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ON EXPANDING LOCALLY FINITE COLLECTIONS

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Introduction. A space X is *m-expandable*, where m is an infinite cardinal, if for every locally finite collection $\{H_\alpha | \alpha \in A\}$ of subsets of X with $|A| \leqq m$ (cardinality of $A \leqq m$) there exists a locally finite collection of open subsets $\{G_\alpha | \alpha \in A\}$ such that $H_\alpha \subseteq G_\alpha$ for every $\alpha \in A$. X is *expandable* if it is m-expandable for every cardinal m. The notion of expandability is closely related to that of collectionwise normality introduced by Bing [**1**]. X is *collectionwise normal* if for every discrete collection of subsets $\{H_\alpha | \alpha \in A\}$ there is a discrete collection of open subsets $\{G_\alpha | \alpha \in A\}$ such that $H_\alpha \subseteq G_\alpha$ for every $\alpha \in A$. Expandable spaces share many of the properties possessed by collectionwise normal spaces. For example, an expandable developable space is metrizable and an expandable metacompact space is paracompact.

In § 2 we study the relationship of expandability with various covering properties and obtain some characterizations of paracompactness involving expandability. It is shown that $\aleph_0$-expandability is equivalent to countable paracompactness. In § 3, countably perfect maps are studied in relation to expandability and various product theorems are obtained. Section 4 deals with subspaces and various sum theorems. Examples comprise § 5.

Definitions of terms not defined here can be found in [**1**; **5**; **16**].

1. Expandability and collectionwise normality. An expandable space need not be regular (Example 5.6), and a completely regular expandable space need not be normal (W in Example 5.3). However, it is not difficult to show that a normal expandable space is collectionwise normal. In fact, we have the following theorem.

THEOREM 1.1 (Katětov [**12**]). *A T_1-space X is normal and expandable if and only if it is collectionwise normal and countably paracompact.*

It is not known whether a collectionwise normal Hausdorff space is countably paracompact. Using Theorem 1.1 one can see that this is equivalent to asking whether a collectionwise normal Hausdorff space is expandable.

In [**1**] Bing showed that collectionwise normality lies strictly between paracompactness and normality and that a collectionwise normal developable space is metrizable. A space X is *developable* if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers such that, for any $x \in X$ and any open set U containing x, there is an integer n such that $\mathrm{St}(x, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n | x \in G\} \subseteq U$. A regular

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developable space is called a *Moore* space. Analogously we have the following theorem.

THEOREM 1.2. *An expandable developable Hausdorff space is metrizable.*

Proof. We will show later (Corollary 2.9.1) that an expandable F_σ-screenable space is paracompact. Since a developable space is F_σ-screenable [**1**], our result follows.

Theorem 1.2 is a generalization of a result of Borges [**2**, Corollary 2.15] (see Theorem 3.9).

A well-known problem in topology is whether a normal Moore space is metrizable. If Theorem 1.2 is true with expandability replaced by $\aleph_0$-expandability, then the problem would be solved since a normal Moore space is $\aleph_0$-expandable.

Question. Is an $\aleph_1$-expandable Moore space metrizable?

2. Relation to covering properties.

Definition 2.1 (Morita [**20**]). Let m be an infinite cardinal. A space X is *m-paracompact* if every open cover of cardinality $\leqq m$ has a locally finite open refinement.

Remark 2.2. Every paracompact space is m-paracompact and $\aleph_0$-paracompactness is just countable paracompactness.

Remark 2.3. It is clear that X is m-expandable if and only if for every locally finite collection of *closed* subsets $\{F_\alpha | \alpha \in A\}$ with $|A| \leqq m$ there exists a locally finite collection of open sets $\{G_\alpha | \alpha \in A\}$ such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.

THEOREM 2.4. *If X is m-paracompact, then X is m-expandable.*

Proof. Let $\mathscr{F} = \{F_\alpha | \alpha \in A\}$ be a locally finite collection of closed subsets of the m-paracompact space X with $|A| \leqq m$. Let Γ be the collection of all finite subsets of A and define

$$V_\gamma = X - \bigcup\{F_\alpha | \alpha \notin \gamma\}, \qquad \gamma \in \Gamma.$$

Now V_γ is open, V_γ meets only finitely many elements of $\mathscr{F}$, and $\{V_\gamma | \gamma \in \Gamma\}$ covers X. Since $|\Gamma| \leqq m$, there is a locally finite open refinement

$$\mathscr{W} = \{W_\delta | \delta \in \Delta\}.$$

Set

$$U_\alpha = \mathrm{St}(F_\alpha, \mathscr{W}) = \bigcup\{W_\delta \in \mathscr{W} | W_\delta \cap F_\alpha \neq \emptyset\}, \qquad \alpha \in A.$$

Clearly $F_\alpha \subseteq U_\alpha$ and U_α is open for each $\alpha \in A$. We claim that $\{U_\alpha | \alpha \in A\}$ is locally finite. Each $x \in X$ belongs to an open set 0 which meets only finitely many members of $\mathscr{W}$. Thus $0 \cap U_\alpha \neq \emptyset$ if and only if $0 \cap W_\delta \neq \emptyset$ and

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$W_\delta \cap F_\alpha \neq \emptyset$ for some $\delta \in \Delta$. But W_δ, since it is contained in some V_γ, meets only finitely many F_α. Thus $\{U_\alpha | \alpha \in A\}$ is locally finite.

COROLLARY 2.4.1. *If X is paracompact, then X is expandable.*

The converse to Theorem 2.4 is false. In fact, there is an expandable normal Hausdorff space which is not $\aleph_1$-paracompact (Example 5.1). However, one has the following theorem.

THEOREM 2.5. X is *$\aleph_0$-expandable if and only if X is countably paracompact.*

Proof. One implication follows from Theorem 2.4. The other follows from [**15**, Theorem 3, the proof that (i) $\Rightarrow$ (a)].

COROLLARY 2.5.1. *An expandable space is countably paracompact.*

There is a normal countably paracompact Hausdorff space which is not $\aleph_1$-expandable (Example 5.2).

Mansfield [**15**] proved Theorem 2.5 under the assumption that X was normal. We have the following analogue of Theorem 2.5.

THEOREM 2.6. *X is countably metacompact if and only if for every locally finite countable collection $\{F_i | i = 1, 2, \ldots\}$ of closed subsets of X there is a point-finite collection of open subsets $\{G_i | i = 1, 2, \ldots\}$ such that $F_i \subseteq G_i$ for each i.*

Proof. A simple modification of the proofs of Theorems 2.4 and 2.5.

It is clear that a space would be expandable if it had the property that every locally finite collection is finite. Thus we state the following theorem and its corollary.

THEOREM 2.7. *The following are equivalent for a space X:*

(a) *X is countably compact;*
(b) *Every locally finite collection of subsets is finite;*
(c) *Every locally finite disjoint collection of subsets is finite;*
(d) *Every locally finite countable collection of subsets is finite;*
(e) *Every locally finite countable disjoint collection of subsets is finite.*

Proof. This constitutes in proving that (e) $\Rightarrow$ (a) $\Rightarrow$ (b); both implications are easy.

COROLLARY 2.7.1. *A countably compact space is expandable.*

Call a space *semiparacompact*† if each of its open covers has a σ-locally finite open refinement. If $\mathscr{H}$ is a collection of subsets, then $\mathscr{H}^* = \bigcup\{H | H \in \mathscr{H}\}$.

Michael showed [**17**] that for regular spaces, paracompactness is equivalent to semiparacompactness.

†This terminology as well as that in Definition 3.1 were suggested by the referee.

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THEOREM 2.8. *X is paracompact if and only if X is $\aleph_0$-expandable and semiparacompact.*

Proof. Only one implication requires proof. Let $\mathscr{U} = \{U_\alpha | \alpha \in A\}$ be an open cover and $\mathscr{V} = \bigcup_{i=1}^\infty \mathscr{V}_i$ a σ-locally finite open refinement where $\mathscr{V}_i = \{V_{\alpha,i} | \alpha \in A_i\}$. $\{\mathscr{V}_i^* | i = 1, 2, \ldots\}$ is a countable open cover, and hence there is a locally finite open refinement $\{G_i | i = 1, 2, \ldots\}$, where we may assume that $G_i \subseteq \mathscr{V}_i^*$ for each i. Then $\{G_i \cap V_{\alpha,i} | \alpha \in A_i, i = 1, 2, \ldots\}$ is a locally finite open refinement of $\mathscr{U}$.

We can weaken the condition of semiparacompactness to obtain the following theorem.

THEOREM 2.9. *The following are equivalent for a* T_1*-space X:*
(a) *X is paracompact;*
(b) *X is expandable and every open cover of X has a σ-locally finite closed refinement;*
(c) *X is expandable and every open cover of X has a σ-locally finite refinement.*

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (a). Let $\mathscr{U} = \{U_\beta | \beta \in B\}$ be an open cover of X and $\mathscr{H} = \bigcup_{i=1}^\infty \mathscr{H}_i$ a σ-locally finite refinement where $\mathscr{H}_i = \{H_{\alpha,i} | \alpha \in A_i\}$ for each i. Since X is expandable, for each i there is a locally finite open collection $\{G_{\alpha,i} | \alpha \in A_i\}$ such that $H_{\alpha,i} \subseteq G_{\alpha,i}$ for every $\alpha \in A_i$. Since $\mathscr{H}$ is a refinement, each $H_{\alpha,i} \subseteq U_{\beta(\alpha,i)}$ for some $\beta(\alpha, i) \in B$. Let

$$W_{\alpha,i} = G_{\alpha,i} \cap U_{\beta(\alpha,i)}, \qquad \alpha \in A_i, \quad i = 1, 2, \ldots,$$

and let

$$\mathscr{W}_i = \{W_{\alpha,i} | \alpha \in A_i\}, \qquad i = 1, 2, \ldots.$$

Then $\mathscr{W} = \bigcup_{i=1}^\infty \mathscr{W}_i$ is a σ-locally finite open refinement, and the result follows from Theorem 2.8.

In [**16**] McAuley showed that paracompactness is equivalent to collectionwise normality plus F_σ-screenability.

COROLLARY 2.9.1. *A* T_1*-space X is paracompact if and only if X is expandable and F_σ-screenable.*

Proof. Theorem 2.9 (b).

There is an $\aleph_0$-expandable normal Hausdorff space which is F_σ-screenable but is not paracompact (Example 5.2).

The following definition is due to Worrell and Wicke [**29**].

Definition 2.10. A collection $\mathscr{W}$ of point sets is *finite at a point p* if $p \in \mathscr{W}^*$ and only a finite number of elements of $\mathscr{W}$ contain p.

A space is *θ-refinable* if for every open cover $\mathscr{U}$ of X there exists an open refinement $\mathscr{V} = \bigcup_{i=1}^\infty \mathscr{V}_i$, where

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($\theta 1$) $\mathscr{V}_i$ is an open cover of X for each i, and
($\theta 2$) For each $p \in X$ there is an integer k such that $\mathscr{V}_k$ is finite at p.
Clearly every metacompact space is θ-refinable.

Michael [**18**] and Nagami [**24**] showed that paracompactness is equivalent to collectionwise normality and metacompactness.

THEOREM 2.11. *X is paracompact if and only if X is expandable and θ-refinable.*

Proof (Modelled after Michael's proof in [**18**]). Only one implication needs proof. Let $\mathscr{W} = \{W_\alpha | \alpha \in A\}$ be an open cover of X and let $\mathscr{V} = \bigcup_{i=1}^\infty \mathscr{V}_i$ be an open refinement satisfying ($\theta 1$) and ($\theta 2$), where $\mathscr{V}_i = \{V_{\alpha,i} | \alpha \in A_i\}$, $i = 1, 2, \ldots$.

We shall construct, for each i, a sequence $\{\mathscr{G}_{i,k} | k = 0, 1, \ldots\}$ of collections of open sets such that:
(1) $\mathscr{G}_{i,k}$ is locally finite for each k,
(2) Each element of $\mathscr{G}_{i,k}$ is a subset of some element of $\mathscr{V}_i$,
(3) If $x \in X$ is an element of at most m elements of $\mathscr{V}_i$, then $x \in \bigcup_{k=0}^m \mathscr{G}_{i,k}{}^*$,
(4) Each $x \in \mathscr{G}_{i,k}{}^*$ belongs to at least k elements of $\mathscr{V}_i$.

Let $\mathscr{G}_{i,0} = \emptyset$. Suppose that $\mathscr{G}_{i,1}, \mathscr{G}_{i,2}, \ldots, \mathscr{G}_{i,n}$ were constructed and let us construct $\mathscr{G}_{i,n+1}$. Let $\mathscr{B}$ be the family of all $B \subseteq A_i$ such that B has exactly $n + 1$ elements. Define

$$Y(B) = \left(X - \bigcup_{j=0}^{n} \mathscr{G}^*_{i,j}\right) \cap (X - \bigcup\{V_{\alpha,i} \in \mathscr{V}_i | \alpha \notin B\}).$$

Clearly $Y(B)$ is closed and we claim that $\{Y(B) | B \in \mathscr{B}\}$ is locally finite (in fact, discrete).

Case 1. x belongs to $n + 1$ or more elements of $\mathscr{V}_i$. Choose $n + 1$ elements, say, with $\alpha = \alpha(1), \alpha(2), \ldots, \alpha(n + 1)$. Then $\bigcap_{j=1}^{n+1} V_{\alpha(j),i}$ is a neighbourhood of x which meets $Y(B)$ only if $B = \{\alpha(1), \alpha(2), \ldots, \alpha(n + 1)\}$.

Case 2. x belongs to less than $n + 1$ elements of $\mathscr{V}_i$. By (3), $x \in \bigcup_{j=0}^n \mathscr{G}_{i,j}{}^*$ which is disjoint from each $Y(B)$.

Then $\{Y(B) | B \in \mathscr{B}\}$ is a locally finite collection of closed sets; thus, by the expandability of X, there is a locally finite collection of open sets $\{H(B) | B \in \mathscr{B}\}$ such that $Y(B) \subseteq H(B)$ for $B \in \mathscr{B}$. Now $Y(B) \subseteq V_{\alpha,i}$ for each $\alpha \in B$. Let

$$T(B) = H(B) \cap (\bigcap\{V_{\alpha,i} | \alpha \in B\}).$$

Then $Y(B) \subseteq T(B)$ for each $B \in \mathscr{B}$. Define

$$\mathscr{G}_{i,n+1} = \{T(B) | B \in \mathscr{B}\}.$$

Then (1) follows since $T(B) \subseteq H(B)$ and $\{H(B) | B \in \mathscr{B}\}$ is locally finite. (2) and (4) follow since $T(B) \subseteq \bigcap\{V_{\alpha,i} | \alpha \in B\}$. To see (3), let $x \in X$ be such that x belongs to no more than $n + 1$ elements of $\mathscr{V}_i$. If $x \in \bigcup_{j=0}^n \mathscr{G}_{i,j}{}^*$, then the result follows. If $x \notin \bigcup_{j=0}^n \mathscr{G}_{i,j}{}^*$, then $x \in Y(B)$ for some $B \in \mathscr{B}$. Now consider $\mathscr{G} = \bigcup_{i=1}^\infty(\bigcup_{j=0}^\infty \mathscr{G}_{i,j})$. Since $\mathscr{V}$ satisfies ($\theta 2$), $\mathscr{G}$ is an open

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cover which refines $\mathscr{V}$. (1) implies that $\mathscr{G}$ is σ-locally finite, which together with expandability implies that X is paracompact by Theorem 2.8.

COROLLARY 2.11.1. *X is paracompact if and only if X is metacompact and expandable.*

COROLLARY 2.11.2. *X is m-paracompact if and only if X is m-metacompact and m-expandable.*

Proof. Making the obvious definition of m-metacompactness, we proceed as in the proof of Theorem 2.11.

Corollary 2.11.1 fails if expandability is replaced by $\aleph_0$-expandability. In fact, there is a normal $\aleph_0$-expandable metacompact Hausdorff space which is the countable union of closed paracompact subspaces but is not paracompact (Example 5.2). However, we do have the following result.

THEOREM 2.12. *If X is an expandable T_1-space and $X = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed and paracompact, then X is paracompact.*

Proof. The proof follows immediately from the following lemma and Corollary 2.9.1.

LEMMA 2.13. *If $X = \bigcup_{i=1}^{\infty} F_i$ and each F_i is closed and paracompact, then X is F_σ-screenable.*

Proof. Let $\mathscr{U} = \{U_\alpha | \alpha \in A\}$ be an open cover of X, where X satisfies the hypothesis of the lemma. Then $\mathscr{U}_i = \{U_\alpha \cap F_i | \alpha \in A\}$ is an open cover of F_i. Thus there is a σ-discrete closed (in F_i and hence in X) refinement of $\mathscr{U}_i$, namely $\mathscr{F}_i = \bigcup_{k=1}^{\infty} \mathscr{F}_{i,k}$, where $\mathscr{F}_{i,k}$ is discrete in F_i and hence in X. Thus $\mathscr{F} = \bigcup_{i=1}^{\infty}(\bigcup_{k=1}^{\infty} \mathscr{F}_{i,k})$ is a σ-discrete closed refinement of $\mathscr{U}$.

3. Mapping and product spaces. By a map we mean a continuous function. Although collectionwise normality is preserved under closed maps [**22**], closed maps do not always preserve expandability (Example 5.3). If the preimages of points are restricted, then we do preserve expandability.

Definition 3.1. A *countably perfect*†† map $f: X \to Y$ is a closed surjective map such that $f^{-1}(y)$ is countably compact for each y in Y.

First we state two lemmas without proofs.

LEMMA 3.2. *Let f be a map from X onto Y. If $\mathscr{H} = \{H_\alpha | \alpha \in A\}$ is a locally finite collection of subsets of Y, then $f^{-1}(\mathscr{H}) = \{f^{-1}(H_\alpha) | \alpha \in A\}$ is a locally finite collection in X.*

LEMMA 3.3 (Okuyama [**25**]). *Let f be a countably perfect map from a space X onto Y. If $\{F_\alpha | \alpha \in A\}$ is a locally finite collection of subsets of X, then $\{f(F_\alpha) | \alpha \in A\}$ is a locally finite collection in Y.*

††It should be noted that the term "quasi-perfect" is also used.

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THEOREM 3.4. *Let f be a countably perfect map from a space X onto Y. Then X is m-expandable if and only if Y is m-expandable.*

Proof. Suppose that X is m-expandable and $\mathscr{F} = \{F_\alpha | \alpha \in A\}$ is a locally finite collection of subsets of Y with $|A| \leqq m$. Then $f^{-1}(\mathscr{F}) = \{f^{-1}(F_\alpha) | \alpha \in A\}$ is a locally finite collection of subsets of X with $|A| \leqq m$, and so there is a locally finite collection $\{G_\alpha | \alpha \in A\}$ of open subsets of X such that $f^{-1}(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in A$. Set $V_\alpha = Y - f(X - G_\alpha)$, $\alpha \in A$. It is easy to see that $F_\alpha \subseteq V_\alpha$; we show that $\{V_\alpha | \alpha \in A\}$ is a locally finite collection of open sets. V_α is open since f is a closed map. Now $V_\alpha \subseteq f(G_\alpha)$, and Lemma 3.3 applies.

Now assume that Y is m-expandable and $\mathscr{F} = \{F_\alpha | \alpha \in A\}$ is a locally finite collection of subsets of X with $|A| \leqq m$. By Lemma 3.3, $\{f(F_\alpha) | \alpha \in A\}$ is also locally finite. Hence there is a locally finite collection of open sets $\{G_\alpha | \alpha \in A\}$ such that $f(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in A$. Then

$$F_\alpha \subseteq f^{-1}(f(F_\alpha)) \subseteq f^{-1}(G_\alpha)$$

and $\{f^{-1}(G_\alpha) | \alpha \in A\}$ is an open locally finite collection by Lemma 3.2.

COROLLARY 3.4.1. *If X is countably paracompact and f is a countably perfect map from X onto Y, then Y is countably paracompact.*

A space is *m-compact* if every open cover of cardinality $\leqq m$ has a finite subcover.

LEMMA 3.5 (Hanai [**6**]). *If Y is m-compact and X is a space such that each of its points has a neighbourhood base of power $\leqq m$, then the projection map $\pi_X: X \times Y \rightarrow X$ is a closed map.*

THEOREM 3.6. *Let m and n be infinite cardinals. If X is an m-expandable space and each of its points has a neighbourhood base of power $\leqq n$ and if Y is an n-compact space, then $X \times Y$ is m-expandable.*

Proof. By Lemma 3.5, $\pi_X: X \times Y \rightarrow X$ is a closed map, and since $\pi_X{}^{-1}(x)$ is n-compact (and hence countably compact), $X \times Y$ is m-expandable by Theorem 3.4.

COROLLARY 3.6.1. *If X is an m-expandable first countable space and Y is a countably compact space, then $X \times Y$ is m-expandable.*

COROLLARY 3.6.2. *If X is m-expandable and Y is a compact space, then $X \times Y$ is m-expandable.*

In regard to Corollaries 3.6.1 and 3.6.2, it should be mentioned that the product of expandable spaces need not be expandable (Example 5.4).

THEOREM 3.7. *Let $f: X \rightarrow Y$ be a closed map of a T_1-space X onto Y such that the boundary of $f^{-1}(y)$ is countably compact for each y in Y. If X is m-expandable, then so is Y.*

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Proof. Ishii [**10**] has shown that if f is a map from X onto Y with the stated properties, then there is a closed subset F of X and a closed map g from F onto Y such that $g^{-1}(y)$ is countably compact for each y in Y. Since m-expandability is hereditary with respect to closed subsets (Theorem 4.1), Y is m-expandable by Theorem 3.4.

Morita has introduced a class of spaces which he terms *M-spaces* [**19**]. He gave the following characterization.

THEOREM 3.8. *X is an M-space if and only if there is a countably perfect map of X onto a metric space Y.*

Using the fact that every metric space is paracompact [**28**] and thus expandable (Corollary 2.4.1), we have the following theorem.

THEOREM 3.9. *Every M-space is expandable.*

The converse to Theorem 3.9 fails (Example 5.4).

4. Subspaces and sum theorems.

THEOREM 4.1. *Each closed subset of an m-expandable space is m-expandable.*

Subspaces of a compact space need not be expandable (Example 5.5). However, we do have the following theorem.

THEOREM 4.2. *X is hereditarily m-expandable if and only if every open subset is m-expandable.*

Proof. Suppose that each open subset of X is m-expandable. Let $B \subseteq X$ and let $\mathscr{H} = \{H_\alpha | \alpha \in A\}$ be a locally finite (in B) collection of subsets of B with $|A| \leqq m$. Define

$$V = \{x \in X | x \text{ belongs to an open set which intersects only finitely many members of } \mathscr{H}\}.$$

Then V is open and $B \subseteq V$. Since V is m-expandable, there is a locally finite (in V and hence in B) collection $\{G_\alpha | \alpha \in A\}$ of open subsets of V such that $H_\alpha \subseteq G_\alpha$ for every $\alpha \in A$. $\{G_\alpha \cap B | \alpha \in A\}$ is the desired collection.

THEOREM 4.3. *Let $\mathscr{U}$ be a disjoint open cover of X. Then X is m-expandable if and only if each element of $\mathscr{U}$ is m-expandable.*

THEOREM 4.4. *Suppose that $\{F_\alpha | \alpha \in A\}$ is a locally finite closed cover of a space X. Then X is m-expandable if and only if every F_α is m-expandable.*

Proof. One implication follows from Theorem 4.1. To prove the other, let Z be the topological disjoint sum of the F_α. Then Z is m-expandable by Theorem 4.3 since each F_α is open and closed in Z. Let $f: Z \to X$ denote the natural map. Then f is a countably perfect map, and so X is m-expandable by Theorem 3.4.

Corollary 4.4.1. *If $\{F_\alpha | \alpha \in A\}$ is a locally finite closed cover of a space X such that each F_α is countably paracompact, then X is countably paracompact.*

Hodel [**9**] showed that if Q is a class of topological spaces which satisfies the following two properties:

(1) Q is hereditary with respect to closed subsets, and

(2) if $\{F_\alpha | \alpha \in A\}$ is a locally finite closed cover of X with each F_α in Q, then X is in Q,

then various sum theorems hold for Q. Since we have Theorems 4.1 and 4.4, we obtain the following two theorems.

Theorem 4.5. *Let $\mathscr{U}$ be a σ-locally finite open cover of a space X such that the closure of each element of $\mathscr{U}$ is m-expandable. Then X is m-expandable.*

Theorem 4.6. *Let X be a regular* T_1*-space. If $\mathscr{U}$ is a σ-locally finite open cover of X, each element of which is m-expandable and has compact boundary, then X is m-expandable.*

5. Examples.

Example 5.1. *An expandable completely normal Hausdorff space which is not $\aleph_1$-paracompact.*

Let Ω denote the first uncountable ordinal. Then $X = [0, \Omega)$ with the usual order topology is countably compact (hence expandable by Corollary 2.7.1) and completely normal. It is well known [**3**] that the open cover $\{[0, \alpha) | \alpha < \Omega\}$ has no open locally finite refinement.

Example 5.2. *A normal $\aleph_0$-expandable metacompact Hausdorff space which is the countable union of closed paracompact subspaces and is not $\aleph_1$-expandable.*

Let F denote the normal but not collectionwise normal space constructed by Bing in [**1**, Example G] where the underlying space P has cardinality $\aleph_1$. In [**18**] Michael proved that a certain subset of F (which he called G) is metacompact and normal but not collectionwise normal. If we let

$$B_k = F_p \cup \{f \in G | f(q) = 0 \text{ except for at most } k \text{ elements } q \text{ in } Q\}, \quad k = 1, 2, \ldots,$$

then $G = \bigcup_{k=1}^{\infty} B_k$ and each B_k is a closed discrete subspace.

The collection $\{\{f\} | f \in F_p\}$ is a locally finite collection, of cardinality $\aleph_1$, of subsets of F, and it cannot be expanded.

Example 5.3. *An expandable Hausdorff space X, a closed map $f: X \to Y$ such that Y is not expandable.*

Let $W = [0, \Omega] \times [0, \Omega)$ and $X = W \times N$, where N denotes the positive integers with the discrete topology. Since W is countably compact, X is expandable. However, Zenor [**30**] has displayed a closed map $f: X \to Y$ such that Y is not $\aleph_0$-expandable.

Example 5.4. *An expandable Hausdorff space X which is not an M-space and also such that $X \times X$ is not expandable.*

Let X be the space in Sorgenfrey's example [**27**] (the reals with the "half-open-interval" topology). Then X is paracompact and hence expandable. If X were an M-space, then $X \times X$ would be paracompact, which it is not.

In $X \times X$, the collection $\{\{(x, -x)\} | x \in X\}$ is locally finite, yet one can show by a category argument that there does not exist a locally finite collection of open sets $\{G_x | x \in X\}$ such that $(x, -x) \in G_x$ for each $x \in X$.

Example 5.5. *A compact Hausdorff space with a subspace that is not expandable.*

Let X be the Tychonoff plank, $[0, \Omega] \times [0, \omega]$, and let G be the subspace $X - \{(\Omega, \omega)\}$. Then G is not $\aleph_0$-expandable [**7**].

Example 5.6. *An expandable Hausdorff space which is not regular.*

Let $Y = [0, \Omega] \times [0, \Omega] - \{(\Omega, \Omega)\}$ and let $F = \{\Omega\} \times [0, \Omega)$. If $X = Y/F$, then X is countably compact (and hence expandable) but not regular.

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