Proceedings of the Edinburgh Mathematical Society (1982) 25, 87–95 ©

THE WEAK-STAR CLOSURE OF THE UNIT BALL IN A SUBSPACE

by G. J. O. JAMESON

(Received 1st October 1980)

Introduction

Let X be a (real or complex) Banach space, and let K be a linear subspace of its dual X^*. Denote by K_1 the unit ball in K. If K is not weak-star closed, then the Krein-Šmul'yan theorem says that K_1 is not weak-star closed. What, however, is its weak-star closure? Inner and outer extimates were obtained in [3] for the special case where K is a hyperplane. In the present paper we generalise these estimates to arbitrary linear subspaces. For f to belong to $w^*(K_1)$ it is sufficient to have $|\varphi(f)| \leqq 1-\|f\|$ for all φ in K^0 (the annihilator of K in X^{**}) with $\operatorname{dist}(\varphi, X) \leqq 1$. It is necessary to have $|\varphi(f)| \leqq 1+\|f\|$ for all such φ. These estimates depend on the action of each φ in K^0 separately, which will often make them hard to apply in practice; in both cases, we derive a second estimate, expressed only in terms of certain constants that describe the relative position of X and K^0 in X^{**}.

When the pair (X, K^0) has "property (G)" (which happens automatically for hyperplanes if X is complete), $w^*(K_1)$ always contains a ball in $w^*(K)$. By contrast, when this property fails, $w^*(K_1)$ is contained in a proper linear subspace L of $w^*(K)$, and does not even contain a ball in L.

The outer estimates can be improved by the removal of the $\|f\|$ term under a certain condition that holds, in particular, when X and X^* have bases with unconditional constant 1. For the special case $X = c_0$ we can do better still and show that the "inner estimate" is the exact answer. However, examples show that in general our estimates are as close as we can expect to get using descriptions of this type.

Notation

Throughout the paper, X will denote a normed linear space (not necessarily complete) and K, K_1 will be as above. We regard X as a subspace of X^{**}. Accordingly, the notations $f(x)$ and $x(f)$ will be used interchangeably when $x \in X$ and $f \in X^*$. If A is a subset of X (or X^*), then A^0 denotes its polar in X^* (or X^{**} respectively). The closed unit ball of X is denoted by U, so that U^0 and U^{00} are the unit balls of X^* and X^{**}. If B is a subset of X^*, then B_0 denotes $B^0 \cap X$, the polar of B in X. Finally, $w^*(B)$ denotes the weak-star closure of B.

Inner estimates

We start with a straightforward generalisation of Theorem 1 of [**3**]. The proof is nearly the same, but it is very short and contains ideas required constantly, so we give it in full.

Proposition 1. *The following condition is sufficient for an element f of X^* to belong to $w^*(K_1)$:*

$$|\varphi(f)| \leqq (1-\|f\|) \operatorname{dist}(\varphi, X)$$

for all $\varphi \in K^0$.

Proof. Note that $w^*(K_1)=[(K_1)_0]^0$. It follows at once from the Hahn-Banach theorem, by extending the restriction of a functional to K, that

$$(K_1)^0 = K^0 + U^{00}.$$

Thus an arbitrary element x of $(K_1)_0$ is expressible as $\varphi+\psi$, where $\varphi \in K^0$ and $\|\psi\| \leqq 1$. Then $\operatorname{dist}(\varphi, X) \leqq 1$, so by hypothesis $|\varphi(f)| \leqq 1-\|f\|$. Hence

$$\begin{aligned} |f(x)| &= |\varphi(f)+\psi(f)| \\ &\leqq |\varphi(f)|+\|f\| \\ &\leqq 1. \end{aligned}$$

To give inner estimates that do not depend on the action of each $\varphi \in K^0$ separately, we need the following concept. Let A, B be linear subspaces of some normed linear space X (with unit ball U). In the terminology of [**4**], the pair (A, B) has "property (G, ρ)" if each element x of $A+B$ is expressible as $a+b$, with $a \in A$, $b \in B$ and $\|a\|+\|b\| \leqq \rho\|x\|$. The pair has "property (G)" if this holds for some ρ. When X is complete and A, B are closed, this is equivalent to $A+B$ being closed. For our present purposes, ρ is of less interest than the two related constants defined unsymmetrically as follows. We say that the *ordered* pair (A, B) has "property (G_1, α)" if

$$(A+B) \cap U \subseteq (A \cap \alpha U)+B,$$

in other words, if each x in $A+B$ is expressible as above with the norm condition replaced by $\|a\| \leqq \alpha\|x\|$. Similarly, we say that (A, B) has "property (G_2, β)" if

$$(A+B) \cap U \subseteq A+(B \cap \beta U).$$

Clearly, property (G) implies that these conditions hold for some α, β, and property (G_1, α) implies property $(\mathrm{G}_2, \alpha+1)$.

Proposition 2. *Suppose that the pair (X, K^0) has property (G_1, α) and property (G_2, β). Then $w^*(K_1)$ contains*

$$\{f \in w^*(K): \|f\| \leqq \alpha^{-1}\}$$

and

$$\{f \in w^*(K): \|f\|+\beta \operatorname{dist}(f, K) \leqq 1\}.$$

In particular, if (X, K^0) has property (G), then $w^*(K_1)$ contains a ball in $w^*(K)$.

Proof. Let x, φ, ψ be as in Proposition 1. By hypothesis, $\psi(=x-\varphi)$ is expressible as $x_1-\varphi_1$, where $x_1 \in X$, $\varphi_1 \in K^0$ and $\|\varphi_1\| \leqq \beta$. Then

$$x-x_1=\varphi-\varphi_1 \in X \cap K^0 = K_0.$$

If $f \in w^*(K)$, it follows that $f(x)=f(x_1)$. Now

$$|\varphi_1(f)| \leqq \beta \operatorname{dist}(f, K),$$

so

$$|f(x)| = |\varphi_1(f)+\psi(f)| \leqq \beta \operatorname{dist}(f, K)+\|f\|.$$

This proves the second statement. An obvious modification gives the proof of the first one.

If α_0 is the infimum of the α for which (X, K^0) has property (G_1, α), then of course we can replace α by α_0 in the conclusion. However, (X, K^0) need not have property (G_1, α_0). It is easily seen that α_0 is the norm of the natural projection of $(X+K^0)/(X \cap K^0)$ onto the image of X.

The second estimate in Proposition 2 has the advantage that, unlike the first one, it at least contains K_1 itself. The first estimate was actually given in **[4]**, where it was also shown that the converse applies: if $w^*(K_1)$ contains the α^{-1}-ball in $w^*(K)$, then (X, K^0) has property (G_1, α') for all $\alpha' > \alpha$.

Outer estimates

Of course, $w^*(K_1)$ is contained in $w^*(K)$ and in U^0.

Proposition 3. *If f is in $w^*(K_1)$, then*
(i) $|\varphi(f)| \leqq (1+\|f\|) \operatorname{dist}(\varphi, X)$ *for all* $\varphi \in K^0$,
(ii) $\operatorname{dist}(f, K) \leqq r(1+\|f\|)$,
where $r=\sup\{\operatorname{dist}(\varphi, X)\colon \varphi \in K^0 \cap U^{00}\}$.
(Note: (ii) only says something when $r<\frac{1}{2}$.)

Proof. If f satisfies (i), then it satisfies (ii), since there exists φ in K^0 with $\|\varphi\|=1$ and $\varphi(f)=\operatorname{dist}(f, K)$.

Suppose, then, that f is in U^0 but does not satisfy (i): we show that f is not in $w^*(K_1)$. For some $\varphi \in K^0$, we have

$$|\varphi(f)| = \rho(1+\|f\|)+3\varepsilon,$$

where $\varepsilon>0$ and $\rho=\operatorname{dist}(\varphi, X)$. There exists x_0 in X with $\|\varphi-x_0\| \leqq \rho+\varepsilon$. Then

$$|(\varphi-x_0)(f)| \leqq (\rho+\varepsilon)\|f\| \leqq \rho\|f\|+\varepsilon,$$

so $|f(x_0)| \geqq \rho+2\varepsilon$.

If g is in K_1, then $\varphi(g)=0$, so

$$|g(x_0)| = |(x_0-\varphi)(g)| \leqq \rho+\varepsilon.$$

Hence $\{h \in X^*\colon |(h-f)(x_0)|<\varepsilon\}$ is a weak-star neighbourhood of f that does not meet K_1.

https://doi.org/10.1017/S0013091500004181 Published online by Cambridge University Press

When specialised to hyperplanes, this is a slight improvement of Theorem 2 of [**3**], in that 2 has been replaced by $1+\|f\|$.

Let L be the linear subspace of X^* generated by $w^*(K_1)$, that is, $\bigcup_{n\in N} nw^*(K_1)$. By Propositions 1 and 3, we have:

Corollary. *L is the set of f for which there exists M such that*

$$|\varphi(f)| \leqq M \operatorname{dist}(\varphi, X) \quad \textit{for all} \quad \varphi \in K^0.$$

Proof. If the condition holds, then Proposition 1 shows that $\delta f \in w^*(K_1)$, where $\delta = (M+\|f\|)^{-1}$. Conversely, if δf is in $w^*(K_1)$, then Proposition 3 shows that $|\varphi(f)| \leqq 2\delta^{-1} \operatorname{dist}(\varphi, X)$ for $\varphi \in K^0$.

If X is complete and separable, then L is the set of points that are weak-star limits of sequences in K. Of course, when (X, K^0) has property (G), L is the whole of $w^*(K)$.

We now introduce a condition that makes it possible to remove the term $\|f\|$ in Proposition 3. Denote by $K(X)$ the set of compact linear mappings of X into itself. We say that X^* satisfies "condition (A^*)" if, given $\varepsilon > 0$ and two elements f, g of X^*, there exists P in $K(X)$ such that, with $Q = I - P$, we have

$$\|Q^*f\| \leqq \varepsilon, \quad \|Q^*g\| \leqq \varepsilon,$$
$$\|Q\| \leqq 1.$$

This condition is satisfied if X has a shrinking basis with $\|I-P_n\| \leqq 1$ for all n, where (P_n) is the associated sequence of projections (in particular, if X has a shrinking basis with unconditional constant 1). If X^* has the bounded approximation property with constant 1, then of course the condition is satisfied except that one has $\|P\| \leqq 1$ (and hence only $\|Q\| \leqq 2$): we shall see below that this is a significant difference. The way we use condition (A^*) is through the following lemma:

Lemma 1. *Suppose that X^* satisfies condition (A^*). Let $x_0 \in X$, $f_0 \in X^*$ and $\varepsilon > 0$ be given. Then there exists P in $K(X)$ such that, with $Q = I - P$, we have*

$$\|Qx_0\| \leqq \varepsilon, \quad \|Q^*f_0\| \leqq \varepsilon, \quad \|Q\| \leqq 1.$$

Proof. Suppose that the condition fails for a certain x_0, f_0, ε, with $\|x_0\| \leqq 1$. Let R be the set of linear mappings Q: $X \to X$ such that $I - Q$ is compact, $\|Q\| \leqq 1$ and $\|Q^*f_0\| \leqq \varepsilon$. If R is empty, then condition (A^*) certainly fails. In any case, R is convex and, by hypothesis, $\|Qx_0\| > \varepsilon$ for all $Q \in R$. Hence there is an element g_0 of X^* such that

$$|g_0(Qx_0)| = |(Q^*g_0)(x_0)| > \varepsilon$$

for all $Q \in R$. Thus condition (A^*) fails for f_0, g_0, ε.

Proposition 4. *If X^* satisfies condition (A^*) and f is in $w^*(K_1)$, then:*

(i) $|\varphi(f)| \leqq \operatorname{dist}(\varphi, X)$ *for all* $\varphi \in K^0$,

(ii) $\operatorname{dist}(f, K) \leqq r$,

where $r = \sup\{\operatorname{dist}(\varphi, X)\colon \varphi \in K^0 \cap U^{00}\}$.

Proof. Again, (i) implies (ii). Suppose that f is in U^0 but does not satisfy (i), so that for some $\varphi \in K^0$ (with $\|\varphi\| = 1$), we have $|\varphi(f)| = \rho + 5\varepsilon$, where $\varepsilon > 0$ and $\rho = \text{dist}(\varphi, X)$. There exists x_0 in X with $\|\varphi - x_0\| \leqq \rho + \varepsilon$. Let P be as in Lemma 1 (for x_0, f, ε). Then

$$|\varphi(P^*f)| \geqq \rho + 4\varepsilon.$$

Since P is compact, $P^*|_{U^0}$ is continuous as a mapping from U^0 with the weak-star topology into X^* with the norm topology. Hence there is a weak-star neighbourhood W of f such that if g is in $W \cap U^0$, then $\|P^*g - P^*f\| \leqq \varepsilon$, and hence

$$|\varphi(P^*g)| \geqq \rho + 3\varepsilon \tag{1}$$

Now suppose that g is in K_1. Then $\varphi(g) = 0$, so

$$|\varphi(P^*g)| = |\varphi(Q^*g)| \tag{2}$$

Now $|Q^*g)(x_0)| = |g(Qx_0)| \leqq \varepsilon$, and since $\|\varphi - x_0\| \leqq \rho + \varepsilon$ and $\|Q^*\| \leqq 1$, we have

$$|(\varphi - x_0)(Q^*g)| \leqq \rho + \varepsilon.$$

Hence $|\varphi(Q^*g)| \leqq \rho + 2\varepsilon$, which contradicts (1) and (2).

It is actually sufficient if P is *weakly* compact, since P^* is then continuous with respect to the weak-star and weak topologies.

The case $X = c_0$

By a modification of the proof of Proposition 4, we show that in this case the inner estimate of Proposition 1 is exact. The new feature is that the P in condition (A^*) can be chosen so that

$$\|g\| = \|P^*g\| + \|Q^*g\|$$

for $g \in X^*$ (we simply take P to be a suitable truncation).

Proposition 5. *Let $X = c_0$. Then $w^*(K_1)$ is precisely the set of f in X^* such that*

$$|\varphi(f)| \leqq (1 - \|f\|)\,\text{dist}(\varphi, X) \quad \textit{for all} \quad \varphi \in K^0.$$

Proof. Suppose that f does not satisfy this condition, so that for some $\varepsilon > 0$ and some $\varphi \in K^0$ (with $\|\varphi\| = 1$), we have

$$|\varphi(f)| + \rho\|f\| = \rho + 6\varepsilon,$$

where $\rho = \text{dist}(\varphi, X)$. Take x_0 and P as in Proposition 4. Then

$$|\varphi(P^*f)| + \rho\|P^*f\| \geqq \rho + 4\varepsilon.$$

Hence there is a weak-star neighbourhood W of f such that if g is in $W \cap U^0$, then

$$|\varphi(P^*g)| + \rho\|P^*g\| \geqq \rho + 3\varepsilon. \tag{1'}$$

Now suppose that g is in K_1. The reasoning of Proposition 4 shows in fact that

$$|\varphi(P^*g)| = |\varphi(Q^*g)| \leqq \varepsilon + (\rho + \varepsilon)\,\|Q^*g\|,$$

https://doi.org/10.1017/S0013091500004181 Published online by Cambridge University Press

so that

$$\begin{aligned}|\varphi(P^*g)|+\rho\,\|P^*g\| &\leqq \varepsilon+(\rho+\varepsilon)(\|Q^*g\|+\|P^*g\|)\\ &=\varepsilon+(\rho+\varepsilon)\,\|g\|\\ &\leqq \rho+2\varepsilon,\end{aligned}$$

contrary to (1′).

Notes. This applies also when X is a c_0-product of finite-dimensional spaces. It will aid the understanding of this characterisation to establish what it amounts to in a particular case. We do so for a class of subspaces that includes an example mentioned in [**1**], ch. IV, §5, ex. 14] in connection with weak-star closures.

Example 1. Let the natural numbers be partitioned into disjoint, infinite sets A_1, $A_2, \ldots$. For each n, let z_n be an element of m with $\text{dist}\,(z_n, c_0)=1$, having non-zero values only on A_n. Regarding z_n as a functional on l_1, let

$$K=\bigcap_{n\in N}\ker z_n.$$

For $z\in m$, let $P_n z$ be the element obtained by changing the values of z to 0 off A_n. Then K^0 is the set of $z\in m$ such that for each n, $P_n z=\lambda_n z_n$ for some λ_n. For such an element, $\text{dist}\,(z, c_0)=\sup|\lambda_n|$ and $z(y)=\sum\lambda_n z_n(y)$ for $y\in l_1$. Note that $K^0\cap X=\{0\}$, so that $w^*(K)=X^*$. One deduces easily that

$$w^*(K_1)=\left\{y:\|y\|+\sum_{n=1}^{\infty}|z_n(y)|\leqq 1\right\}.$$

Hence the linear subspace L generated by $w^*(K_1)$ is the set of y such that $\sum|z_n(y)|$ is convergent. In particular, L contains each basis element e_n, so is norm-dense in $w^*(K)$ $(=X^*)$.

Let a_n be the first element of A_n. The specific example of [**1**] is obtained by letting z_n take the value n at a_n and 1 on the rest of A_n. Then (X, K^0) does not have property (G), and L is not the whole of X^*: for instance, it does not contain the element that takes the value n^{-2} at a_n and 0 elsewhere. We shall see below that this is not accidental.

Further discussion of outer estimates and condition (A*)

An example was given in [**3**] in which $w^*(K_1)$ contains an element f such that $\varphi(f)=2\,\text{dist}\,(\varphi, X)$ for some $\varphi\in K^0$, showing that we cannot do without condition (A^*) in Proposition 4. This was done with X equal to l_1. The same happens when X is $C(S)$:

Example 2. Let $X=C(S)$, where S is compact and infinite. Write $\delta_s(x)=x(s)$, where $s\in S$ and $x\in X$. Choose a non-isolated point s_0 of S, and write δ_0 for δ_{s_0}. Let K be the linear subspace of X^* generated by the elements δ_s for $s\neq s_0$. Clearly, δ_0 is in the weak-star closure of the set of such elements, so $\delta_0\in w^*(K_1)$. Let X^*_+ denote the set of non-negative functionals in X^*. It is easily verified that

$$\text{dist}\,(\delta_0+X^*_+, K)=1.$$

Hence there is a positive element φ of X^{**} that is zero on K and satisfies $\|\varphi\| = \varphi(\delta_0) = 1$. In the ordering of X^{**}, we have $0 \leqq \varphi \leqq e$, where e is the function taking constant value 1 on S. So

$$\text{dist}\,(\varphi, X) \leqq \|\varphi - \tfrac{1}{2}e\| \leqq \tfrac{1}{2}.$$

Note that this applies, in particular, to l_1 regarded as the dual of c, in strong contrast to its nice behaviour as the dual of c_0. Of course, quite different weak-star topologies are induced on l_1 by c_0 and c, neither containing the other.

It is quite easy to see directly why condition (A*) (indeed, a weaker condition) fails for these spaces. Let (B*) be the variant of (A*) obtained by having only *one* element of X^* given. Also, say that X satisfies "condition (B)" if, given $\varepsilon > 0$ and $x_0 \in X$, there exists P in $K(X)$ such that $\|Q\| \leqq 1$ and $\|Qx_0\| \leqq \varepsilon$, where $Q = I - P$. Clearly, if a dual space X^* satisfies (B*), then it satisfies (B). The proof of Lemma 1 shows that if X^* satisfies (B*), then X satisfies (B).

Lemma 2. *Suppose that X satisfies condition (B). Let E be a subset of the unit ball of X, and x_0 a point of the weak closure of E. Then* $\text{dist}\,(x_0, E) \leqq 1$.

Proof. Take $\varepsilon > 0$, and let P be as in condition (B) (for x_0, ε). By the weak-to-norm continuity of $P|_U$, the point Px_0 is in the norm-closure of $P(E)$. Hence $\text{dist}\,(x_0, P(E)) \leqq \varepsilon$. For y in E, $\|Qy\| \leqq 1$ and

$$x_0 - y = (x_0 - Py) - Qy.$$

It follows that $\text{dist}\,(x_0, E) \leqq 1 + \varepsilon$.

A similar statement holds for condition (B*) and weak-star closure.

In the space m, let e denote the sequence with 1 in every place and e_n the sequence with 1 in place n and 0 elsewhere. Then $e_n \to 0$ weakly. Let $f_n = e - 2e_n$. Then $\|f_n\| = 1$, $f_n \to e$ weakly and $\|f_n - e\| = 2$ for all n. Hence m (and also c) fails condition (B). One can easily give a similar example in $C[0, 1]$.

The case where property (G) fails

We have seen that when (X, K^0) has property (G), $w^*(K_1)$ contains a ball in $w^*(K)$. We show now that when this property fails, $w^*(K_1)$ generates a proper linear subspace L of $w^*(K)$ and (if X is complete) does not contain a ball in L. Our proof of the second statement is achieved by a careful dissection of Pryce's direct proof of the Krein-Šmul'yan theorem (for a sketch, see [**2**], p. 50; the idea seems to have originated in [**5**], p. 111–2).

Write $(K_1)_0 = C$, so that $w^*(K_1) = C^0$ and L is the set of elements of X^* that are bounded on C. Note also that $K_0 = \bigcap\{\varepsilon C\colon \varepsilon > 0\}$.

Lemma 3. *If there exists ρ such that* $\text{dist}\,(x, K_0) \leqq \rho$ *for all $x \in C$, then (X, K^0) has property* (G).

Proof. Take any $\rho' > \rho$. Let $\psi = x + \varphi$, where $x \in X$, $\varphi \in K^0$ and $\|\psi\| \leqq 1$. Then x belongs to C. By hypothesis, there exists $y \in K_0$ with $\|x - y\| \leqq \rho'$. Then

$$\psi = (x - y) + (\varphi + y),$$

in which $x - y \in X$, $\varphi + y \in K^0$ and $\|x - y\| \leqq \rho'$. Hence (X, K^0) has property (G).

Note. It follows at once from the hypothesis of Lemma 3, simply by taking polars, that $w^*(K_1)$ contains $w^*(K)\cap(\rho^{-1}U^0)$. This is the final step of Pryce's proof.

Proposition 6. *If* (X, K^0) *does not have property* (G), *then* L *is a proper linear subspace of* $w^*(K)$.

Proof. For $x\in X$, let $\hat{x}$ be the functional on $w^*(K)$ defined by $\hat{x}(f)=f(x)$. Then $\|\hat{x}\|=\text{dist}\,(x, K_0)$. By Lemma 3, $\{\hat{x}: x\in C\}$ is unbounded. By the uniform boundedness theorem, there is an element f of $w^*(K)$ such that $f(C)$ is unbounded (so $f\notin L$).

Proposition 7. *Suppose that* X *is complete and that* (X, K^0) *does not have property* (G). *Then* $w^*(K_1)$ *does not contain* $L\cap(\varepsilon U^0)$ *for any* $\varepsilon>0$.

Proof. Suppose that $w^*(K_1)(=C^0)$ contains $L\cap(\varepsilon U^0)$. This contains the polar of $\frac{1}{2}C+\varepsilon^{-1}U$, so C is contained in $\frac{1}{2}C+\alpha U$ for any $\alpha>\varepsilon^{-1}$. We show that this implies the hypothesis of Lemma 3, and hence property (G).

Choose $x\in C$. There exists $y_1\in\frac{1}{2}C$ with $\|x-y_1\|\leqq\alpha$. Repeating, one obtains a sequence (y_n) with $y_n\in 2^{-n}C$ and

$$\|y_n-y_{n-1}\|\leqq 2^{-n}\alpha$$

for all n. Then (y_n) is a Cauchy sequence, and the limit y belongs to $2^{-n}C$ for all n, and hence to K_0. Clearly, $\|x-y\|\leqq 2\alpha$.

Corollary. *Under these conditions,* L *is not norm-closed in* X^*.

Proof. This follows, by Baire's theorem.

If X is incomplete, then L can be norm-closed. For instance, let φ be an element of $X^{**}\backslash X$ with $\text{dist}\,(\varphi, X)=0$, and let $K=\ker\varphi$. It follows from Proposition 3 (and is well-known) that K_1 is then weak-star closed, so that $L=K$.

We finish with an example to show that even when X is complete, L need not be norm-dense in $w^*(K)$.

Example 3. Consider the specific version of Example 1. Note that for $y\in K$,

$$\|y\|\geqq\sum_{n=1}^{\infty} n\,|y(a_n)| \tag{1}$$

(here $y(r)$ denotes the rth term of y). Define

$$z_0(y)=\sum_{n=1}^{\infty} y(a_n),$$

and let $K'=K\cap(\ker z_0)$. It is a straightforward exercise to show that K' separates points of c_0, so that $w^*(K')=X^*$. However, $w^*(K_1')$ is contained in $\ker z_0$. To show this, let y_0 be such that $z_0(y_0)\neq 0$. Write $|z_0(y_0)|=3\delta$. Take N such that $N\delta>1$ and

$$\left|\sum_{n=1}^{N} y_0(a_n)\right|>2\delta.$$

There is a weak-star neighbourhood W of y_0 such that if $y \in W$, then

$$\left|\sum_{n=1}^{N} y(a_n)\right| > \delta.$$

If y is in K', then $z_0(y) = 0$, so

$$\left|\sum_{n+1}^{\infty} y(a_n)\right| > \delta.$$

By (1), it follows that $\|y\| \geqq N\delta > 1$. Hence W does not meet K_1'.

REFERENCES

1. N. Bourbaki, *Espaces Vectoriels Topologiques* (Hermann, Paris, 1955).

2. M. M. Day, *Normed Linear Spaces*, 3rd ed. (Springer, Berlin, 1973).

3. G. J. O. Jameson, The weak-star closure of the unit ball in a hyperplane, *Proc. Edinburgh Math. Soc.* (2) **18** (1972), 7–11.

4. G. J. O. Jameson, The duality of pairs of wedges, *Proc. London Math. Soc.* (3) **24** (1972), 531–547.

5. A. P. Robertson and W. Robertson, *Topological Vector Spaces* (Cambridge Univ. Press, 1964).

Department of Mathematics
University of Lancaster