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# ON DISTRIBUTION TAILS AND EXPECTATIONS OF MAXIMA IN CRITICAL BRANCHING PROCESSES

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**Abstract**

We derive the limit behaviour of the distribution tail of the global maximum of a critical Galton–Watson process and also of the expectations of partial maxima of the process, when the offspring law belongs to the domain of attraction of a stable law. Thus the Lindvall (1976) and Athreya (1988) results are extended to the infinite variance case. It is shown that in the general case these two asymptotics are closely related to each other, and the latter follows readily from the former. We also discuss a related problem from the theory of general branching processes.

BRANCHING PROCESS; MAXIMA OF GALTON–WATSON PROCESSES; INFINITE VARIANCE; GENERAL BRANCHING PROCESS

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## 1. Introduction and main results

Let $Z_n$, $n=0, 1, 2, \cdots$ be a critical Galton–Watson process (GWP) with $Z_0=1$, and $\nu$ be the 'offspring variable' (the distribution of which coincides with that of $Z_n$ conditioned that $Z_{n-1}=1$) with the generating function (g.f.) $f(s)$:

$$f(s)=\boldsymbol{E}s^{\nu}, \quad \boldsymbol{E}\nu=f'(1-)=1.$$

Denote by $M_n=\max_{0\leq k\leq n} Z_k$ and $M=\max_n M_n$ the partial and global maxima of the process $\{Z_n\}$ respectively. Recall that, in the critical case, $M<\infty$ a.s., for the process becomes extinct in a finite time $\tau_0=\min\{n: Z_n=0\}$ with probability one.

The two problems of (i) studying the limit behaviour of the distribution tail $\boldsymbol{P}(M>x)$ of the global maximum $M$ as $x\to\infty$, and (ii) studying the limit behaviour of the expectations $\boldsymbol{E}M_n$ of partial maxima $M_n$ as $n\to\infty$ have, by now, a rather long history. So far, only the case of a finite second moment $\boldsymbol{E}\nu^2<\infty$ has been considered, and the final results by Lindvall (1976) and Athreya (1988) are that, in this case,

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$$\lim_{x \to \infty} x\boldsymbol{P}(M > x) = 1 \tag{1.1}$$

and

$$\lim_{n \to \infty} \frac{\boldsymbol{E}M_n}{\log n} = 1 \tag{1.2}$$

respectively. In fact, in both papers the results are stated for the processes with $Z_0 = i$, $i \geqq 1$, in which case the units on the right-hand sides of (1.1) and (1.2) should be replaced by $i$. However, an elementary argument shows that the original assertions are equivalent to our (1.1) and (1.2) for $Z_0 = 1$, and therefore we restrict ourselves to a consideration of the case $Z_0 = 1$ only.

We mention here also the papers by Weiner (1984), Kämmerle and Schuh (1986) and Pakes (1987) containing intermediate results on $\boldsymbol{E}M_n$.

Strangely enough, so far the two problems (i) and (ii) have been treated quite separately. However, they are closely connected to each other and, in regular cases (when the g.f. of the process is of the form (1.3)), the solution to problem (ii) follows readily as soon as we can solve problem (i), as our Lemma 2 below shows. Thus it turns out that the lemma also gives, as a by-product, a simple and straightforward alternative proof of (1.2) in the finite variance case.

To solve problem (i) for GWP with infinite variance, we make use of the 'natural' embedding of a GWP into the corresponding random walk (r.w.) of which the jumps are distributed as $\xi = \nu - 1$ (the same approach was employed by Lindvall (1976) when proving (1.1); see also Dwass (1969) and Viskov (1970), where this point of view was adopted to get relations for the total progeny of a branching process, and Borovkov (1985) for the use of this embedding for the diffusion approximation). We show that the distribution tail of the total maximum of *any critical* GWP is in fact equivalent to that of the maximum of the corresponding zero-drift r.w. stopped at the time when it first hits zero (Lemma 1), thus removing a superficial condition of the finiteness of the second moment in the similar assertion in Lindvall (1976). As for the asymptotics of the distribution of the stopped r.w., it turned out to be known for the walks of interest (Pakes (1978); we are grateful to R. A. Doney, whose bibliographical comments helped us to find this reference).

Now we state our main results. Let the g.f. of the GWP $\{Z_n\}$, for some $\alpha \in (1, 2]$, have the form

$$f(s) = s + (1 - s)^{\alpha} L(1 - s), \tag{1.3}$$

where $L$ is a slowly varying (at zero) function. The form of the assertions (1.1) and (1.2) (the value of the variance does not appear there in any form) tempted us to suspect at first that these are true for any critical GWP, and our Lemma 1 below seemed to confirm this suspicion. However, as the final result of Theorem 1 shows (it involves the exponent $\alpha$), one can scarcely expect the asymptotics $x\boldsymbol{P}(M > x) \to$ constant to hold without an assumption of the form (1.3). The same applies seemingly to the asymptotics (still not depending on $\alpha$) of $\boldsymbol{E}M_n$.

*Theorem* 1. If the g.f. $f(s)$ of a GWP has the form (1.3), then

$$\lim_{x\to\infty} x\boldsymbol{P}(M>x)=\alpha-1. \tag{1.4}$$

The basic result relating in the general case the behaviour of the probabilities of interest to the corresponding characteristics of stopped r.w. is of independent interest. We state this result as Lemma 1 below. Let $\xi_n$, $n=1,2,\cdots$ be i.i.d. random variables having the same distribution as $\xi=\nu-1$ and hence having zero means. Put

$$S_0=1,\quad S_n=S_{n-1}+\xi_n,\qquad n\geqq 1. \tag{1.5}$$

Recall that, without loss of generality, we may assume that the process $\{Z_n\}$ is embedded into the r.w. $\{S_n\}$: for $V_{-1}=0$,

$$Z_n=S_{V_{n-1}},\quad V_n=\sum_{k=0}^{n} Z_k,\qquad n\geqq 0. \tag{1.6}$$

Clearly $V_k$ are stopping times for the r.w. $\{S_n\}$, and the latter can be replaced in (1.6) by the stopped r.w.

$$S_n^*=S_{n\wedge\tau},\quad \tau=\min\{k\geqq 1: S_k=0\}. \tag{1.7}$$

Denote by $M^*=\max_{n\geqq 0} S_n^*$ the global maximum of the stopped r.w. $\{S_n\}$. It is obvious from (1.6) that $M\leqq M^*$, and for the maximum $M^*$, it was proved in Pakes (1978) that, if (1.3) holds for $\alpha\in(1,2]$, then

$$\lim_{x\to\infty} x\boldsymbol{P}(M^*>x)=\alpha-1. \tag{1.8}$$

One should only note here that the strong aperiodicity property, assumed throughout by Pakes (1978), was not used there to establish (1.8).

The following lemma shows that, on the other hand, $M^*$ cannot be 'essentially greater' than $M$, and hence the relation (1.8) implies the same asymptotics for the distribution of $M$.

*Lemma* 1. For any critical GWP $\{Z_n\}$, there exists a function $\varepsilon=\varepsilon(x)\to 0$ as $x\to\infty$, such that

$$(1-\varepsilon)\boldsymbol{P}(M^*>(1+\varepsilon)x)\leqq \boldsymbol{P}(M>x)\leqq \boldsymbol{P}(M^*>x)\leqq\frac{1}{x}. \tag{1.9}$$

If $c_r=\boldsymbol{E}|\xi|^r<\infty$ for some $r\in(1,2]$, one can take

$$\varepsilon(x)=\max(1,2c_r)x^{-\beta},\quad \beta=\frac{r-1}{r+1}. \tag{1.10}$$

Now we turn to the exact statement relating problems (i) and (ii) above. The next lemma gives both upper and lower bounds for the expectations $\boldsymbol{E}M_n$ in terms of the tail $\boldsymbol{P}(M>x)$.

Recall that $\tau_0$ is the extinction time of the process $\{Z_n\}$.

*Lemma* 2. For any $r \in (1, 2]$ and $t > 0$,

$$-t\boldsymbol{P}(\tau_0 > n) \leqq \boldsymbol{E}M_n - \int_0^t \boldsymbol{P}(M > x)dx \leqq 2^r \frac{nc_r + 1}{r-1} t^{1-r}. \tag{1.11}$$

The remarkable fact is that Theorem 1 and Lemma 2 yield for $\boldsymbol{E}M_n$ logarithmic asymptotics *not depending on* $\alpha$.

*Theorem* 2. If the g.f. $f(s)$ of a GWP $\{Z_n\}$ has the form (1.3) for $\alpha \in (1, 2]$, then the relation (1.2) holds true.

*Remark.* Clearly, in view of Lemma 1, problem (i) is very close to the boundary problem for the r.w. (1.5) with two zero-slope linear boundaries $y_1 = 0$ and $y_2 = x$:

$$\{M^* > x\} = \{\text{the r.w. } S_n \text{ hits the boundary } y_2 \text{ before hitting } y_1\}.$$

Let $\tau(x) = \min\{k : S_k > x\}$ and $\chi(x) = S_{\tau(x)} - x$ be the first overshoot over the level $x$ in the walk (1.5). Since the stopping time $\tau^* = \min\{k > 0 : S_k > x \text{ or } S_k \leqq 0\}$ is integrable (in fact, its distribution tail tends to zero exponentially fast), we have by the Wald identity that

$$1 = \boldsymbol{E}S_{\tau^*} = (x + \boldsymbol{E}(\chi(x) \mid S_{\tau^*} > 0))\boldsymbol{P}(S_{\tau^*} > 0) - 0 \cdot \boldsymbol{P}(S_{\tau^*} = -1).$$

Hence

$$\boldsymbol{P}(M^* > x) = \boldsymbol{P}(S_{\tau^*} > 0) = \frac{1}{x + \boldsymbol{E}(\chi(x) \mid S_{\tau^*} > 0)} < \frac{1}{x} \tag{1.12}$$

assuming only that the expectation $\boldsymbol{E}\xi_i = 0$.

Now (1.8) implies that, if (1.3) holds, then

$$\lim_{x \to \infty} \frac{1}{x} \boldsymbol{E}(\chi(x) \mid S_{\tau^*} > 0) = \frac{2-\alpha}{\alpha - 1}, \qquad \alpha \in (1, 2]. \tag{1.13}$$

Note that, if we omit conditioning on the event $\{S_{\tau^*} > 0\}$, the expectation $\boldsymbol{E}\chi(1)$ is finite if and only if $\boldsymbol{E}\xi_i^2 < \infty$ (see e.g. Example 2 in Section XVII.18 of Spitzer (1964)). Thus, for unconditional expectations, we have in the cases $\alpha = 2$ and $\alpha < 2$ that $\boldsymbol{E}\chi(x) = o(x)$ (which is essentially the integral renewal theorem for the sequence of i.i.d. random variables of which the distribution coincides with that of $1 + \chi(1)$; see, for example, Appendix 1 in Borovkov (1976)) and $\boldsymbol{E}\chi(x) = \infty$ respectively. Comparing these last relations with (1.13), we see now how large, in the case $\alpha < 2$, is the contribution to $\boldsymbol{E}\chi(x)$ of those 'very large' jumps which carry the walk from below 0 over the (high) level $x$.

## 2. Proofs

*Proof of Lemma* 1. The second inequality in (1.9) is obvious from (1.6), while the last inequality in (1.9) has already been proved in (1.12) (it follows also from the Doob inequality for the stopped r.w. $S_n^*$, which is clearly a martingale with $\boldsymbol{E}S_n^* = \boldsymbol{E}S_0^* = 1$). Thus it remains only to prove the first inequality in (1.9).

Letting $y=(1+\varepsilon)x$, $\varepsilon>0$, we get

$$\begin{aligned} P(M>x) &\geqq P(M>x;\, M^*>y)=P(M>x \mid M^*>y)P(M^*>y) \\ &=(1-P(M\leqq x \mid M^*>y))P(M^*>y). \end{aligned} \tag{2.1}$$

Put $T=\min\{k\geqq 1: S_k^*>y\}$, and $m=\min\{j\geqq 1: V_j>T\}$. Since $\{M^*>y\}=\{T<\infty\}$, we see that

$$\begin{aligned} P(M\leqq x \mid M^*>y) &= P(M\leqq x \mid T<\infty) \\ &\leqq P(Z_m\leqq x,\, Z_{m+1}\leqq x \mid T<\infty) \\ &\leqq P(Z_m\leqq x,\, S_{V_m}-S_T\leqq x-y \mid T<\infty). \end{aligned} \tag{2.2}$$

But $V_m-T\leqq Z_m$ by the definition of $m$ and, on the event $\{Z_m\leqq x\}$, one has $S_{V_m}-S_T\geqq \min_{j\leqq x}(S_{T+j}-S_T)$. By the strong Markov property, the last expression does not depend on $T$ and has, conditioned that $T<\infty$, the same distribution as $\min_{j\leqq x} S_j$. Therefore the right-hand side of (2.2) does not exceed

$$P\left(\min_{j\leqq x} S_j\leqq -x\varepsilon\right)=P\left(x^{-1}\min_{j\leqq x} S_j\leqq -\varepsilon\right). \tag{2.3}$$

Further, by the strong law of large numbers $x^{-1}S_x\to 0$ a.s. as $x\to\infty$, and hence $x^{-1}\max_{j\leqq x}|S_j|\to 0$ a.s. as $x\to\infty$, so that, for any fixed $\varepsilon>0$, the probability (2.3) tends to 0 as $x\to\infty$. This means that, for some positive function $\varepsilon(x)\to 0$ as $x\to\infty$,

$$P\left(\min_{j\leqq x} S_j\leqq -x\varepsilon(x)\right)\leqq \varepsilon(x). \tag{2.4}$$

In view of (2.1) and (2.2) or $y=(1+\varepsilon(x))x$, relation (2.4) gives

$$P(M>x)\geqq (1-\varepsilon(x))P(M^*>(1+\varepsilon(x))x).$$

If $c_r=E|\xi|^r<\infty$ for some $r\in(1,2]$, we take $\varepsilon(x)$ as defined in (1.10) and apply the von Bahr–Esseen inequality (von Bahr and Esseen 1965) to estimate the probability in (2.3) by

$$P\left(\max_{j\leqq x}|S_j|\geqq x^{1-\beta}\right)\leqq 2xc_r x^{-2r/(r+1)}\leqq \varepsilon(x).$$

Lemma 1 is proved.

The proof of Theorem 1 follows immediately from Lemma 1 and relation (1.8).

*Proof of Lemma* 2. For any $t>0$,

$$EM_n=\int_0^\infty P(M_n>x)dx\leqq \int_0^t P(M>x)dx+\int_t^\infty P(M_n>x)dx. \tag{2.5}$$

To estimate the last integral observe that, since $Z_n$ is a martingale, the Doob inequality yields, for $r\geqq 1$,

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(2.6) $$P(M_n > x) \leqq x^{-r} EZ_n^r.$$

Further, put $X_k = Z_k - Z_{k-1}$ and note that by the total probability formula and the von Bahr–Esseen inequality we have, for $r \in (1, 2]$,

$$E|X_k|^r = \sum_{j \geqq 0} E\left(\left|\sum_{i \leqq j} \xi_i\right|^r \,\middle|\, Z_{k-1} = j\right) P(Z_{k-1} = j) \leqq E(2c_r Z_{k-1}) = 2c_r.$$

Therefore, by the same inequality,

$$EZ_n^r = E\left(\sum_{k=1}^n X_k + 1\right)^r \leqq 2^{r-1}\left(E\left|\sum_{k=1}^n X_k\right|^r + 1\right) \leqq 2^r(nc_r + 1),$$

and hence (2.6) implies that

$$\int_t^\infty P(M_n > x)dx \leqq \int_t^\infty 2^r(nc_r + 1)x^{-r}dx = 2^r \frac{nc_r + 1}{r - 1} t^{1-r}.$$

The right inequality in (1.11) is proved. On the other hand

$$\begin{aligned} EM_n &\geqq E(M_n;\, \tau_0 \leqq n) = E(M;\, \tau_0 \leqq n) \\ &\geqq \int_0^t P(M > x;\, \tau_0 \leqq n)dx = \int_0^t P(M > x)dx \\ &\quad - \int_0^t P(M > x;\, \tau_0 > n)dx \geqq \int_0^t P(M > x)dx - tP(\tau_0 > n). \end{aligned}$$

Lemma 2 is proved.

*Proof of Theorem* 2. First we note that, if $t = n^h$ for some $h > 0$, then Theorem 1 yields

(2.7) $$\int_0^t P(M > x)dx = (1 + \theta)(\alpha - 1)h \log n.$$

Here, and in what follows, we denote by $\theta$ (possibly different) quantities tending to zero as $n \to \infty$.

Now let $\delta > 0$ be arbitrarily small. First we choose $t = t_n = n^{1/(\alpha - 1 + \delta)}$. As is well known (Slack (1968); see also Borovkov (1988) for an alternative proof), in the case (1.3) with $\alpha \in (1, 2]$, the non-extinction probability $P(\tau_0 > n) = n^{-1/(\alpha-1)} L_1(n)$, where $L_1(x)$ is slowly varying as $x \to \infty$. Combining this relation with (2.7) and using the left inequality in (1.11), we conclude that, under our choice of $t$, (2.8) $$EM_n \geqq (1+\theta)\frac{\alpha-1}{\alpha-1+\delta}\log n - n^{-\delta'}L_1(n), \quad \delta' = \frac{\delta}{(\alpha-1+\delta)(\alpha-1)} > 0.$$

To make use of the right inequality in (1.11), one has to choose

$$r=\alpha-\delta, \quad t=t_n=n^{1/(r-1)}=n^{1/(\alpha-1-\delta)}.$$

Since $c_r<\infty$ for any $r<\alpha$ when (1.3) holds, we see from (1.11) that

(2.9) $$EM_n \leqq (1+\theta)\frac{\alpha-1}{\alpha-1-\delta}\log n + O(1).$$

Now $\delta>0$ is arbitrarily small, and therefore relations (2.8) and (2.9) mean that, for any positive $\varepsilon>0$,

$$1-\varepsilon \leqq \liminf_{n\to\infty}\frac{EM_n}{\log n} \leqq \limsup_{n\to\infty}\frac{EM_n}{\log n} \leqq 1+\varepsilon,$$

which yields immediately (1.2). Theorem 2 is proved.

## 3. General branching process

In this section we discuss another relevant embedding of discrete time branching processes in left-continuous integer-valued r.w. (i.e. when the jumps are $\geqq -1$), for which the asymptotic (2.4) gives the behaviour of the non-extinction probabilities of the processes. This construction leads not to a GWP, but to a general branching process $\{Y_n\}$ of special type, the study of which is of independent interest. (In fact, this construction is close to that proposed by Dwass (1975) for the simple r.w., in which case it gives a GWP with geometric g.f.)

Let $\{S_n\}$ be the r.w. on integers given by (1.5) and $S_n^*=S_{n\wedge\tau}$ be the stopped version (1.7) of the walk. Put $Y_0=1$. To each positive jump $\xi_{j_0}$ of the walk $\{S_n^*\}$, we relate a particle, say $\mu$, born at time $S_{j_0-1}^*$. This particle lives $\xi_{j_0}$ units of time and is dead at time $S_{j_0}^*$. The evolution of its descendants is described by the evolution of the walk on the segment $[j_0, j_0^*)$, where $j_0^*=\min\{j>j_0: S_{j_0^*}^*=S_{j_0-1}^*\}$.

Note that, since the walk $\{S_n\}$ is left-continuous with zero drift, $j_0^*<\infty$ a.s., and, by the strong Markov property, the evolution of the walk on the time interval $[j_0, j_0^*]$ (and hence that of the descendants of our particle $\mu$) does not depend on what occurs outside the segment $[j_0, j_0^*]$ (i.e. on evolution of the descendants of other particles which do not belong to the progeny of $\mu$). Note also that, to each positive jump $\xi_{j_0}$ 'covering' the time $m$ (so that $m\in[S_{j_0-1}^*, S_{j_0}^*)$, the particle $\mu$ is alive at time $m$), there corresponds one and only one negative jump at the epoch $j_0^*$ returning the walk to the point $S_{j_0-1}^*$, and hence the number $Y_m$ of particles which are alive at time $m$ is

$$Y_m=\#\{j<\tau: S_{j-1}^*=m+1, S_j^*=m\}.$$

All positive jumps of the walk (1.5) (or (1.7)) on the segment $(j_0, j_0^*)$ 'generate' descendants of the particle $\mu$. The particle corresponding to a positive jump $\xi_{j_1}>0$,

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$j_1 \in (j_0, j_0^*)$, is a direct descendant of $\mu$, if $j_1 - 1$ is a weak lower ladder epoch in the segment of our walk starting from the point $(j_0, S_{j_0})$, i.e. if

$$S_{j_1-1} = \min\{S_l : j_0 \leqq l \leqq j_1 - 1\}.$$

Since all the negative jumps in the walk (1.5) are equal to $-1$, it is easy to see that *at each time* $m \in [S_{j_0-1}, S_{j_0})$ (recall that this is just the life interval of the particle $\mu$), *the particle* $\mu$ *gives birth to* $\zeta_m$ direct descendants ($\zeta_m$ is the number of weak lower ladder epochs on the segment of the walk starting from $(j_0, S_{j_0})$ till $j_0^*$, which are on the level $m$ and are followed immediatley by a positive jump of the walk),

$$(3.1) \qquad \boldsymbol{P}(\zeta_m = r) = (1-q)^r q, \quad r = 0, 1, 2, \cdots, \quad \boldsymbol{E}\zeta_m = \frac{1-q}{q} = \frac{\boldsymbol{P}(\xi_1 > 0)}{\boldsymbol{P}(\xi_1 = -1)},$$

where

$$q = \boldsymbol{P}(\xi_1 = -1 \mid \xi_1 \neq 0) = \frac{\boldsymbol{P}(\xi_1 = -1)}{\boldsymbol{P}(\xi_1 > 0) + \boldsymbol{P}(\xi_1 = -1)}.$$

Clearly $\zeta_m$ are independent of each other. Thus the mean total number of direct descendants of the particle $\mu$ is

$$\boldsymbol{E}(\xi_1 \mid \xi_1 > 0)\boldsymbol{E}\zeta = \frac{\boldsymbol{E}(\xi_1; \xi_1 > 0)}{\boldsymbol{P}(\xi_1 > 0)} \cdot \frac{\boldsymbol{P}(\xi_1 > 0)}{\boldsymbol{P}(\xi_1 = -1)} = 1,$$

for $0 = \boldsymbol{E}\xi_1 = \boldsymbol{E}(\xi_1; \xi_1 > 0) - \boldsymbol{P}(\xi_1 = -1)$.

Thus we have constructed a critical general branching process $\{Y_n\}$, $Y_0 = 1$, of which the initial particle dies at time 1 giving birth to $\zeta_1$ particles. In the sequel, all the particles of the process live a random time having distribution of $\xi_1$ conditioned that $\xi_1 > 0$ and generate new particles with constant rate $(1-q)/q$ (see (3.1)) all their lives.

Now clearly the lifetime of the process coincides with $M^*$, the maximum of the stopped r.w. (1.7), and hence we have from (2.4) the following assertion on the non-extinction probabilities $Q(n) = \boldsymbol{P}(Y_n > 0)$ of the process $\{Y_n\}$.

*Theorem* 3. Let $\{Y_n\}$ be a critical discrete time general branching process starting with one particle dying at time $t = 1$, and such that all the other particles in the process live a random time having the g.f. $(f(s) - f(0))/sf(0)$, where $f(s)$ is given by (1.3) for $\alpha \in (1, 2]$. We suppose that at each time epoch during its life, a particle in the process gives birth to a geometric number of particles (following the law (3.1)). Then

$$\lim_{n \to \infty} nQ(n) = \alpha - 1.$$

Thus $Q(n)$ behaves just as the non-extinction probability of a critical GWP with finite variance (equal to $2/(\alpha - 1)$), although the number of direct descendants of a particle in our process has infinite variance if $\alpha < 2$ and can have infinite variance when $\alpha = 2$. For general results on non-extinction probabilities of general branching processes, see Topchii (1987).

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