1
Preliminaries

1.1 Notation

We denote by $\mathcal{L}^n$ the Lebesgue measure in the Euclidean n-space $\mathbb{R}^n$. In a metric space X, $d(A)$ stands for the diameter of A, $d(A, B)$ the minimal distance between the sets A and B, and $d(x, A)$ the distance from a point x to a set A. The closed ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$ and the open ball by $U(x, r)$. In $\mathbb{R}^n$ we sometimes denote $B^n(x, r)$. The unit sphere in $\mathbb{R}^n$ is S^{n-1}. The Grassmannian manifold of linear m-dimensional subspaces of $\mathbb{R}^n$ is $G(n, m)$. It is equipped with an orthogonally invariant Borel probability measure $\gamma_{n,m}$. For $V \in G(n, m)$, we denote by P_V the orthogonal projection onto V.

For $A \subset X$, we denote by $\mathcal{M}(A)$ the set of non-zero finite Borel measures μ on X with support $\operatorname{spt}\mu \subset A$. We shall denote by $f_\#\mu$ the push-forward of a measure μ under a map f: $f_\#\mu(A) = \mu(f^{-1}(A))$. The restriction of μ to a set A is defined by $\mu \llcorner A(B) = \mu(A \cap B)$. The notation $\ll$ stands for absolute continuity.

The characteristic function of a set A is χ_A. By the notation $M \lesssim N$, we mean that $M \leq CN$ for some constant C. The dependence of C should be clear from the context. The notation $M \sim N$ means that $M \lesssim N$ and $N \lesssim M$. By c and C, we mean positive constants with obvious dependence on the related parameters.

1.2 Hausdorff Measures

For $m \geq 0$, the m-dimensional Hausdorff measure $\mathcal{H}^m = \mathcal{H}^m_d$ in a metric space (X, d) is defined by

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$$\mathcal{H}^m(A) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} \alpha(m)2^{-m} d(E_i)^m : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) < \delta \right\}.$$

Then $\mathcal{H}^0$ is the counting measure. Usually m will be a positive integer and then $\alpha(m) = \mathcal{L}^m(B^m(0,1))$, from which it follows by the isodiametric inequality that $\mathcal{H}^m = \mathcal{L}^m$ in $\mathbb{R}^m$. The isodiametric inequality says that among the subsets of $\mathbb{R}^m$ with a given diameter, the ball has the largest volume; see, for example, [203, 2.10.33]. For non-integral values of m the choice of $\alpha(m)$ does not really matter. We denote by dim the Hausdorff dimension. The *spherical Hausdorff measure* $\mathcal{S}^m$ is defined in the same way but using only balls as covering sets.

The lower and upper m-densities of $A \subset X$ are defined by

$$\Theta^m_*(A, x) = \liminf_{r\to 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)),$$

$$\Theta^{*m}(A, x) = \limsup_{r\to 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)).$$

The density $\Theta^m(A, x)$ is defined as their common value if they are equal.

We have

Theorem 1.1 *If A is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$2^{-m} \leq \Theta^{*m}(A, x) \leq 1 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in A,$$

$$\Theta^{*m}(A, x) = 0 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in X \setminus A.$$

When $m \leq 1$ the constant 2^{-m} is sharp; for $m > 1$ the best constant is not known.

We also have

Theorem 1.2 *If $A \subset X$ is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$\lim_{\delta\to 0} \sup\{d(B)^{-m}\mathcal{H}^m(A \cap B) : x \in B, d(B) < \delta\} = 1 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in A.$$

For general measures, we have

Theorem 1.3 *Let $\mu \in \mathcal{M}(X)$, $A \subset X$, and $0 < \lambda < \infty$.*

(1) *If $\Theta^{*m}(A, x) \leq \lambda$ for $x \in A$, then $\mu(A) \leq 2^m \lambda \mathcal{H}^m(A)$.*
(2) *If $\Theta^{*m}(A, x) \geq \lambda$ for $x \in A$, then $\mu(A) \geq \lambda \mathcal{H}^m(A)$.*

For the above results, see [203, 2.10.17–19], [190, Section 2.2] or [321, Chapter 6].

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We say that a closed set E is AD-m-regular (AD for Ahlfors and David) if there is a positive number C such that

$$r^m/C \leq \mathcal{H}^m(E \cap B(x,r)) \leq Cr^m \text{ for } x \in E, 0 < r < d(E).$$

A measure μ is said to be AD-m-regular if

$$r^m/C \leq \mu(B(x,r)) \leq Cr^m \text{ for } x \in \operatorname{spt}\mu, 0 < r < d(\operatorname{spt}\mu),$$

which means that $\operatorname{spt}\mu$ is an AD-m-regular set.

1.3 Lipschitz Maps

Since Lipschitz maps are at the heart of rectifiability, we state here some basic well-known facts about them. We say that a map $f\colon X \to Y$ between metric spaces X and Y is *Lipschitz* if there is a positive number L such that

$$d(f(x), f(y)) \leq Ld(x,y) \text{ for } x, y \in X.$$

The smallest such L is the Lipschitz constant of f, which is denoted by $\operatorname{Lip}(f)$.

Euclidean valued Lipschitz maps $f\colon A \to \mathbb{R}^k, A \subset X$, can be extended: there is a Lipschitz map $g\colon X \to \mathbb{R}^k$ such that $g|A = f$, see [203, 2.10.43–44] or [321, Chapter 7].

Any Lipschitz map $g\colon \mathbb{R}^m \to \mathbb{R}^k$ is almost everywhere differentiable by Rademacher's theorem, see [203, 3.1.6] or [321, 7.3].

There is the Lusin type property: if $f\colon A \to \mathbb{R}^k, A \subset \mathbb{R}^m$ is Lipschitz, then for every $\varepsilon > 0$ there is a C^1 map $g\colon \mathbb{R}^m \to \mathbb{R}^k$ such that

$$\mathcal{L}^m(\{x \in A\colon g(x) \neq f(x)\}) < \varepsilon, \tag{1.1}$$

see [203, 3.1.16].

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