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LOWER BOUND FOR THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC POLYNOMIAL

M. N. MISHRA, N. N. NAYAK and S. PATTANAYAK

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Abstract

Let $X_1, X_2, \ldots, X_n$ be identically distributed independent random variables belonging to the domain of attraction of the normal law, have zero means and $\Pr\{X_r \neq 0\} > 0$. Suppose $a_0, a_1, \ldots, a_n$ are non-zero real numbers and $\max_{0 \leq r \leq n} |a_r| = k_n$, $\min_{0 \leq r \leq n} |a_r| = t_n$ and ε_n is such that as $n \to \infty$, $\varepsilon_n \to 0$, but $\varepsilon_n \log n \to \infty$. If N_n be the number of real roots of the equation $\sum_{r=0}^{n} a_r X_r x^r = 0$ then for $n > n_0$, $N_n > \varepsilon_n \log n$ outside an exceptional set of measure at most $\mu/\varepsilon_n \log n + (k_n/t_n)^{\beta} \exp(-\mu'\beta/\varepsilon_n)$, $(0 < \beta < 2 - \varepsilon, 0 < \varepsilon < 2)$ provided $\lim_{n\to\infty}(k_n/t_n)$ is finite.

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1. Introduction

Let $N_n(u)$ be the number of real roots of a random algebraic equation $\sum_{r=0}^{n} X_r(u)x^r = 0$ where X_r's are independent identically distributed random variables. The problem of finding bounds for $N_n(u)$ has been considered by various authors. Samal (1972) has considered the general case when X_r's have identical distribution with expectation zero, variance and third absolute moment finite and non-zero. A stronger result has been obtained by Littlewood and Offord (1939) in the case where the coefficients are Gaussian variates. Dunnage (1968, 1970, 1972) while dealing with general probability distribution has considered the lower bound of $N_n(u)$. Almost all the previous workers have considered the cases in which the random variables have finite moments of second and higher orders. The exception to those are the studies of Samal and Mishra (1972a, 1972b, 1973) in which they have considered the identically distributed random variables having characteristic function $\exp(-C|t|^{\alpha})$ where C is a positive constant and

$\alpha > 1$. The probability distribution in this case represents a symmetric stable distribution with infinite variance when $1 < \alpha \leqslant 2$.

The lower bound for the number of real roots of a random algebraic equation when X_r's belong to the domain of attraction of the normal law has not yet been studied, though Ibragimov and Maslova (1971) have studied the expectation of the number of real roots for this situation. The object of this paper is to find the lower bound of $N_n(u)$ when the coefficients are not identically distributed and belong to the domain of attraction of the normal law.

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THEOREM 1. *Let $f(x) = \sum_{r=0}^{n} a_r X_r x^r$ be a polynomial of degree n where the X_r's are identically distributed independent random variables which belong to the domain of attraction of the normal law, have zero means and* $\Pr\{X_r \neq 0\} > 0$. *Let a_0, a_1, $a_2, \ldots, a_n$ be non-zero real numbers. Then there exists a positive integer n_0 such that for $n > n_0$, the number of real roots of the equations $f(x) = 0$ is at least $\varepsilon_n \log n$ outside a set of measure at most*

$$\mu/\varepsilon_n \log n + (k_n/t_n)^{\beta} \exp(-\mu'\beta/\varepsilon_n), \qquad (0 < \beta < 2 - \varepsilon, 0 < \varepsilon < 2),$$

provided $\lim_{n \to \infty} (k_n/t_n)$ *is finite, where* $k_n = \max_{0 \leqslant r \leqslant n} |a_r|$, $t_n = \min_{0 \leqslant r \leqslant n} |a_r|$ *and $\varepsilon_n \to 0$, but $\varepsilon_n \log n \to \infty$ as $n \to \infty$.*

In the sequel we need the following definitions, notation and lemmas for the proof of the theorem.

We shall denote μ's for positive constants not necessarily the same in different places of occurrence. $[x]$ denotes the greatest integer $\leqslant x$. We take constants A and D satisfying the relations

$$0 < D < 1, \quad \text{and} \quad A > 1. \tag{2.1}$$

Following the techniques of Samal and Mishra (1972a) we consider $f(x) = \sum_{r=0}^{n} a_r X_r x^r$ at the points

$$x_m = (1 - M^{-2m})^{1/2}, \tag{2.2}$$

for $m = [k/2] + 1, [k/2] + 2, \ldots, k$, where

$$M = \left[\lambda_n^2\left(\sqrt{2} + 1\right)^2 (Ae/D)(k_n/t_n)^2\right] + 1, \tag{2.3}$$

k is determined by

$$M^{2k} \leqslant n < M^{2k+2}, \tag{2.4}$$

and λ_n is a sequence tending to infinity with n.

We express $f(x_m)$ for sufficiently large n as sum of three parts as follows:

$$f(x_m) = \left(\sum_1 + \sum_2 + \sum_3\right) a_r X_r(u) x_m^r,$$

where the index r ranges from $M^{2m-1}+1$ to M^{2m+1} in Σ_1, from 0 to M^{2m-1} in Σ_2 and from $M^{2m+1}+1$ to n in Σ_3.

We write

(2.5) $$U_m = \sum_1 a_r X_r(u) x_m^r,$$

and

(2.6) $$R_m = \left(\sum_2 + \sum_3\right) a_r X_r(u) x_m^r.$$

We say a function h: $\mathbf{R}_+ \to \mathbf{R}_+$ is slowly varying in the neighbourhood of zero if

(2.7) $$\lim_{x\to 0} h(\nu x)/h(x) = 1, \qquad (\nu > 0).$$

LEMMA 1. *A slowly varying function h with property* (2.7) *can be represented in the form*

(2.8) $$h(x) = c(x)\exp\left\{-\int_a^x \frac{\bar{\varepsilon}(u)}{u}\,du\right\},$$

where $\lim_{x\to 0} c(x) = c \neq 0$, $\lim_{x\to 0} \bar{\varepsilon}(x) = 0$ *and* $a > 1$.

The above lemma follows immediately if we put $1/x$ for x in Karamata's theorem (see Ibragimov and Linnik (1972), Appendix 1, page 394).

Since the random variables belong to the domain of attraction of the normal law, their common characteristic function admits the representation

$$\phi(t) = \exp\{-\tfrac{1}{2}t^2 L(|t|)(1+o(1))\}$$

(see Ibragimov and Linnik (1972) Chapter 2 page 91), where, as $t \to 0$, $L(t)$ is a slowly varying function. Since $L(|t|)$ is positive we can write the characteristic function ϕ in the form

(2.9) $$\phi(t) = \exp\{-\tfrac{1}{2}t^2 h(t)\},$$

where $h(t) = L(|t|)(1+o(1))$ with the property that

(2.10) $$\bar{h}(t) = \operatorname{Re} h(t)(1+o(1)).$$

Obviously $h(t)$ is a slowly varying function as $t \to 0$.

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We know $L(1/x)$ is given by the formula

$$L(1/x) = -\int_0^x u^2\, d\psi(u) = \int_{-x}^x u^2\, dG(u),$$

where $\psi(x) = 1 - G(x) + G(-x)$, $G(x)$ being the common distribution function of the random variables X_r's (see Ibragimov and Linnik (1972) Chapter 2, page 90). Hence for infinite variance $\lim_{x\to 0} L(1/x) = \infty$ which gives

$$\lim_{t\to 0} \operatorname{Re} h(t) = \infty. \tag{2.11}$$

Consider the function $h_1(t)$ determined by

$$h_1(t) = \begin{cases} \operatorname{Re} h(t) & \text{if } V(X_r) = \infty, \\ \sigma^2 & \text{if } V(X_r) = \sigma^2 < \infty. \end{cases}$$

Clearly $h_1(t)$ is slowly varying in a neighbourhood of the origin. By (2.10), $h(t) = h_1(t)(1 + o(1))$ in both cases as $t \to 0$.

We define normalising constants V_m starting from the relation

$$\left(1/V_m^2\right) \sum_1 a_r^2 x_m^{2r} h_1(\eta\theta) = 1, \tag{2.12}$$

where $\eta = a_r x_m^r / V_m$, and θ is a small positive number whose final choice will be dealt with later. Proceeding as Ibragimov and Maslova (1971) page 232, we can show that such constants V_m always exist if θ is sufficiently small.

LEMMA 2. *For some constant $b > 1$ we have*

$$M^m < \begin{cases} (Ae/D)^{1/2}(V_m/\sigma t_n) & \text{if } V(X_r) = \sigma^2 < \infty, \\ b^{-1/2}(Ae/D)^{1/2}(V_m/t_n) & \text{if } V(X_r) = \infty. \end{cases}$$

PROOF. If $V(X_r) = \sigma^2 < \infty$, then

$$V_m^2 = \sigma^2 \sum_1 a_r^2 x_m^2 \geqslant \sigma^2 t_n^2 M^{2m} (D/Ae).$$

Or

$$M^m < (Ae/D)^{1/2}(V_m/\sigma t_n). \tag{2.13}$$

Again let $V(X_r) = \infty$. Then by (2.11), $\lim_{t\to 0} h_1(t) = \infty$, so we can choose $\theta_0 > 0$ such that for $\eta\theta < \theta_0$, $h_1(\eta\theta) > b > 1$ where b is a constant and $0 \leqslant r \leqslant n$. Then we have

$$V_m^2 > b \sum_1 a_r^2 x_m^{2r} \geqslant b t_n^2 \sum_1 x_m^{2r} > b t_n^2 M^{2m} (D/Ae).$$

Or

$$M^m < b^{-1/2}(Ae/D)^{1/2}(V_m/t_n). \tag{2.14}$$

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LEMMA 3.

$$\left|\sum_2 a_r X_r(u) x_m^r\right| < \lambda_n W_m$$

except for a set of measure at most $\mu/\lambda_n^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

$$W_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(\eta_1 \theta_1), \tag{2.15}$$

$\eta_1 = a_r x_m^r / W_m$ *and* θ_1 *has similar meaning to* θ.

PROOF. Let $\psi_m(x)$ and $\phi_m(t)$ be respectively the distribution function and the characteristic function of $a_r X_r(u)/W_m$.

So $\phi_m(t) = \exp(-\frac{1}{2}t^2 h_m(t))$, where

$$h_m(t) = (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h(\eta_1 t). \tag{2.16}$$

Since $h_1(t) = L(|t|)(1 + o(1))$ and $L(|t|)$ is a slowly varying function as $t \to 0$, we have by Lemma 1, for $|t| < \theta$,

$$\begin{aligned} h_1(\eta_1 t)(h_1(\eta_1\theta_1))^{-1} &= \frac{L(|\eta_1 t|)(1 + o(1))}{L(|\eta_1 \theta_1|)(1 + o(1))} \\ &= \frac{c(|\eta_1 t|)\exp\{-\int_a^{|\eta_1 t|} (\bar{\varepsilon}(u)/u)\, du\}(1 + o(1))}{c(|\eta_1\theta_1|)\exp\{-\int_a^{|\eta_1\theta_1|} (\bar{\varepsilon}(u)/u)\, du\}(1 + o(1))} \\ &= \frac{c(|\eta_1 t|)(1 + o(1))}{c(|\eta_1\theta_1|)(1 + o(1))} \exp\left\{\int_{|\eta_1 t|}^{|\eta_1\theta_1|} (\bar{\varepsilon}(u)/u)\, du\right\}, \end{aligned}$$

where $\lim_{u\to 0} c(u) = c \neq 0$, $\lim_{u \to 0} \bar{\varepsilon}(u) = 0$ and $a > 0$. Now since $\lim_{u\to 0} \bar{\varepsilon}(u) = 0$, we have for $\varepsilon > 0$, there exists a positive t_0 such that for $|t| \leqslant \theta_1 < t_0^{-1}$,

$$\exp\left\{\int_{|\eta_1 t|}^{|\eta_1\theta_1|} \frac{\bar{\varepsilon}(u)}{u}\, du\right\} < \exp\left\{\int_{|\eta_1 t|}^{|\eta_1\theta_1|} \frac{\varepsilon}{u}\, du\right\} = |t/\theta_1|^{-\varepsilon}.$$

Since $c(t) = (1 + o(1))c$ as $t \to 0$, we get

$$h_1(\eta_1 t)(h_1(\eta_1\theta_1))^{-1} \leqslant |t/\theta_1|^{-\varepsilon}(1 + o(1))$$

which gives

$$h_1(\eta_1 t) \leqslant |t/\theta_1|^{-\varepsilon} h_1(\eta_1\theta_1)(1 + o(1)). \tag{2.17}$$

Hence by virtue of (2.15), (2.16) and (2.17) we have $\operatorname{Re} h_m(t) \leqslant |t/\theta_1|^{-\varepsilon}$. Again by (3.10), we have

$$h_m(t) = \operatorname{Re} h_m(t)(1 + o(1)) \quad \text{as } t \to 0.$$

Therefore, for $|t| < t_0^{-1}$ and $\varepsilon > 0$, we have $|h_m(t)| < \mu_1 |t|^{-\varepsilon}$, whence

$$|\phi_m(t) - 1| = \left|\exp\left(\tfrac{1}{2}t^2 h_m(t)\right) - 1\right| \leqslant \mu_1 |t|^{2-\varepsilon}. \tag{2.18}$$

Now by Gnedenko and Kolmogorov (1968) Chapter 2, page 54, we have

$$\Pr\left\{\left|\sum_2 a_r X_r x_m^r\right| \geqslant \lambda W_m\right\} = 1 - (\psi_m(\lambda_n) - \psi_m(-\lambda_n))$$

$$\leqslant 2 - \left|(\lambda_n/2)\int_{-d}^{d}\phi_m(t)\,dt\right| \quad (\text{where } d = 2/\lambda_n)$$

$$\leqslant (\lambda_n/2)\int_{-d}^{d}|\phi_m(t) - 1|\,dt \leqslant (\lambda_n/2)\mu_1\int_{-d}^{d}|t|^{2-\varepsilon}\,dt < \mu/\lambda_n^{2-\varepsilon}.$$

Hence the result.

LEMMA 4.

$$\left|\sum_3 a_r X_r(u) x_m^r\right| < \lambda_n Z_m$$

except for a set of measure at most $\mu/\lambda_n^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

(2.19) $$Z_m^2 = \sum_3 a_r^2 x_m^{2r} h_1(\eta_1\theta_2),$$

$\eta_2 = a_r x_m^r / Z_m$ *and* θ_2 *has similar meaning to* θ.

The proof of this lemma is exactly similar to that of Lemma 3.

LEMMA 5. *For a given* m, $|R_m| < V_m$ *except for a set of measure at most* $\mu/\lambda_n^{2-\varepsilon}$ *for* $m = [k/2] + 1, [k/2] + 2, \ldots, k$.

Case I. Let $V(X_r) = \infty$. Now we choose θ so that for $0 \leqslant r \leqslant n$, $(\eta\theta) < \min(\eta_1\theta_1, \eta_2\theta_2)$ by which

$$1 < h_1(\eta_1\theta_1) \leqslant b \quad \text{and} \quad 1 < h_1(\eta_2\theta_2) \leqslant b.$$

Now the choice of θ is final and by this choice it follows from Lemma 3 and Lemma 4 that

$$|R_m| < \lambda_n b^{1/2} k_n \left(\left(\sum_2 x_m^{2r}\right)^{1/2} + \left(\sum_3 x_m^{2r}\right)^{1/2}\right)$$

except for a set of measure at most $\mu/\lambda_n^{2-\varepsilon}$.

Proceeding as Samal and Mishra (1972a) page 527 and using Lemma 3, we can show that $|R_m| < V_m$ except for a set of measure at most $\mu/\lambda_n^{2-\varepsilon}$.

Case II. Let $V(X_r) = \sigma^2 < \infty$. Then

$$|R_m| < \lambda_n k_n \sigma\left(\left(\sum_2 x_m^{2r}\right)^{1/2} + \left(\sum_3 x_m^{2r}\right)^{1/2}\right).$$

As in Case I, we can show that $|R_m| < V_m$ except for a set of measure at most $\mu/\lambda_n^{2-\varepsilon}$.

3. Proof of the theorem

By (2.5) we have

$$U_{2m} = \sum_{M^{4m-1}+1}^{M^{4m+1}} a_r X_r x_m^r \quad \text{and} \quad U_{2m+1} = \sum_{M^{rm+1}+1}^{M^{4m+3}} a_r X_r x_m^r.$$

So X_r's in U_{2m} do not occur in U_{2m+1}. Therefore U_{2m} and U_{2m+1} are independent random variables.

We define sets E_m and F_m as follows:

$$E_m = \{U_{2m} \geqslant V_{2m}, U_{2m+1} < -V_{2m+1}\},$$
$$F_m = \{U_{2m} < -V_{2m}, U_{2m+1} \geqslant V_{2m+1}\}.$$

Let $G_m(x)$ and $g_m(t)$ be respectively the distribution function and the characteristic function of (U_m/V_m). Then

$$g_m(t) = \exp\left\{-\tfrac{1}{2}t^2(1/V_m^2)\sum_1 a_r^2 x_m^{2r} h(\eta t)\right\}. \tag{3.1}$$

Let

$$F(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x} \exp(-u^2/2)\, du. \tag{3.2}$$

From Lemma 2 it follows that $V_m \to \infty$ as $m \to \infty$ and so $\eta t \to 0$ as $m \to \infty$. Then $h(\eta t) = h_1(\eta t)(1 + o(1))$. By Lemma 1, we have in a neighbourhood of the origin,

$$h_1(\eta_t) = h_1(\eta\theta)|\theta/t|^{o(1)}(1 + o(1)).$$

So when $m \to \infty$,

$$g_m(t) = \exp\left\{-\frac{1}{2}t^2(1/V_m^2)\sum_1 a_r x_m^{2r} h_1(\eta\theta)|\theta/t|^{o(1)}(1 + o(1))(1 + o(1))\right\}$$
$$= \exp\{-\tfrac{1}{2}t^{2-o(1)}\theta^{o(1)}(1 + o(1))\} \qquad (\text{by definition of } V_m).$$

Therefore as $m \to \infty$, $g_m(t) \to \exp(-\frac{1}{2}t^2)$ in any bounded interval of t-values. Hence

$$\sup_x |G_m(x) - F(x)| = o(1). \tag{3.3}$$

So for $\varepsilon > 0$, we have

$$|G_{2m}(-1) - F(-1)| < \varepsilon, \tag{3.4}$$

and

(3.5) $$|G_{2m+1}(-1) - F(-1)| < \varepsilon.$$

Now since U_{2m} and U_{2m+1} are independent, we have

$$\begin{aligned}\Pr(E_m \cup F_m) &= \Pr(U_{2m} \geqslant V_{2m})\Pr(U_{2m+1} < -V_{2m+1}) \\ &\quad + \Pr(U_{2m} < -V_{2m})\Pr(U_{2m+1} \geqslant V_{2m+1}) \\ &= G_{2m+1}(-1)(1 - G_{2m}(1)) + G_{2m}(-1)(1 - G_{2m+1}(1)).\end{aligned}$$

By (3.4) and (3.5), we have $G_{2m}(-1) > F(-1) - \varepsilon$, $G_{2m+1}(-1) > F(-1) - \varepsilon$, and $1 - G_{2m}(1) > 1 - F(1) - \varepsilon$, $1 - G_{2m+1}(1) > 1 - F(1) - \varepsilon$. Hence $\Pr(E_m \cup F_m) \geqslant 2(F(-1) - \varepsilon)(1 - F(1) - \varepsilon)$. Thus $\Pr(E_m \cup F_m)$ is greater than a quantity which tends to $2F(-1)(1 - F(1))$ as $m \to \infty$ with n. This limit being positive we conclude that for large m, $\Pr(E_m \cup F_m) > \delta > 0$ where δ is an absolute constant. Hence $\Pr(E_m \cup F_m) = \delta_m > \delta > 0$.

Let us define random variables y_m such that it takes value 1 on $E_m \cup F_m$ and 0 elsewhere. Define

$$z_m = \begin{cases} 0 & \text{if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1}, \\ 1 & \text{otherwise.} \end{cases}$$

Proceeding as in Samal and Mishra (1972b) page 560, we can show that the number of roots in the interval (x_{2m_0}, x_{2k+1}) must exceed $\sum_{m=m_0}^{k}(y_m - y_m z_m)$, where $m_0 = [k/2] + 1$ and also

$$\begin{aligned}E\left(\sum y_m z_m\right) &\leqslant \sum \Pr\left\{\left(|R_{2m}| \geqslant V_{2m}\right) \cup \left(|R_{2m+1}| \geqslant V_{2m+1}\right)\right\} \\ &< (k+1)\left(\mu/\lambda_n^{2-\varepsilon}\right) \qquad \text{(by Lemma 5).}\end{aligned}$$

Hence for $0 < \beta < 2 - \varepsilon$,

$$\Pr\left\{\sum y_m z_m > (k+1)\lambda_n^{\beta}\left(\mu/\lambda_n^{2-\varepsilon}\right)\right\} < \frac{E(\sum y_m z_m)}{(k+1)\lambda_n^{\beta}\left(\mu/\lambda_n^{2-\varepsilon}\right)} < 1/\lambda_n^{\beta}.$$

Therefore,

(3.6) $$\sum_{m=m_0}^{k} y_m z_m < (k+1)\left(\mu/\lambda_n^{2-\varepsilon}\right)\lambda_n^{\beta} < \mu' k/\lambda_n^{2-\varepsilon-\beta}.$$

As in Samal and Mishra (1972a) page 525, we can show here that $\sum_{m=m_0}^{k} y_m > \mu k$ outside a set of measure at most μ'/k. Therefore

$$N_n > \sum_{m_0}^{k}(y_m - y_m z_m) > \mu k - \left(\mu' k/\lambda_n^{2-\varepsilon-\beta}\right) > \mu_0 k$$

outside an exceptional set of measure at most $\mu/k + 1/\lambda_n^{\beta}$. From (2.3) we have

(3.7) $$\mu_1(\lambda_n(k_n/t_n))^2 \leqslant M \leqslant \mu_2(\lambda_n(k_n/t_n))^2,$$

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and from (2.4), we have

$$\mu_3 \frac{\log n}{\log M} < k < \mu_4 \frac{\log n}{\log M}. \tag{3.8}$$

Now from (3.7) and (3.8) we have

$$\frac{\mu_5 \log n}{\log(\lambda_n(k_n/t_n))} < k < \mu_6 \frac{\log n}{\log(\lambda_n(k_n/t_n))}. \tag{3.9}$$

Set

$$\lambda_n = (t_n/k_n)\exp(1/\varepsilon'_n), \tag{3.10}$$

where ε'_n has the same meaning as ε_n. Obviously $\lambda_n \to \infty$ as $n \to \infty$. From (3.10) it follows that

$$\mu_5 \varepsilon'_n \log n < k < \mu_6 \varepsilon'_n \log n. \tag{3.11}$$

Therefore $k \to \infty$ as $n \to \infty$. Again by (3.11), we have

$$N_n > \mu_7 \varepsilon'_n \log n = \varepsilon_n \log n, \qquad \text{where } \mu_7 \varepsilon'_n = \varepsilon_n.$$

If G is the exceptional set then

$$\begin{aligned} \Pr(G) &< \mu'/k + 1/\lambda_n^\beta < \mu'/(\mu_5 \varepsilon'_n \log n) + (k_n/t_n)^\beta \exp\left(-\frac{\beta}{\varepsilon'_n}\right) \\ &= \mu'/((\mu_5/\mu_7)\varepsilon_n \log n) + (k_n/t_n)^\beta \exp(-(\mu_7 \beta \lambda \varepsilon_n)) \\ &< \mu/\varepsilon_n \log n + (k_n/t_n)^\beta \exp(-(\mu' \beta/\varepsilon_n)). \end{aligned}$$

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School of Mathematical Sciences
Sambalpur University
Jyoti Vihar, Burla, 768017
Orissa India

College of Basic Science and Humanities,
O.U.A.T. Bhubaneswar 751005
Puri, Orissa India