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Sidon Sets are Proportionally Sidon with Small Sidon Constants

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Abstract. In his seminal work on Sidon sets, Pisier found an important characterization of Sidonicity: A set is Sidon if and only if it is proportionally quasi-independent. Later, it was shown that Sidon sets were proportionally "special" Sidon in several other ways. Here, we prove that Sidon sets in torsion-free groups are proportionally n-degree independent, a higher order of independence than quasi-independence, and we use this to prove that Sidon sets are proportionally Sidon with Sidon constants arbitrarily close to one, the minimum possible value.

1 Introduction

Let G be a compact abelian group and let Γ be its discrete dual. A subset $E \subseteq \Gamma$ is called a *Sidon set* if there is a constant C such that every bounded E-function ϕ is the restriction of the Fourier Stieltjes transform of a finite measure on G of measure norm at most $C\|\phi\|_\infty$. The least such C is called the *Sidon constant* of E. Sidon sets are well known to be plentiful. Indeed, infinite examples can be found in every infinite subset of Γ and include lacunary sets (in $\Gamma = \mathbb{Z}$) and independent sets.

Sidon sets have been extensively studied, yet fundamental questions remain open. As the class of Sidon sets is closed under finite unions, it is natural to ask whether every Sidon set is the finite union of a "nicer", *i.e.,* more restricted, class of interpolation sets. Important progress on this general problem was made when Pisier [12] characterized Sidon sets as those that are "proportionally" quasi-independent (special Sidon sets that are independent-like). Later, Ramsey [15] proved that Sidon sets are proportionally I_0 (special Sidon sets where the interpolating measure can be chosen to be discrete) in a uniform sense, and subsequently one of the authors with Graham [4] showed that they are proportionally ε-Kronecker (special Sidon sets defined by an approximation property) under the assumption that Γ has no elements of finite order.

In this paper we prove that if Γ has no elements of finite order, then every Sidon set is proportionally Sidon with Sidon constants arbitrarily close to one. This will be established by generalizing Pisier's proportional quasi-independent characterization of Sidon to higher degrees of independence. Of course, every Sidon set has Sidon constant at least one, and this is the Sidon constant for an independent set in the case

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where Γ has no elements of finite order. But many groups, including $\mathbb{Z}$, have no non-trivial independent sets, or even any subsets with Sidon constant equal to one, other than sets with one or two elements.

Our result does not hold, in general, for groups that admit elements of finite order, as there are such groups with the property that the Sidon constant of every non-trivial independent set is bounded away from one. However, we do show that the proportionality result holds when Γ is a product of finite groups with prime order tending to infinity.

2 Definitions and Basic Properties

We begin by recalling some well-known equivalent definitions of Sidonicity. For proofs of these facts and other properties of Sidon sets mentioned below, we refer the reader to [5] or [9].

Definition 2.1 A subset $E \subseteq \Gamma$ is called a *Sidon set* if whenever $\phi: E \to \mathbb{C}$ is a bounded function, there is a measure μ on G, the *interpolating measure,* with the property that $\widehat{\mu}(\gamma) = \phi(\gamma)$ for every $\gamma \in E$ and $\|\mu\|_{M(G)} \leq C\|\phi\|_\infty$. The least such constant C is known as the *Sidon constant* of E. The set E is called I_0 if the measure μ can be chosen to be discrete.

Proposition 2.2 *The following are equivalent:*

(i) *E is Sidon.*

(ii) *There are constants C and $0 \leq \delta < 1$ such that for every $\phi \in \mathrm{Ball}(\ell^\infty(E))$, there is a measure μ on G with $\|\mu\|_{M(G)} \leq C$ and satisfying*

$$\sup_{\gamma\in E}\{|\phi(\gamma) - \widehat{\mu}(\gamma)|\} \leq \delta.$$

(iii) *For every $\phi: E \to \{\pm 1\}$, there is a measure μ on G such that*

$$\sup_{\gamma\in E}\{|\phi(\gamma) - \widehat{\mu}(\gamma)|\} < 1.$$

(iv) *There is a constant C such that whenever f is a trigonometric polynomial with $\operatorname{supp}\widehat{f} \subseteq E$,*

$$\sum_{\gamma\in E}|\widehat{f}(\gamma)| \leq C\sup\{|f(x)| : x \in G\}.$$

A duality argument shows that the minimal constant C satisfying (iv) is (also) the Sidon constant; see [5, Thm. 6.2.3]. An iterative argument, such as given in [5, Prop. 1.3.2], can be used to show that if item (ii) is satisfied, then the Sidon constant of E is at most $C/(1-\delta)$.

Definition 2.3 A subset $\{\gamma_j\} \subseteq \Gamma$ is called *independent* if, whenever $k \in \mathbb{N}$, $\prod_{j=1}^k \gamma_j^{n_j} = 1$ for $n_j \in \mathbb{Z}$ implies $\gamma_j^{n_j} = \mathbf{1}$ for all $j = 1, \dots, k$ (where $\mathbf{1}$ denotes the identity in Γ).

If $\phi \in \mathrm{Ball}(\ell^\infty(E))$, then $\phi = (\psi_1+\psi_2)/2$ for $\psi_j: E \to \mathbb{T}$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. If E is an independent set in a group with no elements of finite order, then there are

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elements $x_j \in G$ such that for all $\gamma \in E$, we have $\psi_j(\gamma) = \gamma(x_j)$ for $j = 1, 2$. Thus, if we let δ_x denote the point mass measure at x and put $\mu = (\delta_{x_1} + \delta_{x_2})/2$, then $\|\mu\| = 1$ and $\phi(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \in E$. Hence, E satisfies (ii) with $\delta = 0$ and $C = 1$, and therefore E has Sidon constant 1. More generally, if E is any independent set and $\phi: E \to \{\pm 1\}$, then choose x such that $\widehat{\delta_x}(\gamma) = \phi(\gamma)$ if $\gamma \in E$ has infinite order and $\widehat{\delta_x}(\gamma)$ is the n-th root of unity nearest $\phi(\gamma)$ if $\gamma \in E$ has order n. It follows from (iii) that E is Sidon.

Finite sets F are always Sidon sets with Sidon constant at most $\sqrt{|F|}$. Subsets of $\mathbb{Z}$ consisting of one or two elements have Sidon constant one, but this is never the case for subsets of three or more elements; see [11]. A classical example of an infinite Sidon set is the subset $E = \{3^n\}_{n=1}^{\infty} \subseteq \mathbb{Z}$. Indeed, given $\phi \in \ell^{\infty}(E)$ with $\|\phi\|_{\infty} = 1/2$, we can take as the interpolating measure the Riesz product measure

$$\mu = \prod_{j=1}^{\infty} \left(1 + 2\mathcal{R}(\phi(3^n) e^{i3^n x})\right),$$

where the infinite product notation means μ is the weak $*$ limit in $M(\mathbb{T})$. As $\|\mu\|_{M(\mathbb{T})} = 1$, the Sidon constant of E is at most 2. In fact, the set $\{3^n\}$, or more generally any lacunary set $\{n_j\} \subseteq \mathbb{Z}^+$ (meaning $\inf n_{j+1}/n_j = q > 1$), is an example of an I_0 set (although the Riesz product measure argument does not show this).

The class of I_0 sets is a proper subset of the Sidon sets, since the class of Sidon sets is closed under finite unions, but the class of I_0 sets is not. But I_0 sets are also plentiful, and every infinite subset of Γ contains an infinite I_0 set. It is of interest to understand the relationship between Sidon and I_0 sets since I_0 sets are known not to cluster at any continuous character in the Bohr topology, while it is unknown whether Sidon sets can (even) be dense in the Bohr compactification of Γ.

Another interesting class of Sidon sets is that of the ε-Kronecker sets: $E \subseteq \Gamma$ is ε-Kronecker if for every $\phi: E \to \mathbb{T}$ there exists $x \in G$ such that $|\phi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in E$. Any lacunary set $\{n_j\}$ with $\inf n_{j+1}/n_j > 2$ is ε-Kronecker for some $\varepsilon < 2$, and every set that is $(2 - \varepsilon)$ -Kronecker is Sidon [7]. There are examples of Sidon sets that are not $(2 - \varepsilon)$-Kronecker for some groups Γ, but it is unknown if such examples can be found in $\mathbb{Z}$.

A weakened version of independence is the following notion.

Definition 2.4 Let $n \in \mathbb{N}$. We say that $E \subseteq \Gamma$ is *n-degree independent* if whenever $k \in \mathbb{N}, \gamma_1, \dots, \gamma_k$ are distinct elements in E and $m_1, \dots, m_k$ are integers with $|m_i| \le n$, then $\prod_{i=1}^{k} \gamma_i^{m_i} = \mathbf{1}$ implies $\gamma_i^{m_i} = \mathbf{1}$ for all $i = 1, \dots, k$. A 1-degree independent set is usually called *quasi-independent*, and a 2-degree independent set is called *dissociate*.

The set E is said to be *n-length independent* if whenever $\gamma_1, \dots, \gamma_n$ are distinct elements in E and $m_1, \dots, m_n \in \{0, \pm 1\}$, then $\prod_{i=1}^{n} \gamma_i^{m_i} = \mathbf{1}$ implies $\gamma_i^{m_i} = \mathbf{1}$ for all i.

Clearly, 1-degree independence implies n-length independence, and a set is independent if and only if it is n-degree independent for every n. The set $E = \{3^n\}$ is a dissociate set, and a Riesz product construction shows that any dissociate set is Sidon. A modification of this argument can be given to show that quasi-independent sets are also Sidon.

Significant efforts have been made to characterize Sidon sets in terms of these more restricted classes of sets. Towards this end, Malliavin and Malliavin [10] showed that any Sidon set not containing the identity, in the group $\bigoplus \mathbb{Z}_p$ where p is a given prime, is a finite union of independent sets, while Bourgain [1] proved that every Sidon set $E \subseteq \Gamma \backslash \{\mathbf{1}\}$ is a finite union of n -length independent sets. However, it is unknown if every Sidon set is one of the following:

- a finite union of I_0 sets;
- a finite union of ε-Kronecker sets;
- a finite union of quasi-independent sets.

Pisier introduced probabilistic techniques to study these and related questions and obtained important "proportional" characterizations of Sidon sets (see Theorem 2.5). These characterizations inspired a number of other such characterizations and are the motivation for this paper. Here are some examples of these "proportional" characterizations.

Terminology Given two classes of sets $\mathcal{A}$, $\mathcal{B}$, we will say that $E \in \mathcal{A}$ is *proportionally* $\mathcal{B}$ if there is some constant $\delta > 0$ such that for every finite $F \subseteq E$, there is some $H \subseteq F$ such that $|H| \geq \delta |F|$ and $H \in \mathcal{B}$.

Theorem 2.5

(i) *The following are equivalent for $E \subseteq \Gamma \backslash \{\mathbf{1}\}$.*
  - (a) *E is Sidon;*
  - (b) *E is proportionally quasi-independent;*
  - (c) *There exists a constant C such that E is proportionally Sidon with Sidon constant at most C;*
  - (d) *There exists an integer M such that E is proportionally $I_0(M)$.* [1]

(ii) *If Γ has only finitely many elements that are of order 2^k for some k and E has no elements of order two, then E is Sidon if and only if E is proportionally ε-Kronecker for some $\varepsilon < \sqrt{2}$.*

The equivalence of (a)–(c) is a deep result of Pisier (see [12–14]) with later proofs given by Bourgain in [2,3]. The equivalence of (d) was shown by Ramsey in [15], while (ii) was established in [4] along with other related proportional equivalences. We also refer the reader to [5, ch. 7,9] and [8, pp. 482–499] for expositions of these results.

In this paper, we will modify Pisier's technique to prove that if Γ has no elements of finite order, then E is Sidon if and only if E is proportionally n-degree independent for each n, if and only if for every constant $C > 1$, E is proportionally Sidon with Sidon constant C. Partial results are obtained in the case that Γ has elements of finite order.

3 Proportional Sidon Subsets in Torsion-free Groups

In this section our main focus will be on torsion-free, discrete abelian groups Γ, groups that have no elements of finite order. These are the groups whose dual groups G are

[1] E is $I_0(M)$ if for every $\phi \in \mathrm{Ball}(\ell^\infty(E))$, there is a discrete measure $\mu = \sum_{j=1}^{M} a_j \delta_{x_j}$ with $|a_j| \leq 1$ and $\sup_{\gamma \in E} |\phi(\gamma) - \widehat{\mu}(\gamma)| \leq 1/2$.

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connected. We will first extend Pisier's proportional quasi-independent characterization of Sidon to n-degree independence and then use this to deduce that Sidon sets are proportionally Sidon with constants arbitrarily close to 1.

We begin with some preliminary lemmas that hold for general discrete abelian groups.

Lemma 3.1 *Suppose $E \subseteq \Gamma \setminus \{\mathbf{1}\}$ is Sidon. There is a constant K, depending only on the Sidon constant of E, such that for all finite subsets $A \subseteq E$ and real numbers $(a_\gamma)_{\gamma\in A}$, we have*

$$\int_G \exp\Big(\sum_{\gamma\in A} a_\gamma \mathcal{R}(\gamma)\Big) \le \exp\Big(K\sum_{\gamma\in A} a_\gamma^2\Big).$$

Proof This is a straightforward argument using the power series expansion of the exponential function and the well-known fact that if E is a Sidon set with Sidon constant S, then

$$\|f\|_p \le 2S\sqrt{p}\|f\|_2$$

for any integer $p \ge 2$ and trigonometric polynomial f with $\operatorname{supp}\widehat{f} \subseteq E$ ([5, Thm. 6.3.9]).

Let $A \subset E$ be a finite set. We will let $f = \sum_{\gamma\in A} \alpha_\gamma \gamma$ and let S denote the Sidon constant of E. With this notation,

$$\begin{aligned}
&\int_G \exp\Big(\sum_{\gamma\in A} \alpha_\gamma \mathcal{R}(\gamma)\Big)\\
&= \sum_{k\ge 0}\int \frac{(\sum_{\gamma\in A}\alpha_\gamma\mathcal{R}(\gamma))^k}{k!} = 1 + \sum_{k\ge 2}\int \frac{(\mathcal{R}(\sum_{\gamma\in A}\alpha_\gamma\gamma))^k}{k!}\\
&\le 1 + \sum_{k\ge 2}\int_G \frac{|f|^k}{k!} = 1 + \sum_{k\ge 2}\frac{\|f\|_k^k}{k!} \le 1 + \sum_{k\ge 2}\frac{(2S\sqrt{k}\|f\|_2)^k}{k!}\\
&= 1 + \sum_{p\ge 1}\frac{(2S\sqrt{2p}\|f\|_2)^{2p}}{(2p)!} + \sum_{p\ge 1}\frac{(2S\sqrt{2p+1}\|f\|_2)^{2p+1}}{(2p+1)!}.
\end{aligned}$$

If we let $L = \max\{2S+1, 4\}$, then since $p^p \le (2p)(2p-1)\cdots(p+1)$, we have

$$\frac{(2S\sqrt{2p}\|f\|_2)^{2p}}{(2p)!} \le \frac{(8LS^2\|f\|_2^2)^p}{(2S+1)p!}.$$

Thus,

$$1 + \sum_{p\ge 1}\frac{(2S\sqrt{2p}\|f\|_2)^{2p}}{(2p)!} \le \frac{1}{2S+1}\big(\exp(8LS^2\|f\|_2^2) + 2S\big).$$

Similarly,

$$\sum_{p\ge 1}\frac{(2S\sqrt{2p+1}\|f\|_2)^{2p+1}}{(2p+1)!} \le \frac{2S\|f\|_2}{2S+1}\big(\exp(8LS^2\|f\|_2^2) - 1\big),$$

and therefore

$$\int_G \exp\Big(\sum_{\gamma\in A}\alpha_\gamma\mathcal{R}(\gamma)\Big) \le \exp(16LS^2\|f\|_2^2) = \exp\Big(16LS^2\sum_{\gamma\in A} a_\gamma^2\Big).$$

■

Notation Given $E \subseteq \Gamma$ and $k \in \mathbb{N}$, we let $E_k = \{\gamma^k : \gamma \in E\}$.

Lemma 3.2 *Let $n \in \mathbb{N}$ and assume that Γ contains no non-trivial elements of order $\le n$. Suppose $E \subseteq \Gamma \setminus \{\mathbf{1}\}$ and E_k is Sidon for each $k = 1, \dots, n$. Then there is a constant K_n, depending only on n and the Sidon constants of the sets E_k, $k = 1, \dots, n$, such that for all $0 < \lambda < 1/n$ and finite subsets $A \subseteq E$, we have*

$$\int \prod_{\gamma \in A} \Big(1 + \lambda \sum_{k=1}^{n} \mathcal{R}(\gamma^k)\Big) \le \exp\big(K_n |A| n^3 \lambda^2\big).$$

(Here $|A|$ denotes the cardinality of the set A.)

Proof Let $A \subseteq E$ be finite. Since $|\sum_{k=1}^{n} \mathcal{R}(\gamma^k)| \le n$ and $\lambda < 1/n$, we have

$$\prod_{\gamma \in A} \Big(1 + \lambda \sum_{k=1}^{n} \mathcal{R}(\gamma^k)\Big) \le \exp\Big(\lambda \sum_{\gamma \in A} \sum_{k=1}^{n} \mathcal{R}(\gamma^k)\Big).$$

Put $A^{(n)} = \bigcup_{k=1}^{n} A_k$. We can write

$$\sum_{\gamma \in A} \sum_{k=1}^{n} \mathcal{R}(\gamma^k) = \sum_{\beta \in A^{(n)}} a_\beta \mathcal{R}(\beta).$$

Note that the coefficients a_β satisfy $0 \le a_\beta \le 2n$, since the assumption that Γ contains no elements of order $\le n$ ensures that $\mathcal{R}(\gamma^k) = \mathcal{R}(\chi^k)$ for $\gamma, \chi \in A$ and $k \le n$ only if $\gamma = \chi$ or $\overline{\chi}$.

Since a finite union of Sidon sets is Sidon with Sidon constant depending only on the Sidon constants of the individual sets and the number of sets in the union, $A^{(n)}$ is Sidon with Sidon constant depending only on that of the sets E_k and n. Thus, Lemma 3.1 and the fact that $|A^{(n)}| \le n|A|$ implies that there is a constant k_n with

$$\begin{aligned}\int \exp\Big(\lambda \sum_{\gamma \in A} \sum_{k=1}^{n} \mathcal{R}(\gamma^k)\Big) &= \int \exp\Big(\lambda \sum_{\beta \in A^{(n)}} a_\beta \mathcal{R}(\beta)\Big) \\ &\le \exp\Big(k_n \sum_{\beta \in A^{(n)}} \lambda^2 a_\beta^2\Big) \le \exp\big(4 k_n n^3 \lambda^2 |A|\big). \qquad \blacksquare\end{aligned}$$

Next, we upgrade Pisier's proportional quasi-independent characterization of Sidon to n-degree independent proportional sets. Our proof follows his strategy that can be found in [13, Thm. 2.11].

Proposition 3.3 *Let $n \in \mathbb{N}$ and assume that Γ contains no non-trivial elements of order $\le n$. Suppose $E \subseteq \Gamma \setminus \{\mathbf{1}\}$ and E_k is Sidon for each $k = 1, \dots, n$. There exists $\delta_n > 0$ such that for each finite set $F \subseteq E$ that there is a further finite subset $H \subseteq F$ that is n-degree independent and satisfies $|H| \ge \delta_n |F|$.*

Proof We will say that a finite set $A \subseteq \Gamma$ is an n-relation set if there exists $(\xi_\gamma)_{\gamma \in A} \in \{\pm 1, \dots, \pm n\}^A$ with $\prod_{\gamma \in A} \gamma^{\xi_\gamma} = 1$. The first step of the proof is to use probabilistic arguments and the Sidon assumption to show that no finite subset of E contains "too many" n-relations for any integer n. Using combinatorial arguments, we will then deduce that there must be large subsets that are complements of n-relation sets and hence are n-degree independent.

To begin, fix an integer n and for any finite subset $F \subseteq E$, let $\mathcal{C}_n(F)$ denote the cardinality of the set of n-relations of F; that is, $\mathcal{C}_n(F)$ is the cardinality of the set

$$\Big\{(\xi_\gamma)_{\gamma\in F} \in \{0,\pm1,\dots,\pm n\}^F : \prod_{\gamma\in F} \gamma^{\xi_\gamma} = 1\Big\}.$$

First, we will show that there are constants $\delta = \delta_n, \alpha = \alpha_n > 0$ such that for each finite $F \subseteq E$, there is a further subset $H \subseteq F$ with $|H| \geq \delta|F|$ and $\mathcal{C}_n(H) \leq 2\cdot 2^{\alpha|H|}$. To see this, fix such F and let $\lambda \in (0,1/n)$. Let $(\varepsilon_\gamma)_{\gamma\in F}$ be a collection of independent 0,1-valued random variables on a probability space $(\Omega,\mathbb{P})$ such that $\mathbb{P}\{\varepsilon_\gamma = 1\} = \lambda/2$. An application of Fubini's theorem, independence, and Lemma 3.2 gives

$$\begin{aligned}\mathbb{E}\int \prod_{\gamma\in F}\Big(1+\varepsilon_\gamma\sum_{k=1}^n(\gamma^k+\gamma^{-k})\Big) &= \int \mathbb{E}\prod_{\gamma\in F}\Big(1+\varepsilon_\gamma\sum_{k=1}^n(\gamma^k+\gamma^{-k})\Big)\\ &= \int \prod_{\gamma\in F}\Big(1+\lambda\sum_{k=1}^n\mathcal{R}(\gamma^k)\Big)\\ &\leq \exp\big(K_n n^3\lambda^2|F|\big).\end{aligned}$$

If we let $F(\omega) = \{\gamma\in F : \varepsilon_\gamma(\omega) = 1\}$, then

$$\mathbb{E}(\mathcal{C}_n(F(\omega))) = \mathbb{E}\int\prod_{\gamma\in F}\Big(1+\varepsilon_\gamma\sum_{k=1}^n(\gamma^k+\gamma^{-k})\Big) \leq \exp\big(K_n n^3\lambda^2|F|\big).$$

By Markov's inequality, with probability at least 1/2, we have

$$\mathcal{C}_n(F(\omega)) \leq 2\exp\big(K_n n^3\lambda^2|F|\big).$$

Notice that if $\gamma_1 \neq \gamma_2$, then $\mathbb{E}((\varepsilon_{\gamma_1} - \mathbb{E}\varepsilon_{\gamma_1})(\varepsilon_{\gamma_2} - \mathbb{E}\varepsilon_{\gamma_2})) = 0$, and thus

$$\begin{aligned}\mathbb{E}\big(|F(\omega)| - \mathbb{E}|F(\omega)|\big)^2 &= \mathbb{E}\Big(\sum_{\gamma\in F}(\varepsilon_\gamma - \mathbb{E}\varepsilon_\gamma)\Big)^2\\ &= \sum_{\gamma\in F}\mathbb{E}(\varepsilon_\gamma - \mathbb{E}\varepsilon_\gamma)^2\\ &= |F|(\lambda/2 - \lambda^2/4) \leq |F|\lambda/2.\end{aligned}$$

Also, since $\mathbb{E}(|F(\omega)|) = |F|\lambda/2$, it follows from Chebyshev's inequality that

$$\begin{aligned}\mathbb{P}\{|F(\omega)| \leq |F|\lambda/4\} &\leq \mathbb{P}\big\{\big(|F(\omega)| - \mathbb{E}|F(\omega)|\big)^2 \geq |F|^2\lambda^2/16\big\}\\ &\leq \frac{|F|\lambda/2}{|F|^2\lambda^2/16} = \frac{8}{|F|\lambda}.\end{aligned}$$

Choose $\lambda = \lambda_n > 0$ so small that $\exp(4K_n n^3\lambda) < 2$ and let $\alpha \in (0,1)$ be given by $2^\alpha = \exp(4K_n n^3\lambda)$. With this choice of λ, $\mathbb{P}\{\,|F(\omega)| > |F|\lambda/4\} > 1/2$ if $|F|$ is sufficiently large, and for any such ω,

$$2 \cdot 2^{\alpha|F(\omega)|} = 2\exp\left(4K_n n^3 \lambda |F(\omega)|\right) \geq 2\exp\left(K_n n^3 \lambda^2 |F|\right).$$

Thus, $|F(\omega)| > |F|\lambda/4$ and $\mathcal{C}_n(F(\omega)) \leq 2 \cdot 2^{\alpha|F(\omega)|}$ with positive probability.

This proves that there are constants $\delta = \lambda/4$ and $0 < \alpha < 1$ such that for any finite $F \subseteq E$ there is a subset $H = F(\omega) \subseteq E$ with $|H| \geq \delta|F|$ and $\mathcal{C}_n(H) \leq 2 \cdot 2^{\alpha|H|}$.

Given A, we let $\mathcal{M}(A)$ denote a maximal (with respect to inclusion) subset of A that is an n-relation set. The maximality ensures that $A \setminus \mathcal{M}(A)$ is an n-degree independent set. To complete the proof of the proposition, we will establish the following claim.

Claim *Given a sufficiently large finite set F satisfying $\mathcal{C}_n(F) \leq 2 \cdot 2^{\alpha|F|}$ for some $\alpha > 0$, there exists a constant $0 < \theta < 1$, depending only on α, and a subset $H_1 \subseteq F$ with $|H_1| \geq |F|/2$ and having $|\mathcal{M}(H_1)| \leq \theta|H_1|$.*

Of course, in this case, $H = H_1 \setminus \mathcal{M}(H_1)$ is an n -degree independent subset of F with cardinality at least $(1-\theta)|F|/2$.

A technical fact we will use in proving the claim is that if we let

$$s(\theta) = \left(\frac{1-\theta}{2}\right)\log_2\left(\frac{2e}{1-\theta}\right) \text{ for } \theta \in (0,1),$$

then, since $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, we have

$$\binom{n}{\frac{n(1-\theta)}{2}} \leq 2^{s(\theta)n}.$$

Assume the claim is false. Then whenever H_1 is a subset of F with $|H_1| = |F|/2$ (without loss of generality we can assume that F has an even number of elements), we must have $|\mathcal{M}(H_1)| > \theta|H_1|$.

As $\lim_{\theta\to 1} s(\theta) = 0$, we can choose θ sufficiently close to 1 such that $1 - s(\theta) > \alpha$. A combinatorial argument shows that if $H_0 \subseteq F$ and $\theta|F|/2 < |H_0| < |F|/2$, then the number of subsets $H_1 \subseteq F$ containing H_0 and having cardinality $|F|/2$ is

$$\left|\{H_1 \subseteq F : |H_1| = |F|/2, H_1 \supseteq H_0\}\right| = \binom{|F| - |H_0|}{|F|/2 - |H_0|} \leq \binom{|F|}{|F|/2 - |H_0|} \leq \binom{|F|}{|F|(1-\theta)/2} \leq 2^{s(\theta)|F|}. \tag{3.1}$$

Here the first inequality holds, because $\binom{N}{n}$ increases if n is fixed and N increases, and the second inequality holds, since $|F|/2 - |H_0| \leq |F|(1-\theta)/2 \leq |F|/2$.

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We let $\mathcal{F}$ denote the collection of all subsets $H_0 \subseteq F$ such that there exists $H_1 \subseteq F$ with $|H_1| = |F|/2$ and $\mathcal{M}(H_1) = H_0$. Of course, $\mathcal{C}_n(F) \geq |\mathcal{F}|$. Thus,

$$\begin{aligned}
\binom{|F|}{|F|/2} &= |\{H_1 \subseteq F : |H_1| = |F|/2\}| \\
&= \sum_{H_0 \subseteq F} |\{H_1 \subseteq F : |H_1| = |F|/2, \mathcal{M}(H_1) = H_0\}| \\
&\leq \sum_{H_0 \in \mathcal{F}} |\{H_1 \subseteq F : |H_1| = |F|/2, H_1 \supseteq H_0\}| \\
&\leq |\mathcal{F}| 2^{|F| s(\theta)} \leq \mathcal{C}_n(F) 2^{|F| s(\theta)},
\end{aligned}$$

where the second inequality comes from (3.1). Since $1 - s(\theta) > \alpha$, this implies

$$\mathcal{C}_n(F) \geq \binom{|F|}{|F|/2} 2^{-|F| s(\theta)} \geq c \frac{1}{\sqrt{|F|}} 2^{|F|} 2^{-|F| s(\theta)} > 2c \cdot 2^{\alpha |F|}$$

for some constant $c > 0$ if $|F|$ is sufficiently large, and that is a contradiction. ∎

Lemma 3.4 *Assume that Γ is a torsion-free group. If $E \subseteq \Gamma$ is a Sidon set, then for all positive integers n, the set $E_n = \{\gamma^n : \gamma \in E\}$ is also a Sidon set with the same Sidon constant as E.*

Proof Assume $f(x) = \sum_{\gamma \in E} a_\gamma \gamma^n(x)$ is a trigonometric polynomial with $\operatorname{supp} \widehat{f} \subseteq E_n$. Choose $x_0 \in G$ such that $|\sum_{\gamma \in E} a_\gamma \gamma(x_0)| = \| \sum a_\gamma \gamma \|_\infty$ and pick $y \in G$ such that $y^n = x_0$. (We can do this, since Γ torsion-free implies G is a divisible group.) As $\gamma^n(y) = \gamma(x_0)$,

$$|f(y)| = \Big| \sum_{\gamma \in E} a_\gamma \gamma(x_0) \Big| = \Big\| \sum_{\gamma \in E} a_\gamma \gamma \Big\|_\infty \geq \frac{1}{S} \sum_{\gamma \in E} |a_\gamma|,$$

where S is the Sidon constant of E. Hence, E_n is a Sidon set with constant at most S.

It is even easier to see that the Sidon constant of E is at most the Sidon constant of E_n, hence we have equality. ∎

We are now ready to prove our main result.

Theorem 3.5 *Assume that Γ is a torsion-free group. The following are equivalent for $E \subseteq \Gamma \backslash \{\mathbf{1}\}$.*

(i) *E is Sidon.*

(ii) *For each positive integer n, E is proportionally n-degree independent.*

(iii) *For each $\varepsilon > 0$, E is proportionally Sidon with Sidon constant at most $1 + \varepsilon$.*

Proof The fact that (ii) and (iii) each imply (i) is a consequence of Pisier's proportional characterizations Theorem 2.5.

The fact that (i) implies (ii) follows directly from the previous lemma and Proposition 3.3.

We turn now to the proof that (i) implies (iii). Fix $\varepsilon > 0$ and choose $\eta > 0$ so that $(1 - \eta)/(1 + \eta) \geq 1/(1 + \varepsilon)$. Pick n such that $|e^{2\pi i t} - 1| < \eta/2$ on $[-1/n, 1/n]$ and consider the continuous, even function $f : \mathbb{T} = [-1/2, 1/2] \to \mathbb{R}$ given by $f(x) = n - n|x|$

for $x \in [-1/n, 1/n]$ and $f(x) = 0$ otherwise. Obviously, $f \geq 0$ and $\widehat{f}(0) = \|f\|_1 = 1$. An easy calculation shows $\widehat{f}(\pm 1) \geq 1 - \eta/2$.

Select an even, real-valued trigonometric polynomial q such that $\|f - q\|_\infty < \eta/2$ and let p be the even, positive, trigonometric polynomial given by

$$p = \frac{q + \eta/2}{\widehat{q}(0) + \eta/2}. \tag{3.2}$$

This normalization ensures that $\widehat{p}(0) = 1$ and $\widehat{p}(\pm 1) \geq (1-\eta)/(1+\eta) \geq 1/(1+\varepsilon)$. Let N be the degree of p.

Since Sidon sets are proportionally n-degree independent for each n, there exists $\delta > 0$ such that each finite $F \subseteq E$ admits an $(N+1)$-degree independent subset H with $|H| \geq \delta|F|$.

We now give a Riesz product construction to bound the Sidon constant of H. Let $\phi: H \to \mathbb{C}$ with $\|\phi\|_\infty = 1/(1+\varepsilon)$. Let $u_\gamma = \phi(\gamma)/|\phi(\gamma)|$ be a complex number of modulus one, and define the trigonometric polynomial P_γ on G by

$$P_\gamma(x) = \frac{|\phi(\gamma)|}{\widehat{p}(1)} \sum_{n=-N}^{N} \widehat{p}(n)(u_\gamma \gamma(x))^n + 1 - \frac{|\phi(\gamma)|}{\widehat{p}(1)} \text{ for } x \in G.$$

Then

$$\widehat{P_\gamma}(\mathbf{1}) = \frac{|\phi(\gamma)|}{\widehat{p}(1)} \widehat{p}(0) + 1 - \frac{|\phi(\gamma)|}{\widehat{p}(1)} = 1$$

and

$$\widehat{P_\gamma}(\gamma) = \frac{|\phi(\gamma)|}{\widehat{p}(1)} \widehat{p}(1) u_\gamma = \phi(\gamma).$$

Since $|\phi(\gamma)|/\widehat{p}(1) \leq 1$, $P_\gamma \geq 0$, and therefore $\|P_\gamma\|_1 = 1$.

Let $P = \prod_{\gamma \in H} P_\gamma$. Since each P_γ is a polynomial in γ and γ^{-1} of degree N and H is $(N+1)$ -degree independent, standard arguments show that $\|P\|_1 = \widehat{P}(\mathbf{1}) = 1$ and $\widehat{P}(\gamma) = \phi(\gamma)$ for all $\gamma \in H$.

Now suppose ψ is any bounded E-function. If $\|\psi\|_\infty = 0$, we can take the zero measure as the interpolating measure. So assume otherwise and set $\phi = \psi/(\|\psi\|_\infty(1+\varepsilon))$. As $\|\phi\| = 1/(1+\varepsilon)$, we can construct P as above for ϕ. Put $Q = (1+\varepsilon)\|\psi\| P$. Then $\widehat{Q}(\gamma) = \psi(\gamma)$ for all $\gamma \in H$ and $\|Q\|_1 = (1+\varepsilon)\|\psi\|$. This proves that H is a Sidon set with Sidon constant bounded by $1+\varepsilon$, as we desired to show. ∎

Remark 3.6 (i) An antisymmetric Sidon set that has the additional property that the interpolating measure can always be chosen to be positive is called a Fatou-Zygmund set with the Fatou-Zymund constant defined in the obvious way. As the Riesz product measure P constructed in the proof of the Theorem is a positive measure, we actually have shown that any Sidon set in a torsion-free group is proportionally Fatou-Zygmund with Fatou-Zygmund constant arbitrarily close to 1.

(ii) Since finite sets have the same Sidon and I_0 constants ([6]), it also follows that E is Sidon if and only if for each $\varepsilon > 0$, E is proportionally I_0 with I_0 constant at most $1+\varepsilon$.

4 Sidon Sets in Torsion Groups

When the group Γ has elements of finite order, the situation is quite different. In [1], Bourgain proved that every Sidon set in $\Gamma = \mathbb{Z}_n^{\mathbb{N}}$, where n has no repeated prime factors, is a finite union of independent sets. His methods actually show the following.

Proposition 4.1 *Suppose $\Gamma = \bigoplus_{i=1}^{N} \mathbb{Z}_{p_i}^{\mathbb{N}}$, p_i prime and assume $p_1 = \min\{p_j\}_{j=1}^{N}$. Then any Sidon set in Γ is a finite union of sets that are $(p_1 - 1)$-degree independent.*

However, such Sidon sets are not necessarily proportionally Sidon with Sidon constants arbitrarily close to 1. Indeed, it is easy to see using Proposition 2.2(iv) that if, for example, $\Gamma = \mathbb{Z}_p^{\mathbb{N}}$ for a prime number p, then any subset of Γ consisting of two elements (even if an independent set) has Sidon constant bounded below by

$$\sup_{\alpha,\beta}\Big(\min_{\xi\ p\text{-root unity}} \frac{|\alpha| + |\beta|}{|\alpha + \beta\xi|}\Big) \geq \sec(\pi/(2p)).$$

We can, however, obtain our "small constants" proportionality result for products of cyclic groups $\mathbb{Z}_{p_i}$ where (p_i) tends to infinity.

Proposition 4.2 *Suppose $\Gamma = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i}$ where $(p_i)_i$ is a sequence of prime numbers tending to infinity. If $E \subseteq \Gamma$ is Sidon, then for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all finite $F \subseteq E$, there exists a further finite subset $H \subseteq F$ with Sidon constant bounded by $1 + \varepsilon$ and satisfying $|H| \geq \delta|F|$.*

Proof Fix $\varepsilon > 0$ and suppose that F is a finite subset of E. Let p be the polynomial defined in (3.2) and put $N = \deg p$. Choose n_0 such that $p_i > N + 1$ for all $i > n_0$. Let $\Gamma_1 = \bigoplus_{i=1}^{n_0} \mathbb{Z}_{p_i}$ and $M = |\Gamma_1|$. Choose $F_1 \subseteq F$ such that $F_1 = \gamma \cdot Y$ where $\gamma \in \Gamma_1$, $Y \subseteq \bigoplus_{i>n_0} \mathbb{Z}_{p_i}$ and $|F_1| \geq |F|/M$. Since translation preserves Sidon constants, Y is a Sidon set with constant at most that of E.

Now consider $Y_k = \{\chi^k : \chi \in Y\}$ for $k \leq N$. Since the elements of $\mathbb{Z}_{p_i}$ for $i > n_0$ have prime order exceeding N, essentially the same argument as in the proof of Lemma 3.4 shows that each Y_k is Sidon with Sidon constant the same as E.

Applying Proposition 3.3, we see that there is a constant $\delta > 0$ (depending on N) and an $(N + 1)$-degree independent set $Y_0 \subseteq Y$ such that $|Y_0| \geq \delta|Y|$. For Y_0, being $(N + 1)$-degree independent is the same as saying $\prod_{i=1}^{k} \gamma_i^{m_i} = \mathbf{1}$ for $|m_i| \leq N + 1$ only if $\gamma_i = \mathbf{1}$ for all i. That fact allows us to apply the Riesz product construction of the proof of Theorem 3.5 (with the polynomial p), and, as in that proof, we deduce that the Sidon constant of Y_0 is at most $1 + \varepsilon$. Of course, this is also a bound on the Sidon constant of $H = \gamma \cdot Y_0$ and this subset of F has cardinality at least $(\delta/M)|F|$, completing the proof. ∎

References

[1] J. Bourgain, *Propriétés de décomposition pour les ensembles de Sidon*. Bull. Soc. Math. France 111(1983), 421–428.

[2] J. Bourgain, *Subspaces of ℓ_N^∞, arithmetical diameter and Sidon sets*. In: *Probability in Banach spaces V*, Lecture Notes in Math., 1153, Springer, Berlin, 1985, pp. 96–127. https://doi.org/10.1007/BFb0074947.

[3] J. Bourgain, *Sidon sets and Riesz products*. Ann. Inst. Fourier (Grenoble) 35(1985), 137–148.
[4] C. C. Graham and K. E. Hare, *Characterizing Sidon sets by interpolation properties of subsets*. Colloq. Math. 112(2008), 175–199. https://doi.org/10.4064/cm112-2-1.
[5] C. C. Graham and K. E. Hare, *Interpolation and Sidon sets for compact groups*, CMS Books in Math., Springer, New York, 2013. https://doi.org/10.1007/978-1-4614-5392-5.
[6] D. Grow, *Sidon sets and I_0 sets*. Colloq. Math. 53(1987), 269–270. https://doi.org/10.4064/cm-53-2-269-270.
[7] K. E. Hare and L. T. Ramsey, *The relationship between ε-Kronecker and Sidon sets*. Canad. Math. Bull. 59(2016), 521–527. https://doi.org/10.4153/CMB-2016-002-3.
[8] D. Li and H. Queffélec, *Introduction à l'etude des espaces de Banach. Analyse et probabilités*, Cours Spécalisés, 12, Société Mathématique de France, Paris, 2004.
[9] J. López and K. Ross, *Sidon sets*, Lecture Notes in Pure and Applied Mathematics, 13, Marcel Dekker, New York, 1975.
[10] M. P. Malliavin-Brameret and P. Malliavin, *Caractérisation arithmétique des ensembles de Helson*. C.R.Acad. Sci. Paris Sér. A–B 264(1967), A192–A193.
[11] S. Neuwirth, *The maximum modulus of a trigonometric trinomial*. J. Anal. Math. 104(2008), 371–396. https://doi.org/10.1007/s11854-008-0028-2.
[12] G. Pisier, *De nouvelles caractérisations des ensembles de Sidon*. In: Mathematical analysis and applications, Part B, Adv. Math. Suppl. Studies, 7b, Academic Press, New York-London, 1981, pp. 685–726.
[13] G. Pisier, *Conditions d' entropie et caractérisations arithmétiques des ensembles de Sidon*. Proc. Conf. on Modern Topics in Harmonic Analysis (Torino/Milano). Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983, pp. 911–944.
[14] G. Pisier, *Arithmetic characterizations of Sidon sets*. Bull. Amer. Math. Soc. 8(1983), 87–89. https://doi.org/10.1090/S0273-0979-1983-15092-9.
[15] L. T. Ramsey, *Comparisons of Sidon and I_0 sets*. Colloq. Math. 70(1996), 103–132. https://doi.org/10.4064/cm-70-1-103-132.

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