20

The Dirac field

The Klein–Gordon equation's negative energy solutions and corresponding negative probabilities prompted Dirac to look for a relativistically invariant equation of motion which was linear in time. His equation was formally the square-root of the Klein–Gordon equation.

The Dirac equation leads naturally to the existence of spin $\frac{1}{2}$. It is the basic starting point for the study of spin-$\frac{1}{2}$ particles such as the electron and quarks. It also appears in condensed matter physics as the relevant low-energy degrees of freedom in the strong-coupling limit of the Hubbard model [92], and has been used as an alternative formulation of gravity [84].

20.1 The action

The action for the Dirac field is given by

$$S_{\mathrm{D}} = \int (\mathrm{d}x) \Big\{ -\frac{1}{2}\mathrm{i}\hbar c \overline{\psi} (\gamma^\mu \overrightarrow{\partial_\mu} - \gamma^\mu \overleftarrow{\partial_\mu})\psi \\ + (mc^2 + V)\overline{\psi}\psi - \overline{J}\psi - \overline{\psi}J \Big\}, \tag{20.1}$$

where ψ and $\overline{\psi} = \psi^\dagger \gamma^0$ are d_{R}-component spinors. The γ^μ are $d_{\mathrm{R}} \times d_{\mathrm{R}}$ matrices, defined below. All quantities here are implicitly matrix-valued. They have hidden 'spinor' indices, which we shall write explicitly at times using Greek letters $\alpha, \beta, \ldots$.

The variation of the action with respect to a dynamical change in the field $\overline{\psi}$ gives the equation of motion for ψ is found by varying the action with respect to $\overline{\psi}$, and is given by

$$(-\mathrm{i}\hbar c\gamma^\mu \partial_\mu + mc^2 + V)\psi = J. \tag{20.2}$$

If we drop the source term J, this can also be written

$$\mathrm{i}\hbar c\partial_0 \psi = \gamma^0(-\mathrm{i}\hbar c\gamma^i \partial_i + mc^2 + V)\psi = H_{\mathrm{D}}\psi, \tag{20.3}$$

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where $H_{\rm D}$ is the differential Hamiltonian operator (to be distinguished from the field theoretical Hamiltonian below). The conjugate equation is found by varying the action with respect to ψ and may be written as

$$\overline{\psi}(\mathrm{i}\hbar c\gamma^{\mu}\overleftarrow{\partial}_{\mu} + mc^2 + V) = 0, \tag{20.4}$$

or in terms of the differential Hamiltonian operator $H_{\rm D}$,

$$-\mathrm{i}\hbar c(\partial_0\overline{\psi}) = \overline{\psi}\gamma^0 H_{\rm D}\gamma^0. \tag{20.5}$$

The free Dirac equation may be viewed as essentially the square-root of the Klein–Gordon equation. In the massless limit, the linear combination of derivatives is a Lorentz-scalar-representation of the square-root of $\Box$. This may be verified by squaring the Dirac operator and separating the product of γ-matrices into symmetric and anti-symmetric parts:

$$(\gamma^{\mu}\partial_{\mu})^2 = \gamma^{\mu}\gamma^{\nu}\,\partial_{\mu}\partial_{\nu} \tag{20.6}$$

$$= \frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\}\,\partial_{\mu}\partial_{\nu} + \frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]\partial_{\mu}\partial_{\nu} \tag{20.7}$$

$$= -g^{\mu\nu}\partial_{\mu}\partial_{\nu} + \frac{1}{2}\gamma^{\mu}\gamma^{\nu}[\partial_{\mu},\partial_{\nu}] \tag{20.8}$$

$$= -\Box\,. \tag{20.9}$$

The commutator of two partial derivatives vanishes when the derivatives act on any non-singular function. Since the fields are non-singular, except in the presence of certain exceptional interactions which do not apply here, the Dirac operator can be identified as the square-root of the d'Alambertian.

20.2 The γ-matrices

In order to satisfy eqn. (20.9), the γ-matrices must satisfy the relation

$$\{\gamma^{\mu},\gamma^{\nu}\} = -2g^{\mu\nu} \tag{20.10}$$

$$(\gamma^0)^2 = -(\gamma^i)^2 = I. \tag{20.11}$$

The matrices satisfy a *Clifford algebra*. The set of matrices which satisfies this constraint is of fundamental importance to the Dirac theory. They are not unique, but may have several representations. The form of the γ^{μ} matrices is dependent on the dimension of spacetime and, since they carry a spacetime index, on the Lorentz frame [61, 103].

Products of the γ^{μ} form a group of matrices Γ_a, where $a = 1,\ldots,d_G$, and the dimension of the group is $d_G = 2^{(n+1)}$. The elements Γ_a are proportional to

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the unique combinations:

$$\begin{gathered} I \\ \gamma^0, \quad \gamma^i, \quad \gamma^i\gamma^j, \quad \gamma^0\gamma^i \\ \gamma^0\gamma^i\gamma^j, \quad \gamma^1\gamma^2\ldots\gamma^n \\ \gamma^0\gamma^1\gamma^2\ldots\gamma^n. \end{gathered} \quad (i \neq j)$$

The usual case is $n + 1 = 4$, where one has Γ_a,

$$\begin{gathered} I \\ \gamma^0, \quad \mathrm{i}\gamma^1, \quad \mathrm{i}\gamma^3, \quad \mathrm{i}\gamma^3, \\ \gamma^0\gamma^1, \quad \gamma^0\gamma^2, \quad \gamma^0\gamma^3, \quad \mathrm{i}\gamma^2\gamma^3, \quad \mathrm{i}\gamma^3\gamma^1, \quad \mathrm{i}\gamma^2\gamma^3 \\ \mathrm{i}\gamma^0\gamma^2\gamma^3, \quad \mathrm{i}\gamma^0\gamma^1\gamma^3, \quad \mathrm{i}\gamma^0\gamma^1\gamma^2, \quad \gamma^1\gamma^2\gamma^3 \\ \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \end{gathered}$$

Factors of i have been introduced so that each matrix squares to the identity (see ref. [112]). These may also be grouped differently, in the more suggestive Lorentz-covariant form:

1	scalar
γ^μ	vector
$\sigma_{\mu\nu}$	anti-symmetric tensor
$\gamma^5\gamma^\mu$	pseudo-(axial) vector
γ^5	pseudo-scalar

where

$$\sigma^{\mu\nu} = \frac{1}{2\mathrm{i}}[\gamma^\mu, \gamma^\nu]. \tag{20.12}$$

For each Γ_a, with the exception of the identity element, it is possible to find a suitably defined Γ_a, such that

$$\Gamma_a\Gamma_b\Gamma_a = -\Gamma_b \quad (b \neq 1). \tag{20.13}$$

By taking the trace of this relation, and noting that

$$\mathrm{Tr}(\Gamma_a\Gamma_b\Gamma_a) = \mathrm{Tr}(\Gamma_a\Gamma_a\Gamma_b) = \mathrm{Tr}(\Gamma_b) \tag{20.14}$$

one obtains

$$\mathrm{Tr}(\Gamma_b) = -\,\mathrm{Tr}(\Gamma_b) = 0. \tag{20.15}$$

From this, it follows that the 2^{n+1} elements are linearly independent, since, if one attempts to construct a linear combination which is zero,

$$\sum_a \lambda_a\Gamma_a = 0, \tag{20.16}$$

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then taking the trace of this implies that each of the components $\lambda_a = 0$. This establishes that each component is linearly independent and that a matrix representation for the γ^μ must have at least this number of elements, in order to satisfy the algebra constraints. This divides the possibilities into two cases, depending on whether the dimension of spacetime is even or odd.

For even $n+1$, the γ^μ are most simply represented as $d_R \times d_R$ matrices, where

$$d_R = 2^{(n+1)/2}. \tag{20.17}$$

d_R^2 then contains exactly the right number of elements. Although the matrices are not unique (they can be transformed by similarity transformations), all such sets of matrices of this dimension are *equivalent* representations. Moreover, since there is no redundancy in the matrices, the $d_R \times d_R$ representations are also *irreducible*, or *fundamental*. In this case, the identity is the only element of the group which commutes with every other element (the group is said to have a trivial centre). Another common way of expressing this, in the literature, is to observe that other matrices, typically $\gamma^0\gamma^1\ldots\gamma^n$, anti-commute with an arbitrary element γ^μ. There are thus more elements in the centre of the group than the identity. This is a sign of reducibility or multiple equivalent representations.

For odd $n + 1$, it is not possible to construct a matrix with exactly the right number of elements. This is a symptom of the existence of several *inequivalent* representations of the algebra. In this case, one must either construct several sets of smaller matrices (which are inequivalent), or combine these into matrices of larger dimension, which are *reducible*. The reducible matrices reduce to block-diagonal representations, in which the blocks are the multiple, inequivalent, irreducible representations. In this case, the identity is not the only element of the group which commutes with every other element (the group is said to have a non-trivial centre), and the matrix $\gamma^0\gamma^1\ldots\gamma^n$ anti-commutes with an arbitrary element γ^μ. Spinors in $n + 1$ dimensions are discussed in ref. [10].

20.2.1 Example: $n + 1 = 4$

In $3 + 1$ dimensions, the dimension of the algebra is $2^4 = 16$, and one has the standard representation,

$$\gamma^0 = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^{\mathbf{i}} \\ \sigma^{\mathbf{i}} & \mathbf{0} \end{pmatrix} \tag{20.18}$$

where σ^i are the Pauli matrices, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20.19}$$

The product of two γ-matrices may be evaluated by separating into even and odd parts, in terms of the commutator and anti-commutator:

$$\begin{aligned}\gamma_\mu\gamma_\nu &= \frac{1}{2}[\gamma_\mu, \gamma_\nu] + \frac{1}{2}\{\gamma_\nu, \gamma_\nu\} \\ &= \mathrm{i}\sigma_{\mu\nu} - g_{\mu\nu},\end{aligned} \tag{20.20}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2\mathrm{i}}[\gamma_\mu, \gamma_\nu]. \tag{20.21}$$

The product of all the γ's is usually referred to as γ^5, and is defined by

$$\begin{aligned}\gamma^5 &= \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.\end{aligned} \tag{20.22}$$

Clearly, this notation is poorly motivated in spacetime dimensions other than $3+1$. In $3+1$ dimensions, it is straightforward to show that

$$(\gamma^5)^2 = 1, \qquad \{\gamma^\mu, \gamma^5\} = 0. \tag{20.23}$$

The cyclic nature of the trace can be used together with the anti-commutativity of γ^5 to prove that the trace of an odd number of γ-matrices vanishes in $3+1$ dimensions. To see this, one notes that

$$\mathrm{Tr}(\gamma_5 A {\gamma_5}^{-1}) = \mathrm{Tr}(A). \tag{20.24}$$

Thus, choosing a product of m such matrices $A = \gamma_\mu\gamma_\nu \ldots \gamma_\sigma$, such that

$$\gamma_5 A = (-1)^m\ A\gamma_5, \tag{20.25}$$

it follows immediately that

$$\mathrm{Tr}(A) = (-1)^m\, \mathrm{Tr}(A), \tag{20.26}$$

and hence the trace of an odd number m of the matrices must vanish. The hermiticity properties of the matrices are contained by the relation

$$\gamma^{\mu\dagger} = \gamma^0\,\gamma^\mu\,\gamma^0, \tag{20.27}$$

which summarizes

$$\gamma^{0^\dagger} = \gamma^0 \tag{20.28}$$

$$\gamma^{i^\dagger} = -\gamma^i. \tag{20.29}$$

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20.2.2 Example: $n + 1 = 3$

In $2 + 1$ dimensions, the dimension of the algebra is $2^3 = 8$, and thus the minimum representation is is terms of either two irreducible sets of 2 matrices, or a single set of reducible 4×4 matrices, with redundant elements.

The fundamental $\vec{2}$ representation is satisfied by

$$\gamma^0 = \sigma_3, \quad \gamma^i = -\mathrm{i}\sigma_i \tag{20.30}$$

for $i = 1, 2$. This representation breaks parity invariance, thus there are two *inequivalent* representations which differ by a sign.

$$\gamma^\mu, -\gamma^\mu \tag{20.31}$$

The $\vec{4}$ representation is a symmetrized direct sum of these, padded with zeros:

$$\gamma^\mu(\vec{4}) = \begin{pmatrix} +\gamma^\mu(\vec{2}) & 0 \\ 0 & -\gamma^\mu(\vec{2}) \end{pmatrix}. \tag{20.32}$$

The matrices of the $\vec{2}$ representation satisfy

$$\gamma^\mu \gamma^\nu = -g^{\mu\nu} - \mathrm{i}\epsilon^{\mu\nu\rho}\gamma^c \tag{20.33}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\mu \gamma^\rho) = 2\mathrm{i}\epsilon^{\mu\nu\rho} \tag{20.34}$$

$$\gamma_5 = \gamma^0\gamma^1\gamma^2 \tag{20.35}$$

$$\left[\gamma_5, \gamma_\mu\right] = 0, \tag{20.36}$$

where the first of these relations is found by splitting into a commutator and anti-commutator and using the $su(2)$ Lie algebra relation for the Pauli matrices:

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k. \tag{20.37}$$

In the $\vec{4}$ representation in $2 + 1$ dimensions the results are as for $\vec{2}$ except that

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho)_{\vec{4}} = 0. \tag{20.38}$$

Note that the product of all elements is usually referred to as γ^5 in the literature, rather than γ^4, by analogy with the $(3 + 1)$ dimensional case.

$$\begin{aligned} \gamma_5 = \gamma^5 = \gamma^4 &= \mathrm{i}\gamma^0\gamma^1\gamma^2 \\ &= \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}. \end{aligned} \tag{20.39}$$

Since this is a multiple of the identity matrix, it commutes with every element in the algebra. Thus there are two elements to the centre of the group: I and $-I$. The centre is the discrete group Z_2, and the complete fundamental representation of the algebra is

$$\gamma^\mu(\vec{2}) \otimes Z_2. \tag{20.40}$$

These two inequivalent representations correspond to the fact that, in a two-dimensional plane, spin up and spin down cannot be continuously rotated into one another (not even classically), and thus these two physical possibilities are disconnected regions of the rotation group. In much of the literature on two-dimensional physics, it is common to adopt either a spin up, or spin down 2×2 representation for the γ-matrices, not the complete $\vec{4}$ representation.

20.3 Transformation properties of the Dirac equation

Consider a Lorentz transformation of the Dirac spinor by a matrix representation of the Lorentz group:

$$\psi'(x') = S(L)\,\psi(x) = L_R\,\psi(x). \tag{20.41}$$

The matrix, usually denoted $S(L)$ in the literature, is just an example of a non-adjoint representation of the Lorentz group from section 9.4.3. This representation has to carry spinor indices α, β, which are suppressed above, in the usual way. These spinor indices correspond to the representation indices A, B of section 9.4.3.

A transformation of the free Dirac equation may be written as

$$(\gamma^\mu p'_\mu c + mc^2)\psi'(x') = 0 \tag{20.42}$$

$$(\gamma^\mu (L^{-1})^\mu{}_\nu p'_\mu c + mc^2)(L_R \psi(x)) = 0, \tag{20.43}$$

where one recalls that

$$p^\mu = L^\mu{}_\nu\, p^\nu \quad \rightarrow \quad p_\mu = (L^{-1})\mu\,{}^\nu p_\nu. \tag{20.44}$$

Multiplying on the left hand side by L_R^{-1}, and comparing with the untransformed equation, leads to a condition

$$L_R^{-1}\,\gamma^\mu\,L_R = L^\mu{}_\nu \gamma^\nu, \tag{20.45}$$

which is an identity, provided $L = L_{\rm adj}$, the adjoint representation of the group.

The infinitesimal form of the spinor representation may be written in terms of the generators of this representation T_R:

$$L_R = S(L) = I + \theta^a T_R^a, \tag{20.46}$$

or, with spinor (representation) indices intact,

$$S(L)^\alpha{}_\beta = \delta^\alpha{}_\beta + \theta^a (T_R^a)^\alpha{}_\beta. \tag{20.47}$$

Consider an infinitesimal transformation

$$x'^\mu = x^\mu + \epsilon\omega^\mu{}_\nu x^\nu, \tag{20.48}$$

so that

$$L_R(I+\epsilon\omega) = I + \epsilon T_R. \tag{20.49}$$

The adjoint transformation can thus be expressed in two equivalent forms:

$$\begin{aligned} S^{-1}\gamma^\mu S &= (1-\epsilon T)\gamma^\mu(1-\epsilon T) \\ &= \gamma^\mu + \epsilon(\gamma^\mu T - T\gamma_\mu) \\ &= (L_{\text{adj}})^\mu_\nu(1+\epsilon_\omega)\gamma^\nu \\ &= \gamma^\mu + \epsilon\omega^\mu_{\ \nu}\gamma^\nu. \end{aligned} \tag{20.50}$$

Thus,

$$\gamma^\mu T - T\gamma^\mu = \omega^\mu_{\ \nu}\gamma^\nu, \tag{20.51}$$

which defines T up to a multiple of the identity matrix. Choosing unit determinant $\det(I+\epsilon T) = 1 + \epsilon\,\mathrm{Tr}\,T$, we have that $\mathrm{Tr}\,T = 0$, and one may write

$$(T_R)^\alpha_{\ \beta} = \frac{1}{8}\omega^{\mu\nu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)^\alpha_{\ \beta}, \tag{20.52}$$

or, compactly,

$$(T_R)^\alpha_{\ \beta} = \frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}. \tag{20.53}$$

20.3.1 Rotations

An infinitesimal rotation by angle ϵ about the x^1 axis has

$$\omega^{23} = -\omega^{32} = 1, \tag{20.54}$$

and all other components zero. Thus the generator in the spinor representation is

$$T^1 = \frac{1}{2}\gamma_2\gamma_3, \tag{20.55}$$

and the exponentiated finite element becomes

$$\begin{aligned} S(R_1) = \mathrm{e}^{\theta_1 T_R} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta\Sigma_1}, \\ &= I\,\cos\frac{\theta}{2} + \mathrm{i}\Sigma_1\sin\frac{\theta}{2}, \end{aligned} \tag{20.56}$$

where

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{20.57}$$

The half-angles are characteristic of the double-valued nature of spin:

$$S(\theta_1 + 2\pi) = -S(\theta_1). \tag{20.58}$$

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20.3.2 Boosts

For a boost in the x^1 direction, $\omega^{10} = -\omega^{01} = 1$,

$$S(B_1) = \frac{1}{2}\gamma^0\gamma^1 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \equiv \frac{1}{2}\alpha_1. \tag{20.59}$$

The finite, exponentiated element is thus

$$S(B_1) = \mathrm{e}^{\frac{\alpha}{2}\alpha_1} = I\cosh\frac{\alpha}{2} + \alpha_1 \sinh\frac{\alpha}{2}, \tag{20.60}$$

where $\tanh\alpha = v/c$. Notice that the half-valued arguments have no effect on translations.

20.3.3 Parity and time reversal

The meaning of parity invariance is intrinsically linked to the number of spacetime dimensions, since an even number of reflections about spatial axes is equivalent to a rotation, and is therefore simply connected to the infinitesimally generated group. In that case, spatial reflection is defined by a reflection in an odd number of axes. In odd numbers of spatial dimensions, reflections in all axes lead to a 'large' transformation which cannot be generated by exponentiated infinitesimal generators.

Consider the case in 3 + 1 dimensions. For a space inversion, one has

$$\begin{aligned} L_R^{-1}\gamma^0 L_R &= \gamma^0 \\ L_R^{-1}\gamma^i L_R &= -\gamma^i. \end{aligned} \tag{20.61}$$

Thus, the parity transformation can be represented by:

$$S(P) = \mathrm{e}^{\mathrm{i}\phi}\,\gamma^0 \tag{20.62}$$

in Dirac space. This exchanges the upper and lower spinor contributions. Similarly, a time inversion

$$\begin{aligned} L_R^{-1}\gamma^0 L_R &= -\gamma^0 \\ L_R^{-1}\gamma^i L_R &= \gamma^i \end{aligned} \tag{20.63}$$

can be given the form

$$S(T) = \mathrm{e}^{\mathrm{i}\phi}\,\gamma^5. \tag{20.64}$$

20.3.4 Charge conjugation

Charge conjugation transforms a positive energy solution with charge q into a negative energy solution with charge $-q$. One searches for a unitary

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transformation $\mathcal{C}$ with the following properties:

$$\mathcal{C}\ \psi\ \mathcal{C}^{-1} = \eta\ C\overline{\psi}^{\mathrm{T}}$$
$$\mathcal{C}\ A_\mu\ \mathcal{C}^{-1} = -A_\mu. \tag{20.65}$$

η is a possible intrinsic property of the field, $\eta^2 = 1$. The two features of this transformation are that it exchanges positive for negative energies and that it reflects the vector field like an axial vector. Since A_μ always multiplies the charge q, in the covariant derivative, this is equivalent to changing the sign of the charge. The action for the gauged Dirac equation is ($\hbar = c = 1$),

$$S = \int (\mathrm{d}x)\ \overline{\psi}(\mathrm{i}\gamma^\mu D_\mu + m)\psi, \tag{20.66}$$

where $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$. In order to find a transformation which exchanges ψ with $\overline{\psi}^{\mathrm{T}}$, one begins by integrating by parts:

$$S = \int (\mathrm{d}x)\ \overline{\psi}(-\mathrm{i}\gamma^\mu(\overleftarrow{\partial_\mu} - \mathrm{i}qA_\mu) + m)\psi, \tag{20.67}$$

then, taking the transpose:

$$S = \int (\mathrm{d}x)\ \psi^{\mathrm{T}}(-\mathrm{i}\gamma^{\mathrm{T}\mu}(\overrightarrow{\partial_\mu} - \mathrm{i}qA_\mu))\overline{\psi}^{\mathrm{T}}. \tag{20.68}$$

This has almost the same form as the original, untransposed action, with opposite charge. In order to make it identical, we require a matrix which has the property

$$C\ \gamma^{\mathrm{T}\mu}\ C^{-1} = -\gamma^\mu. \tag{20.69}$$

Introducing such a matrix, one has

$$S = \int (\mathrm{d}x)\ (\gamma^{\mathrm{T}}C^{-1})(\mathrm{i}\gamma^\mu D^*_\mu + m)(C\overline{\psi}^{\mathrm{T}})$$
$$= \int (\mathrm{d}x)\ \overline{\psi}^{\mathrm{c}}(\mathrm{i}\gamma^\mu D^*_\mu + m)\psi^{\mathrm{c}}, \tag{20.70}$$

where the charge conjugated field is $\psi^{\mathrm{c}} = C\overline{\psi}^{\mathrm{T}}$.

The existence of a matrix C, in $3+1$ dimensions, possessing the above properties can be determined as follows [112]. Taking the transpose of the Clifford algebra relation,

$$\{\gamma^{\mathrm{T}\mu}, \gamma^{\mathrm{T}\nu}\} = -2g^{\mu\nu}, \tag{20.71}$$

one sees that the transposed γ-matrices also satisfy the algebra, and must therefore be related to the untransposed ones by a similarity transformation

$$\gamma^{\mathrm{T}\mu} = B^{-1}\ \gamma^\mu\ B. \tag{20.72}$$

In $3+1$ dimensions, the 4×4 γ-matrices are irreducible, and thus the existence of a non-singular, unitary B is guaranteed. Taking the transpose of eqn. (20.72) and re-using the relation to replace for γ^μ, one obtains

$$\gamma^{\mathrm{T}\mu} = (B^{-1}B^{\mathrm{T}})\,\gamma^{\mathrm{T}\mu}\,(B^{-1\mathrm{T}}B), \tag{20.73}$$

thus establishing that $B^{-1}B^{\mathrm{T}}$ commutes with all the γ-matrices. From Schur's lemma, it follows that this must be a multiple of the identity:

$$B^{-1}B^{\mathrm{T}} = cI. \tag{20.74}$$

Taking the inverse and then the complex conjugate of this relation, one finds

$$\begin{aligned}\frac{1}{c^*} &= B^*(B^{\mathrm{T}})^{-1*} \\ &= B^*B^{**} && (20.75)\\ &= B^*B. && (20.76)\end{aligned}$$

where we have used the unitarity $B^\dagger B = I$. This means that c is real, and furthermore that $c = \pm 1$, i.e.

$$B = \pm B^{\mathrm{T}}, \tag{20.77}$$

so, from this, the matrix is either symmetrical or anti-symmetrical. An additional constraint comes from the number of symmetrical and anti-symmetrical degrees of freedom in the 4×4 γ-matrices. If B is anti-symmetric, then the six matrices $\gamma^\mu B, \gamma^5 B, B$ are also anti-symmetric, whereas the ten matrices $B\gamma^5\gamma^\mu$, $B\sigma^{\mu\nu}$ are symmetrical. This matches the number of anti-symmetrical degrees of freedom in a 4×4 matrix representation. Conversely, if one takes B to be symmetrical, then the numbers are reversed and it does not match. One concludes, then, that B is an anti-symmetric, unitary matrix. This result was shown by Pauli in 1935. It has now been shown that it is possible to construct a similarity transformation which turns γ-matrices into their transposes. The matrix we require is now

$$C = -\mathrm{i}\gamma^5 B. \tag{20.78}$$

With this definition, we have

$$\begin{aligned}C^{-1}\,\gamma^\mu\,C &= -B^{-1}\,\mathrm{i}\gamma_5\,\gamma^\mu\,\mathrm{i}\gamma_5 B = -B^{-1}\gamma^\mu B \\ &= -\gamma^{\mathrm{T}\mu}. && (20.79)\end{aligned}$$

C is thus a charge conjugation matrix for Dirac spinors.

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20.4 Chirality in 3 + 1 dimensions

The field equations of massless spinors are, from eqn. (20.101),

$$\begin{pmatrix} -p_0 + & -\sigma^i p_i \\ \sigma^i p_i & p_0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{20.80}$$

or

$$\begin{aligned} \sigma^i \hat{p}_i \; \chi_{\mathrm{L}} &= 2\lambda \chi_{\mathrm{L}} = -\chi_{\mathrm{L}} \\ \sigma^i \hat{p}_i \; \chi_{\mathrm{R}} &= 2\lambda = +\chi_{\mathrm{R}}, \end{aligned} \tag{20.81}$$

where $\lambda = \frac{1}{2}\sigma^i \hat{p}_i$, and

$$\begin{aligned} \chi_{\mathrm{L}} &= \chi_1 + \chi_2 \\ \chi_{\mathrm{R}} &= \chi_1 - \chi_2. \end{aligned} \tag{20.82}$$

These equations are referred to as the Weyl equations, and χ_{L} and χ_{R} are Weyl-spinors. For such massless particles, the eigenvalue of γ^5 is referred to as the *chirality* of the solution:

$$\begin{aligned} \gamma^5 u(p, \lambda) &= 2\lambda u(p, \lambda) \\ \gamma^5 v(p, \lambda) &= -2\lambda v(p, \lambda). \end{aligned} \tag{20.83}$$

A projection operator for the chirality states is thus

$$\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm \gamma^5). \tag{20.84}$$

Particles with helicity $+\frac{1}{2}$ are referred to as right handed, while particles with helicity $-\frac{1}{2}$ are referred to as left handed. Only left handed neutrinos interact by the weak interaction and appear in the Standard Model. Symmetry under the continuous transformation

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\lambda\gamma_5}\psi \tag{20.85}$$

is known as chiral symmetry.

20.5 Field continuity

The variation of the action leads to surface terms,

$$\hbar \int \mathrm{d}\sigma^\mu \left(\delta\overline{\psi}\gamma_\mu\psi\right), \tag{20.86}$$

for ψ variations, and

$$\hbar \int \mathrm{d}\sigma^\mu \left(\overline{\psi}\gamma_\mu\delta\psi\right), \tag{20.87}$$

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for $\overline{\psi}$ variations. This provides us with definitions for the conjugate momenta across spacelike hyper-surfaces σ:

$$\hbar\overline{\Pi}_{\sigma}^{A} = \gamma_{\sigma}\psi^{A}, \tag{20.88}$$

for the variable conjugate to $\overline{\psi}$, and

$$\Pi_{\sigma}^{A} = \hbar\overline{\psi}^{A}\gamma_{\sigma}, \tag{20.89}$$

for the variable conjugate to ψ. The canonical values for these momenta are

$$\begin{aligned} \overline{\Pi} &= \hbar\gamma_0\psi \\ \Pi &= \hbar\overline{\psi}\gamma_0 = \psi^{\dagger}. \end{aligned} \tag{20.90}$$

20.6 Conserved norm and probability

The linear nature of the Dirac action implies that the conserved current is independent of derivatives. This means that the sign of the energy $\mathrm{i}\hbar c\partial_0$ cannot change the sign of the conserved probability, thus the Dirac equation does not suffer the problem of negative norms or probabilities as does the Klein–Gordon equation.

To determine the conserved current, one considers the effect of an infinitesimal x-independent phase transformation δ:

$$\begin{aligned} \delta S = \int (\mathrm{d}x)\Big\{ & (\overline{\psi}\mathrm{e}^{-\mathrm{i}s}(-\mathrm{i}\delta s)\gamma^{\mu}\partial_{\mu}(\mathrm{e}^{\mathrm{i}s}\psi)) \\ & + (\overline{\psi}\mathrm{e}^{-\mathrm{i}s}(-\mathrm{i}\delta s)\gamma^{\mu}\partial_{\mu}((-\mathrm{i}\delta s)\mathrm{e}^{\mathrm{i}s}\psi)) \\ & - (\partial_{\mu}(\overline{\psi}\mathrm{e}^{-\mathrm{i}s}(-\mathrm{i}\delta s))\gamma^{\mu}(\mathrm{e}^{\mathrm{i}s}\psi)) \\ & - (\partial_{\mu}(\overline{\psi}\mathrm{e}^{-\mathrm{i}s}(-\mathrm{i}\delta s))\gamma^{\mu}((-\mathrm{i}\delta s)\mathrm{e}^{\mathrm{i}s}\psi))\Big\} \end{aligned} \tag{20.91}$$

Integrating by parts to remove derivatives from δs, and using the equations of motion, one arrives at the simple expression

$$\delta S = \hbar \int \mathrm{d}\sigma^{\mu}(\overline{\psi}\gamma_{\mu}\psi)\delta s, \tag{20.92}$$

which defines a conserved current $\delta S = \int \mathrm{d}\sigma^{\mu} J_{\mu}\delta s$. This motivates the definition of an inner product given by

$$(\psi_1, \psi_2) = -\frac{1}{2}\int \mathrm{d}\sigma^{\mu}(\overline{\psi}_1\gamma_{\mu}\psi_2 + \overline{\psi}_2\gamma_{\mu}\psi_1), \tag{20.93}$$

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giving the norm of the field as

$$(\psi, \psi) = -\int d\sigma^{\mu} \overline{\psi} \gamma_{\mu} \psi. \tag{20.94}$$

The canonical interpretation of this is

$$(\psi_1, \psi_2) = -\frac{1}{2} \int d\sigma \, (\overline{\psi}_1 \gamma_0 \psi_2 + \overline{\psi}_2 \gamma_0 \psi_1), \tag{20.95}$$

which means that the norm may also be written

$$(\psi, \psi) = -\int d\sigma \, \psi^{\dagger} \gamma^{0} \gamma_0 \psi = \int d\sigma \, \psi^{\dagger} \psi. \tag{20.96}$$

The norm of the field is defined separately on the manifold of positive and negative energy solutions.

20.7 Free-field solutions in $n = 3$

The free-field equation is

$$(-\mathrm{i}\hbar c \gamma^{\mu} \partial_{\mu} + mc^2)_{\alpha\beta} \psi_{\beta}(x) = 0, \tag{20.97}$$

where $\psi_{\alpha}(x)$ is a $2l$-component vector for some $l \geq n/2$, which lives on spinor space (usually these indices are suppressed). In a given number of dimensions, we may express this equation in terms of a representation of the γ-matrices. In three dimensions we may use eqn. (20.18) to write

$$\begin{pmatrix} \mathrm{i}\hbar \partial_t + mc^2 & \mathrm{i}\hbar c \sigma^i \partial_i \\ -\mathrm{i}\hbar c \sigma^i \partial_i & -\mathrm{i}\hbar \partial_t + mc^2 \end{pmatrix} \psi = 0, \tag{20.98}$$

where we suppress the α, β spinor indices. The blocks are now 2×2 matrices, and the spinor may also be written in terms of two two-component spinors u:

$$\psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{20.99}$$

If we transform the spinors to momentum space,

$$\psi(x) = \int \frac{d^{n+1}k}{(2\pi)^{n+1}} e^{\mathrm{i}kx} \psi(k), \tag{20.100}$$

then the field equations may be written as

$$\begin{pmatrix} -p_0 c + mc^2 & -c\sigma^i p_i \\ c\sigma^i p_i & p_0 c + mc^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \tag{20.101}$$

where $p_\mu = \hbar k_\mu$. This matrix equation has non-zero solutions for ψ only if the determinant of the operator vanishes. Thus,

$$\det = p^2c^2 + m^2c^4 = 0. \tag{20.102}$$

Here one makes use of the fact that

$$\begin{aligned}(\sigma^i p_i)^2 &= \sigma^i\sigma^j p_i p_j \\ &= \left(\frac{1}{2}[\sigma^i,\sigma^j] + \frac{1}{2}\{\sigma^i,\sigma^j\}\right) p_i p_j \\ &= (\mathrm{i}\epsilon^{ijk}\sigma_k + \delta^{ij}) p_i p_j \\ &= p^i p_i.\end{aligned} \tag{20.103}$$

Eqn. (20.102) indicates that the solutions of the Dirac equation must satisfy the relativistic energy relation. Thus the Dirac field also satisfies a Klein–Gordon equation, which may be seen by operating on eqn. (20.97) with the conjugate of the Dirac operator:

$$\begin{aligned}(\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)(-\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)\psi(x) &= 0 \\ (-\hbar^2c^2\Box + m^2)\psi &= 0.\end{aligned} \tag{20.104}$$

The last line follows from eqn. (20.20). The vanishing of the determinant also gives us a relation which will be useful later, namely

$$\frac{(p_0c + mc^2)}{c\sigma^i p_i} = \frac{-c\sigma^i p_i}{(-p_0c + mc^2)}. \tag{20.105}$$

The 2×2 components of eqn. (20.101) are now

$$\begin{aligned}(-p_0c + mc^2)u_1 - c(\sigma^i p_i)u_2 &= 0 \\ c(\sigma^i p_i)u_1 + (p_0c + mc^2)u_2 &= 0,\end{aligned} \tag{20.106}$$

which implies that the two-component spinors u are linearly dependent:

$$\begin{aligned}u_1 &= \frac{c(\sigma^i p_i)}{(-p_0c + mc^2)}u_2, \\ u_1 &= -\frac{(p_0c + mc^2)}{c(\sigma^i p_i)}u_2.\end{aligned} \tag{20.107}$$

The consistency of these apparently contradictory relations is secured by the determinant constraint in eqn. (20.105).

In spite of the linear (first-order) derivative in the Dirac action, the determinant condition for non-trivial solutions leads us straight back to a quadratic constraint on the allowed spectrum of energy and momenta. This means that

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both positive and negative energies are allowed in the Dirac equation, exactly as in the Klein–Gordon case. The linear derivative does cure the negative probabilities, however, as we show below.

The solutions of the Dirac equation may be written in various forms. A direct attempt to apply the field equation constraint in a delta function, by analogy with the scalar field, cannot work directly, since the delta function cannot have a matrix argument. However, by introducing a projection operator $(-\gamma^\mu p_\mu c + mc^2)/2|p_0|$, it is possible to write

$$\psi(x) = \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}kx}(-\gamma^\mu p_\mu c + mc^2)\delta(p^2c^2 + m^2c^4)\, u(k), \tag{20.108}$$

where $p = \hbar k$ and $u(k)$ is a mass shell spinor. The projection term ensures that application of the Dirac operator leads to the squared mass shell constraint. By inserting $\theta(\pm k_0)$ alongside the delta function, one can also restrict this to the manifold of positive or negative energy solutions, i.e.

$$\begin{aligned}\psi(x)^{(\pm)} &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}kx}\theta(\mp k_0)(-\gamma^\mu p_\mu c + mc^2)\delta(p^2c^2 + m^2c^4)\, u_\pm(k), \\ &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \mathrm{e}^{\mathrm{i}kx} \frac{mc^2}{|k_0|}\gamma^0\, u_\pm(k),\end{aligned} \tag{20.109}$$

since

$$(-\gamma^\mu p_\mu c + mc^2) = 2mc^2, \tag{20.110}$$

when p_μ is on the mass shell $\gamma^\mu p_\mu c + mc^2 = 0$. In the literature it is customary to proceed by examining the positive and negative energy cases separately. As we shall see below, solutions of the Dirac equation can be normalized on either the positive or negative energy solution spaces.

It is more usual to consider positive and negative energy solutions to the Dirac equation separately. To this end, there is sufficient freedom in the expression

$$\begin{aligned}\psi^{(\pm)}(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}kx}\delta(p^2c^2 + m^2c^4)\theta(\mp k_0)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} N_\pm(k) \\ &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega t)}}{2|E|c\hbar} \begin{pmatrix} \frac{c\sigma^i p_i}{(\pm E + mc^2)} \\ 1 \end{pmatrix} N_\pm(k) u.\end{aligned} \tag{20.111}$$

The two-component spinors u are taken to be a linear combination of the spin eigenfunctions for spin up and spin down, as measured conventionally along the z axis

$$u_i = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{20.112}$$

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where $i = 1, 2$ and $c_1^2 + c_2^2 = 1$. Unlike, the case of the Klein–Gordon equation, both positive and negative energy solutions can be normalized to unity, although this is not necessarily an interesting choice of normalization. An example: consider the normalization of the positive energy solutions ($-p_0 c = E$),

$$\begin{aligned} 1 &= (\psi^{(+)}(x), \psi^{(+)}(x)) \\ &= \int \mathrm{d}\sigma \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \frac{\mathrm{d}^{n+1}k'}{(2\pi)^{n+1}} \frac{\mathrm{e}^{\mathrm{i}(k-k')x}}{4E^2c^2\hbar^2} N_+^2 \left(\frac{c^2(\sigma^i p_i)^2}{(E+mc^2)} + 1 \right) |u|^2 \\ &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{N_+^2}{4E^2c^2\hbar^2} \left(\frac{2E}{E+mc^2} \right). \end{aligned} \tag{20.113}$$

Assuming a box normalization, where $\mathrm{d}^n k/(2\pi)^n \sim L^{-n} \sum_k$, we have

$$N_+ = \sqrt{2L^n c^2 \hbar^2 (E + mc^2)}, \tag{20.114}$$

and hence

$$\psi^{(+)}(k) = L^{n/2} \mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega t)} \sqrt{\frac{(E+mc^2)}{2E}} \begin{pmatrix} \frac{c\sigma^i p_i}{(E+mc^2)} \\ 1 \end{pmatrix} \chi^{(s)}, \tag{20.115}$$

where

$$\chi^{(\frac{1}{2})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{20.116}$$

20.8 Invariant normalization in p-space

The normalization of Dirac fields is a matter of some subtlety. Different invariant normalizations are used for different purposes. The usual case is to consider plane wave solutions, or wave-packets. Consider the probability on a spacelike hyper-surface, transforming as the zeroth component of a vector:

$$\begin{aligned} \overline{\psi}(x)\gamma_0\psi(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} u(k)^\dagger\, u(k)\, \theta(\mp k_0) \\ &\quad \times (-\gamma^\mu p_\mu c + mc^2)\, \delta(p^2c^2 + m^2c^4) \\ &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{2mc^2}{2|p_0|} u(k)^\dagger u(k) = 1. \end{aligned} \tag{20.117}$$

The factor of $2mc^2/2|p_0|$ is required to ensure that the spinors satisfy the equations of motion for the Dirac field. This indicates that the invariant normalization for the spinors should be

$$u^\dagger(k)u(k) = \frac{|p_0|}{mc^2}. \tag{20.118}$$

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Consider now what this means for the product $\overline{u}(k)u(k)$. The field equations for these, in momentum space, are:

$$\begin{aligned}(\gamma^0 p_0 c + \gamma^i p_i c + mc^2)u(k) &= 0\\ \overline{u}(k)(\gamma^0 p_0 c + \gamma^i p_i c + mc^2) &= 0.\end{aligned} \tag{20.119}$$

Multiplying the first of these by $u^\dagger$, and using the fact that $\overline{u} = u^\dagger \gamma^0$, gives:

$$\overline{u}(\gamma^0 p_0 c + \gamma^0 \gamma^i p_i c + mc^2 \gamma^0)u(k) = 0. \tag{20.120}$$

Now, multiplying the second (adjoint) equation on the right hand side by $\gamma^0 u$ and commuting γ^0 through the left hand side, one has:

$$\overline{u}(p_0 c - \gamma^0 \gamma^i p_i c + mc^2 \gamma^0)u(k) = 0. \tag{20.121}$$

Thus, adding eqns. (20.121) and (20.120), leaves

$$2p_0 c\, \overline{u}u + 2mc^2\, u^\dagger u = 0. \tag{20.122}$$

Taking the normalization for $u^\dagger u$ in eqn. (20.118), we find that

$$\overline{u}u = \frac{-p_0}{|p_0|} = \frac{E}{|E|}. \tag{20.123}$$

Thus, a positive energy spinor is normalized with positive norm, whilst a negative energy spinor has a negative norm, in momentum space. It is custom to refer to the positive and negative energy spinors as $u(k)$ and $v(k)$, respectively. Accordingly, one takes the invariant normalization to be

$$\begin{aligned}\overline{u}_r u_s &= \delta_{rs}\\ \overline{v}_r v_s &= -\delta_{rs},\end{aligned} \tag{20.124}$$

with spinor indices shown.

20.9 Formal solution by Green functions

The formal solution to the free equations of motion ($V = 0$) may be written

$$\psi(x) = \int (\mathrm{d}x') S(x, x') J(x'), \tag{20.125}$$

and the conjugate form

$$\psi^\dagger(x) = \int (\mathrm{d}x') J^\dagger(x') S(x', x). \tag{20.126}$$

20.10 Expressions for the Green functions

The Green functions can be obtained from the corresponding Green functions for the scalar field; see section 5.6:

$$(-\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)(\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2) = -\hbar^2c^2\Box + m^2c^4 + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\partial_\mu\partial_\nu, \tag{20.127}$$

and the latter term vanishes when operating on non-singular objects. It follows for the free field that

$$(\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)G^{(\pm)}(x,x') = S^{(\pm)}(x,x') \tag{20.128}$$

$$(\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)G_{\mathrm{F}}(x,x') = S_{\mathrm{F}}(x,x') \tag{20.129}$$

$$(-\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)S^{(\pm)}(x,x') = 0 \tag{20.130}$$

$$(-\mathrm{i}\hbar c\gamma^\mu\partial_\mu + mc^2)S_{\mathrm{F}}(x,x') = \delta(x,x'). \tag{20.131}$$

20.11 The energy–momentum tensor

The application of Noether's theorem for spacetime translations leads to a symmetrical energy–momentum tensor. In accordance with the other fields, the zero–zero component of the energy–momentum tensor has the interpretation of an energy density or Hamiltonian density. This is to be distinguished from the differential Hamiltonian operator, which generates the time evolution of the field. We have,

$$\begin{aligned}\theta_{00} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\psi)}(\partial_0\psi) + (\partial_0\overline{\psi})\frac{\partial\mathcal{L}}{\partial(\partial^0\overline{\psi})} - \mathcal{L}g_{00}\\ &= -\frac{\mathrm{i}\hbar c}{2}\overline{\psi}\gamma_0(\partial_0\psi) + \frac{\mathrm{i}\hbar c}{2}(\partial_0\overline{\psi})\gamma_0\psi + \mathcal{L}.\end{aligned} \tag{20.132}$$

Using the equation of motion (20.2), the integral of this quantity over all space may be written as

$$\begin{aligned}\int \mathrm{d}\sigma\,\theta_{00} &= \int \mathrm{d}\sigma\,\overline{\psi}(-\mathrm{i}\hbar c\gamma^i\partial_i + mc^2 + V)\psi\\ &= (\psi, H_{\mathrm{D}}\psi),\end{aligned} \tag{20.133}$$

where we have used $(\gamma^0)^2 = 1$. This expression is formally the expectation value of the differential Hamiltonian operator, but it is also used as the definition of a 'field theoretical' Hamiltonian. In the second quantization, where the fields are operator-valued, this expression is referred to as the Hamiltonian operator and may be thought of as generating the time evolution of the fully quantized field.

The spacetime components of the energy–momentum tensor are not explicitly symmetrical. This is a consequence of the linear derivative in the field equation (20.2). However, the off-diagonal components can be shown to be equal provided the field satisfies the equations of motion. We have

$$\begin{aligned}\theta_{0i} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\psi)}(\partial_i\psi) + (\partial_i\overline{\psi})\frac{\partial\mathcal{L}}{\partial(\partial^0\overline{\psi})}\\ &= -\frac{\mathrm{i}\hbar c}{2}\overline{\psi}\gamma_0(\partial_i\psi) + \frac{\mathrm{i}\hbar c}{2}(\partial_i\overline{\psi})\gamma_0\psi. \end{aligned}\tag{20.134}$$

Taking the integral over all space allows us to integrate by parts, giving

$$\begin{aligned}\int \mathrm{d}\sigma\,\theta_{0i} &= -\mathrm{i}\hbar c\int \mathrm{d}\sigma\,\overline{\psi}\gamma_0\partial_i\psi\\ &= -(\psi, p_i c\psi),\end{aligned}\tag{20.135}$$

where $p_i = -\mathrm{i}\hbar\partial_i$. Thus, this component is identified with the momentum in the field. Switching the order of the indices, we have

$$\begin{aligned}\theta_{0i} &= \frac{\partial\mathcal{L}}{\partial(\partial^i\psi)}(\partial_0\psi) + (\partial_0\overline{\psi})\frac{\partial\mathcal{L}}{\partial(\partial^i\overline{\psi})}\\ &= -\frac{\mathrm{i}\hbar c}{2}\overline{\psi}\gamma_i(\partial_0\psi) + \frac{\mathrm{i}\hbar c}{2}(\partial_0\overline{\psi})\gamma_i\psi. \end{aligned}\tag{20.136}$$

This is clearly not the same as eqn. (20.134). However on using the field equation and its conjugate in eqns. (20.2) and (20.5), it may be shown that

$$\theta_{i0} = \frac{\mathrm{i}\hbar c}{2}\overline{\psi}(\gamma_i\gamma^0\gamma^j\,\overrightarrow{\partial}_j - \gamma^j\,\overleftarrow{\partial}_j\,\gamma^0\gamma_i)\psi - \frac{1}{2}\{\gamma_i,\gamma^0\}(mc^2+V)\overline{\psi}\psi,\tag{20.137}$$

so that the integral over all space can be integrated by parts to give

$$\int \mathrm{d}\sigma\,\theta_{i0} = \int \mathrm{d}\sigma\left\{\frac{\mathrm{i}\hbar c}{2}\overline{\psi}\gamma^0\{\gamma_i,\gamma^j\}\partial_j\psi - \frac{1}{2}\{\gamma_i,\gamma^0\}(mc^2+V)\overline{\psi}\psi\right\}.\tag{20.138}$$

On using the anti-commutation relations for the γ-matrices, we find

$$\int \mathrm{d}\sigma\,\theta_{i0} = -\mathrm{i}\hbar c\int \mathrm{d}\sigma\,\overline{\psi}\gamma_0\partial_i\psi = \int \mathrm{d}\sigma\,\theta_{0i}.\tag{20.139}$$

The diagonal space components are given by

$$\begin{aligned}\theta_{ii} &= \frac{\partial\mathcal{L}}{\partial(\partial^i\psi)}(\partial_i\psi) + (\partial_i\overline{\psi})\frac{\partial\mathcal{L}}{\partial(\partial^i\overline{\psi})}\\ &= -\frac{\mathrm{i}\hbar c}{2}\overline{\psi}\gamma_i(\partial_i\psi) + \mathcal{L}, \end{aligned}\tag{20.140}$$

where i is not summed. The off-diagonal space components are

$$\begin{aligned}\theta_{ij} &= \frac{\partial\mathcal{L}}{\partial(\partial^i\psi)}(\partial_j\psi) + (\partial_j\overline{\psi})\frac{\partial\mathcal{L}}{\partial(\partial^i\overline{\psi})} \\ &= -\frac{i\hbar c}{2}\left(\overline{\psi}\gamma_i\partial_j\psi - (\partial_i\overline{\psi})\gamma_j\psi\right),\end{aligned} \tag{20.141}$$

where $i \neq j$. Although not explicitly symmetrical in this form, the integral over all space of this quantity is symmetrical by partial integration. Note that the trace of the space components is given in $n + 1$ dimensions by

$$\sum_i \theta_{ii} = \mathcal{H} + (mc^2 + V)\overline{\psi}\psi + (n-1)\mathcal{L}, \tag{20.142}$$

so that the total trace of the energy–momentum tensor is

$$\theta^\mu_{\ \mu} = g^{\mu\nu}\theta_{\mu\nu} = (mc^2 + V)\overline{\psi}\psi + (n-1)\mathcal{L}. \tag{20.143}$$

This vanishes for $m = V = 0$ in $1 + 1$ dimensions.

20.12 Spinor electrodynamics

The action for spinor electrodynamics is

$$\begin{aligned}S_{\mathrm{QED}} = \int (\mathrm{d}x)\Big\{\overline{\psi}\Big(-\frac{1}{2}i\hbar c(\gamma^\mu \overrightarrow{D}_\mu - \gamma^\mu \overleftarrow{D}_\mu^{\dagger}) + mc^2\Big)\psi \\ + \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\Big\}.\end{aligned} \tag{20.144}$$

This is the basis of the quantum theory of electrodynamics for electrons (QED).

Pauli [104] has shown that the Dirac action may be modified by a term of the form

$$S \to S + \int (\mathrm{d}x)\,\overline{\psi}\frac{1}{2}\frac{\mu c^2}{\hbar}\sigma^{\rho\lambda}F_{\rho\lambda}\psi, \tag{20.145}$$

whereupon the field behaves as though it has an additional (anomalous) magnetic moment $e\hbar/2m$. Later, Foldy investigated generalizations of the Dirac action which preserve Lorentz invariance and gauge invariance [51]. One makes two restrictions: linearity in A_μ (weak field) and finiteness in the zero momentum limit (independent of $\partial_\mu\psi$). The result is

$$S \to S + \int (\mathrm{d}x)\overline{\psi}\left[c\sum_{i=0}^{\infty}\left(\alpha_i\Box^i\,\gamma^\mu A_\mu + \frac{1}{2}\beta_n\sigma^{\mu\nu}\Box^i F_{\mu\nu}\right)\right]\psi, \tag{20.146}$$

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where α_i, β_i are constants representing anomalous charge and magnetic moments respectively.

There is a number of problems for which spinor electrodynamics can be solved exactly. These include:

- the spherically symmetrical Coulomb potential [31, 38, 62, 75, 98];
- the homogeneous magnetic field [73, 81, 106, 109];
- the field of an electromagnetic plane wave [131].

A review of these is given in many books. See, for example, ref. [8].