of the sets $G(n)$ have a void intersection.*

Proof. From the remarks above, it is clear that we need only show that (c) implies (d) and that (d) implies (e) in order to establish the lemma.

Assume that (c) holds and let A be a closed subset of $X \times S$ with $A_0 = \emptyset$. Then $\mathscr{G} = \{(X \times S) - A, X_1, X_2, \ldots\}$ is a countable, open cover of $X \times S$. Let $\{H(n) : n = 0, 1, 2, \ldots\}$ be an open, locally finite refinement of $\mathscr{G}$ such that $H(0) \subset (X \times S) - A$ and $H(n) \subset X_n$ for $n = 1, 2, \ldots$. Let $H = \bigcup \{H(n) : n = 1, 2, \ldots\}$. Then $A \subset H$ and $\mathrm{cl}(H)$ contains no points of X_0, that is, (d) holds.

Now assume that (d) holds. Let $\{F(n)\}$ be a decreasing sequence of closed subsets of X such that $\bigcap F(n) = \emptyset$. Let

$$A = \bigcup \{F(n) \times \{1/n\} : n = 1, 2, \ldots\}.$$

A is closed in $X \times S$ and $A_0 = \emptyset$. Let H be open in $X \times S$ with $A \subset H$ and $(\mathrm{cl}(H))_0 = \emptyset$. Let $\pi : X \times S \to X$ denote the natural coordinate projection and let $G(n) = \pi(H_n)$. Since S is compact, $\mathrm{cl}(G(n)) = \pi(\mathrm{cl}(H_n))$. Consequently, $\bigcap \mathrm{cl}(G(n)) = \emptyset$; since $F(n) \subset G(n)$ for $n = 1, 2, \ldots$, (e) holds, completing the proof.

Using the equivalence of (a) and (d) in Lemma 1, we may now establish the following:

THEOREM 2. *Let $f : X \to Y$ be a closed map from a countably paracompact space X onto a space Y. If $f \times i$ is hereditarily quotient, then Y is countably paracompact.*

Proof. Let A be a closed subset of $Y \times S$ such that $A_0 = \emptyset$. Let $B = (f \times i)^{-1}[A]$. Since B is closed in $X \times S$ and $B_0 = \emptyset$, by Lemma 1 there exists an open subset H of $X \times S$ with $B \subset H$ and $(\mathrm{cl}(H))_0$. Let $G = (Y \times S) - (f \times i)[(X \times S) - H]$. Since f is closed, G is open in $Y \times S$. Moreover, $(f \times i)^{-1}[G] \subset H$. Let $y \in Y$. Since $f \times i$ is hereditarily quotient, there exists an open subset V of $X \times S$ with

$$f^{-1}(y) \times \{0\} \subset V, \quad V \cap \mathrm{cl}(H) = \emptyset, \quad \text{and} \quad (y, 0) \in \mathrm{int}((f \times i)[V]).$$

Then $(f \times i)[V] \cap G = \emptyset$, so that $(\mathrm{cl}(G))_0 = \emptyset$. Since $A \subset G$, Y is countably paracompact by Lemma 1, completing the proof.

The converse of Theorem 2 is not true. The following is an example, suggested to me by Peter Harley, which is similar to Example 7.4 of [**4**]. Let X denote the Euclidean plane and Y be the indentification space obtained from X by identifying the x-axis to a point p. Let $f : X \to Y$ be the identification map.

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f is a closed map and X is, of course, countably paracompact. The space $Y \times S$ is paracompact, but $f \times i$ fails to be hereditarily quotient because of the point $(p, 0)$ in $Y \times S$.

An arbitrary product of bi-quotient maps is bi-quotient [**10**]. Since bi-quotient maps are clearly hereditarily quotient, we have the following:

COROLLARY 3. *The continuous, closed, bi-quotient image of a countably paracompact space is countably paracompact.*

In particular, open maps are bi-quotient, so that countable paracompactness is preserved under an open and closed map.

In [**13**], Zenor showed that a first countable space which is the image of a countably paracompact T_1-space under a closed map is countably paracompact. In [**5**], P. Harley generalized this result by replacing first countability by the q-spaces of E. Michael, [**9**]. Corollary 4 is closely related to these results.

X is a *Frechét space* if whenever $A \subset X$ and $x \in \mathrm{cl}(A)$, there exists a sequence in A which converges to x. A quotient map onto a Hausdorff Frechét space is necessarily hereditarily quotient, as seen in the proof of Proposition 2.3 of [**3**]. In [**12**], G. T. Whyburn introduced a new class of topological spaces called accessibility spaces, and proved the following: Let Y be a T_1-space; Y is an accessibility space if and only if every quotient map onto Y is hereditarily quotient. Consequently, we have the following:

COROLLARY 4. *Let $f : X \to Y$ be a closed map from a countably paracompact space X onto a space Y. If $Y \times S$ is either Frechét Hausdorff space or an accessibility space, then Y is countably paracompact.*

Unfortunately, the product of a Frechét Hausdorff space and the space S may fail to be an accessibility space, as seen by the example just following Theorem 2.

A map is said to be *proper* if it is closed and point inverses are compact, i.e., bi-compact. A closed map having countably compact point inverses is said to be *quasi-proper*. Countable paracompactness is preserved both ways under proper maps [**6**, Theorem 2.2]. The quasi-proper image of a countably paracompact space is countably paracompact [**8**, Corollary 3.4.1]. More completely, we have the following:

COROLLARY 5. *Let $f : X \to Y$ be a quasi-proper map. Then X is countably paracompact if and only if Y is countably paracompact.*

Proof. $f \times i$ is closed by Corollary 3.5 of [**11**]. If X is countably paracompact, then Y is countably paracompact by Theorem 2. Now assume that Y is countably paracompact. Let A be closed in $X \times S$ with $A_0 = \emptyset$. Let $B = (f \times i)[A]$. Then B is closed in $Y \times S$ and $B_0 = \emptyset$. By Lemma 1, there exists an open set H in $Y \times S$ with $B \subset H$ and $(\mathrm{cl}(H))_0 = \emptyset$. Let $G = (f \times i)^{-1}[H]$. Then, $A \subset G$ and $(\mathrm{cl}(G))_0 = \emptyset$. The countable paracompactness of X now follows by Lemma 1.

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