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NATURALLY REDUCTIVE HOMOGENEOUS REAL HYPERSURFACES IN QUATERNIONIC SPACE FORMS

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Abstract

We determine the naturally reductive homogeneous real hypersurfaces in the family of curvature-adapted real hypersurfaces in quaternionic projective space $\mathbb{H}P^n$ ($n \geq 3$). We conclude that the naturally reductive curvature-adapted real hypersurfaces in $\mathbb{H}P^n$ are Q-quasiumbilical and vice-versa. Further, we study the same problem in quaternionic hyperbolic space $\mathbb{H}H^n$ ($n \geq 3$).

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1. Introduction

A Riemannian manifold whose isometry group acts transitively on it is called a *Riemannian homogeneous space*. In the class of homogeneous Riemannian manifolds, naturally reductive spaces have good geometrical properties. They are defined by:

DEFINITION 1.1. Let $M = G/K$ be a Riemannian homogeneous space and g its metric tensor, where G is a transitive group of isometries of M and K its isotropy subgroup at some point $p \in M$. Then (M, g) is said to be a *naturally reductive Riemannian homogeneous space* if there exists a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G which satisfies the following conditions:

(i) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$,
(ii) $Ad(K)\mathfrak{m} \subset \mathfrak{m}$,
(iii) $g([X, Y]_{\mathfrak{m}}, Z) + g(Y, [X, Z]_{\mathfrak{m}}) = 0, \quad X, Y, Z \in \mathfrak{m}$,

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where $\mathfrak{k}$ is the Lie algebra of K and $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X, Y]$. (In the following we call these spaces naturally reductive homogeneous spaces.)

There is a criterion for homogeneity of a Riemannian manifold due to Ambrose and Singer [1].

THEOREM 1.1 ([1]). *A connected, complete and simply connected Riemannian manifold M is homogeneous if and only if there exists a tensor field T of type* $(1, 2)$ *on M such that*

(i) $g(T_X Y, Z) + g(Y, T_X Z) = 0$,
(ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z)$,
(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$

for $X, Y, Z \in \mathfrak{X}(M)$.

Here ∇ denotes the Levi Civita connection, R is the Riemannian curvature tensor of (M, g) and $\mathfrak{X}(M)$ is the Lie algebra of all C^∞ vector fields over M.

On the other hand, there is a criterion for naturally reductivity due to Tricerri and Vanhecke [7].

THEOREM 1.2 ([7]). *Under the same topological conditions for M, (M, g) is naturally reductive homogeneous if and only if there exists a tensor field T of type* $(1, 2)$ *on M such that*

(i) $g(T_X Y, Z) + g(Y, T_X Z) = 0$,
(ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z)$,
(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$,
(iv) $T_X X = 0$

for $X, Y, Z \in \mathfrak{X}(M)$.

In Theorem 1.1 and Theorem 1.2, if we put $\tilde{\nabla} := \nabla - T$, then the conditions (i), (ii) and (iii) are equivalent to $\tilde{\nabla} g = 0$, $\tilde{\nabla} R = 0$ and $\tilde{\nabla} T = 0$, respectively. Further, in both theorems, without the topological conditions of completeness and simply connectedness, 'the only if' part is always true. Furthermore, without these topological conditions, the conditions (i)–(iii) of Theorem 1.1 give a criterion for local homogeneity of M (for more details see [1] and [7]).

In all these cases, T is called a *homogeneous structure*.

Let $\overline{M}^n(c)$ be an n-dimensional ($n \geq 2$) quaternionic Kähler manifold of constant quaternionic sectional curvature $c \in \mathbb{R} - \{0\}$. The standard models for such spaces are the quaternionic projective space $\mathbb{H}P^n(c)$ (for $c > 0$) and the quaternionic hyperbolic space $\mathbb{H}H^n(c)$ (for $c < 0$). A connected real hypersurface M of $\overline{M}^n(c)$ is said to be

Q-quasiumbilical if its shape operator A is locally of the form

$$AX = \lambda X + \mu \sum_{k=1}^{3} \eta_k(X)\xi_k, \quad X \in TM, \tag{1.1}$$

for some real-valued C^∞ functions λ, μ (for definitions of ξ_k and η_k see Section 2). Pak ([6, Theorem 4]) proved that on every Q-quasiumbilical real hypersurface M in a non-flat quaternionic space form $\overline{M}^n(c)$ the functions λ and μ are constant and satisfy $\lambda\mu + c/4 = 0$. All Q-quasiumbilical real hypersurfaces in $\mathbb{H}P^n(c)$ and $\mathbb{H}H^n(c)$ are classified as follows:

THEOREM 1.3 ([2, 5]). *Let M be a Q-quasiumbilical real hypersurface in $\mathbb{H}P^n(c)$ or $\mathbb{H}H^n(c)$. Then M is locally congruent to one of the following spaces:*

- *a geodesic hypersphere of radius $r \in (0, \pi/\sqrt{c})$ in $\mathbb{H}P^n(c)$;*
- *a geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{H}H^n(c)$;*
- *a horosphere in $\mathbb{H}H^n(c)$;*
- *a tube of radius $r \in \mathbb{R}_+$ about the standard totally geodesic embedding of $\mathbb{H}H^{n-1}(c)$ in $\mathbb{H}H^n(c)$.*

Berndt and Vanhecke ([3]) proved

THEOREM 1.4 ([3, Theorem 3]). *Let M be a Q-quasiumbilical real hypersurface in $\mathbb{H}P^n(c)$ or $\mathbb{H}H^n(c)$. Then the tensor field T on M, which is locally given by*

$$\begin{aligned} T_X Y = \lambda \sum_{k=1}^{3} \big(\eta_k(Y)\phi_k X - \eta_k(X)\phi_k Y - g(\phi_k X, Y)\xi_k\big) \\ - \mu \sum_{k=1}^{3} \big(\eta_{k+1}(X)\eta_{k+2}(Y) - \eta_{k+2}(X)\eta_{k+1}(Y)\big)\xi_k, \end{aligned} \tag{1.2}$$

is a naturally reductive homogeneous structure on M.

There is a special class of real hypersurfaces in $\mathbb{H}P^n(c)$ and $\mathbb{H}H^n(c)$ formed by the so-called curvature-adapted real hypersurfaces (for definitions see section 2). This class includes all Q-quasiumbilical real hypersurfaces and they are classified by Berndt [2].

In this paper we classify naturally reductive homogeneous real hypersurfaces in the class of all curvature-adapted real hypersurfaces in $\mathbb{H}P^n(c)$ and $\mathbb{H}H^n(c)$. We prove

THEOREM 4.1. *Let M be a simply connected curvature-adapted real hypersurface in a quaternionic space form $\overline{M}^n(c)$ ($c \neq 0$, $n \geq 3$). In the case $c < 0$, we further assume that M has constant principal curvatures. Then M is naturally reductive homogeneous space if and only if M is Q-quasiumbilical.*

Furthermore, we obtain homogeneous structures on some curvature-adapted real hypersurfaces in $\overline{M}^n(c)$ $(c \neq 0)$. They include T of (1.2) as a special case. We establish

THEOREM 4.2. *On the real hypersurfaces $P_1^l(r)$, $H_1^l(r)$ and H_3 in $\overline{M}^n(c)$, the following tensors T define homogeneous structures for all $\sigma \in \mathbb{R}$:*

$$(4.5)\qquad T_X Y = \sum_{k=1}^{3} \{\eta_k(Y)\phi_k AX + \sigma\eta_k(X)\phi_k Y - g(\phi_k AX, Y)\xi_k\} + (\alpha+\sigma)\sum_{k=1}^{3}(\eta_{k+1}(X)\eta_{k+2}(Y) - \eta_{k+2}(X)\eta_{k+1}(Y))\xi_k,$$

where $\alpha = 2\cot 2r$ for $P_1^l(r)$, $\alpha = 2\coth 2r$ for $H_1^l(r)$ and $\alpha = 2$ for H_3 (for definitions of the spaces $P_1^l(r)$, $H_1^l(r)$ and H_3 see Section 2).

2. Preliminaries

Let $\overline{M}^n(c)$ be an n-dimensional quaternionic Kähler manifold of constant quaternionic sectional curvature $c \in \mathbb{R} - \{0\}$. Let $\bar{g}$ be the Riemannian metric, $\overline{\nabla}$ the Levi Civita connection and $\mathfrak{J}$ the quaternionic Kähler structure of $\overline{M}^n(c)$. The Riemannian curvature tensor $\overline{R}$ of $\overline{M}^n(c)$ is locally of the form

$$(2.1)\quad \overline{R}(X, Y)Z = \frac{c}{4}\left\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \sum_{k=1}^{3}\left(\bar{g}(J_kY, Z)J_kX - \bar{g}(J_kX, Z)J_kY - 2\bar{g}(J_kX, Y)J_kZ\right)\right\},$$

where (J_1, J_2, J_3) is a canonical local basis of $\mathfrak{J}$ and $X, Y, Z \in T\overline{M}$ (for more details see [4]).

Let M be a connected real hypersurface of $\overline{M}^n(c)$. We denote by g the induced Riemannian metric on M, by ∇ the Levi Civita connection of M and ν a local unit normal vector field along M in $\overline{M}^n(c)$.

The Gauss and Weingarten formulas are:

$$(2.2)\qquad \overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)\nu \quad\text{and}\quad \overline{\nabla}_X \nu = -AX$$

for vector fields X, Y on M.

We define on M local vector fields ξ_k $(k = 1, 2, 3)$, their dual 1-forms η_k and tensor fields ϕ_k of type (1, 1) as follows:

$$(2.3)\quad \xi_k = -J_k\nu,\quad \eta_k(X) = g(X, \xi_k),\quad J_kX = \phi_kX + \eta_k(X)\nu,\quad \text{for } X \in TM.$$

The tangent bundle TM of M is orthogonally decomposed by

$$TM = D \oplus D^{\perp}, \quad D^{\perp} = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}. \tag{2.4}$$

In the following, the index k has to be taken modulo three.

By the definition (2.3) we get the following relations:

$$\begin{aligned} &\phi_k\xi_k = 0, \quad \phi_k\xi_{k+1} = \xi_{k+2}, \quad \phi_k\xi_{k+2} = -\xi_{k+1}, \\ &\phi_k^2 X = -X + \eta_k(X)\xi_k, \\ &\phi_k\phi_{k+1}X = \phi_{k+2}X + \eta_{k+1}(X)\xi_k, \\ &\phi_k\phi_{k+2}X = -\phi_{k+1}X + \eta_{k+2}(X)\xi_k. \end{aligned} \tag{2.5}$$

The equation of Gauss and the equation of Codazzi are locally of the form

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\Bigg\{ & g(Y, Z)X - g(X, Z)Y \\ & + \sum_{k=1}^{3}\big(g(\phi_k Y, Z)\phi_k X - g(\phi_k X, Z)\phi_k Y - 2g(\phi_k X, Y)\phi_k Z\big)\Bigg\} \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2.6}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\sum_{k=1}^{3}\big(\eta_k(X)\phi_k Y - \eta_k(Y)\phi_k X - 2g(\phi_k X, Y)\xi_k\big). \tag{2.7}$$

Here R and A are the curvature tensor and the shape operator of M.

Since $\mathfrak{J}$ is parallel, there exist local one-forms q_1, q_2, q_3 on $\overline{M}^n(c)$ such that

$$\overline{\nabla}_X J_k = q_{k+2}(X)J_{k+1} - q_{k+1}(X)J_{k+2} \tag{2.8}$$

for all vector fields X on $\overline{M}^n(c)$. Using (2.2), (2.3) and (2.8), we have

$$(\nabla_X\phi_k)Y = \eta_k(Y)AX - g(AX, Y)\xi_k + q_{k+2}(X)\phi_{k+1}Y - q_{k+1}(X)\phi_{k+2}Y, \tag{2.9}$$

$$\nabla_X\xi_k = q_{k+2}(X)\xi_{k+1} - q_{k+1}(X)\xi_{k+2} + \phi_k AX \tag{2.10}$$

and

$$(\nabla_X\eta_k)Y = q_{k+2}(X)\eta_{k+1}(Y) - q_{k+1}(X)\eta_{k+2}(Y) + g(\phi_k AX, Y). \tag{2.11}$$

For a real hypersurface M of $\overline{M}^n(c)$, the normal Jacobi operator K_ν is defined by $K_\nu := \overline{R}(\cdot, \nu)\nu \in \mathrm{End}(TM)$. The definition of a curvature-adapted real hypersurface is the following:

DEFINITION 2.1. Let M be a real hypersurface of $\overline{M}^n(c)$. Then M is said to be a *curvature-adapted* if the equation $K_\nu \circ A = A \circ K_\nu$ holds.

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Berndt [2] classified curvature-adapted real hypersurfaces in $\mathbb{H}P^n$ as follows:

THEOREM 2.1 (Berndt [2]). *Let M be a connected curvature-adapted real hypersurface in $\mathbb{H}P^n$ $(n \geq 2)$. Then M is congruent to an open part of one of the following real hypersurfaces in $\mathbb{H}P^n$:*

$P_1^l(r)$: *a tube of some radius $r \in (0, \pi/2)$ around the canonically (totally geodesic) embedded quaternionic projective space $\mathbb{H}P^l$ for some $l \in \{0, \dots, n-1\}$;*
$P_2(r)$: *a tube of some radius $r \in (0, \pi/4)$ around the canonically (totally geodesic) embedded complex projective space $\mathbb{C}P^n$.*

For real hypersurfaces in $\mathbb{H}H^n$, Berndt [2] obtained:

THEOREM 2.2 (Berndt [2]). *Let M be a connected curvature-adapted real hypersurface in $\mathbb{H}H^n$ $(n \geq 2)$ with constant principal curvatures. Then M is congruent to an open part of one of the following real hypersurfaces in $\mathbb{H}H^n$:*

$H_1^l(r)$: *a tube of some radius $r \in \mathbb{R}_+$ around the canonically (totally geodesic) embedded quaternionic hyperbolic space $\mathbb{H}H^l$ for some $l \in \{0, \dots, n-1\}$;*
$H_2(r)$: *a tube of some radius $r \in \mathbb{R}_+$ around the canonically (totally geodesic) embedded complex hyperbolic space $\mathbb{C}H^n$;*
H_3: *a horosphere in $\mathbb{H}H^n$.*

The eigenvalues (that is, the principal curvatures) λ_1, λ_2, α_1, α_2, α_3 of the shape operators A and their multiplicities $m(\lambda_1)$, $m(\lambda_2)$, $m(\alpha_1)$, $m(\alpha_2)$, $m(\alpha_3)$ of the spaces in Theorem 2.1 and Theorem 2.2 are:

	$P_1^l(r)$	$P_2(r)$	$H_1^l(r)$	$H_2(r)$	H_3
λ_1	$\cot r$	$\cot r$	$\coth r$	$\coth r$	1
λ_2	$-\tan r$	$-\tan r$	$\tanh r$	$\tanh r$	—
$\alpha_1 = \alpha_2 = \alpha_3$	$2\cot 2r$	$2\cot 2r$	$2\coth 2r$	$2\coth 2r$	2
$\alpha_2 = \alpha_3 \neq \alpha_1$	—	$-2\tan 2r$	—	$2\tanh 2r$	—
$m(\lambda_1)$	$4(n-l-1)$	$2(n-1)$	$4(n-l-1)$	$2(n-1)$	$4(n-1)$
$m(\lambda_2)$	$4k$	$2(n-1)$	$4k$	$2(n-1)$	—
$m(\alpha_1)$	3	1	3	1	3
$m(\alpha_3)$	—	2	—	2	—

Here λ_1 and λ_2 (α_1, α_2 and α_3, respectively) belong to $A|_D$ ($A|_{D^\perp}$, respectively) (for more details see [2] and [5]).

The model spaces in Theorem 2.1 and Theorem 2.2 are all homogeneous real hypersurfaces in $\overline{M}^n(c)$. $P_1^0(r)$ and $H_1^0(r)$ are called *geodesic hyperspheres*.

For the second fundamental forms A of $P_1^l(r)$ and $H_1^l(r)$ we know the following:

LEMMA 2.1 (Pak [6]). *The second fundamental tensor A of $P_1^l(r)$ satisfies*

$$\phi_k A = A\phi_k, \quad k = 1, 2, 3, \tag{2.12}$$

$$A^2 - \alpha_1 A - I = -\sum_{k=1}^{3} \eta_k \otimes \xi_k, \tag{2.13}$$

$$(\nabla_X A)Y = -\sum_{k=1}^{3} \left\{\eta_k(Y)\phi_k X + g(\phi_k X, Y)\xi_k\right\}. \tag{2.14}$$

Here I denotes the identity transformation of the tangent bundle TM.

LEMMA 2.2 (Pak [6]). *The second fundamental tensor A of $H_1^l(r)$ and H_3 satisfies*

$$\phi_k A = A\phi_k, \quad k = 1, 2, 3, \tag{2.15}$$

$$A^2 - \alpha_1 A + I = \sum_{k=1}^{3} \eta_k \otimes \xi_k, \tag{2.16}$$

$$(\nabla_X A)Y = \sum_{k=1}^{3} \left\{\eta_k(Y)\phi_k X + g(\phi_k X, Y)\xi_k\right\}. \tag{2.17}$$

3. Lemmas

In this section we prove some lemmas. In the following we assume that M is a curvature-adapted real hypersurface in a quaternionic space form $\overline{M}^n(c)$ ($c = \pm 4$, $n \geq 3$). Further, we assume that M has constant principal curvatures when $c < 0$.

LEMMA 3.1. *If M is a naturally reductive homogeneous space, then $(\widetilde{\nabla}_W A)\xi_k = 0$ for all $W \in TM$.*

PROOF. Since the condition (ii) of Theorem 1.2 is satisfied, we have

$$\begin{aligned}
0 = (\widetilde{\nabla}_W R)(X, Y)Z = \sum_{k=1}^{3} \{ & g((\widetilde{\nabla}_W \phi_k)Y, Z)\phi_k X + g(\phi_k Y, Z)(\widetilde{\nabla}_W \phi_k)X \\
& - g((\widetilde{\nabla}_W \phi_k)X, Z)\phi_k Y - g(\phi_k X, Z)(\widetilde{\nabla}_W \phi_k)Y \\
& - 2g((\widetilde{\nabla}_W \phi_k)X, Y)\phi_k Z - 2g(\phi_k X, Y)(\widetilde{\nabla}_W \phi_k)Z\} \\
& + g((\widetilde{\nabla}_W A)Y, Z)AX + g(AY, Z)(\widetilde{\nabla}_W A)X \\
& - g((\widetilde{\nabla}_W A)X, Z)AY - g(AX, Z)(\widetilde{\nabla}_W A)Y.
\end{aligned} \tag{3.1}$$

By Theorem 2.1 and Theorem 2.2 we only have to consider the following two cases.

Case I. $\alpha_k \neq 0$ ($k = 1, 2, 3$).

Substituting $Y = Z = \xi_k$ in (3.1), we get

$$
\begin{aligned}
(3.2) \qquad 0 = {} & 3g((\widetilde{\nabla}_W \phi_{k+1})X, \xi_k)\xi_{k+2} - 3g((\widetilde{\nabla}_W \phi_{k+2})X, \xi_k)\xi_{k+1} \\
& - 3\eta_{k+2}(X)(\widetilde{\nabla}_W \phi_{k+1})\xi_k + 3\eta_{k+1}(X)(\widetilde{\nabla}_W \phi_{k+2})\xi_k \\
& + \alpha_k(\widetilde{\nabla}_W A)X - \alpha_k g((\widetilde{\nabla}_W A)X, \xi_k)\xi_k - \alpha_k \eta_k(X)(\widetilde{\nabla}_W A)\xi_k.
\end{aligned}
$$

Here we use the fact that $g((\widetilde{\nabla}_W A)\xi_k, \xi_k) = g(\widetilde{\nabla}_W \xi_k, (\alpha_k I - A)\xi_k) = 0$.

Substituting a vector $X \in D$ and taking the inner product of both sides of (3.2) with $Y \in D$, we are led to

$$
(3.3) \qquad g((\widetilde{\nabla}_W A)X, Y) = 0, \quad \text{for } X, Y \in D, \ W \in TM,
$$

because $\alpha_k \neq 0$.

Next, substituting $Y = Z \in D$, $X = \xi_k$ and $W \in TM$ in (3.1) and using (3.3), we arrive at

$$
-3\sum_{l=1}^{3} g((\widetilde{\nabla}_W \phi_l)\xi_k, Y)\phi_l Y + g(AY, Y)(\widetilde{\nabla}_W A)\xi_k - g((\widetilde{\nabla}_W A)\xi_k, Y)AY = 0.
$$

Suppose $(\widetilde{\nabla}_W A)\xi_k \neq 0$. Then we can choose a principal vector $Y \in D$ such that $(\widetilde{\nabla}_W A)\xi_k$ does not belong to $\operatorname{span}\{Y, \phi_1 Y, \phi_2 Y, \phi_3 Y\}$, since $n \geq 3$. By the table in Section 2, $g(AY, Y) \neq 0$ is satisfied for all our model spaces. This is a contradiction. So, we obtain $(\widetilde{\nabla}_W A)\xi_k = 0$.

Case II. $\alpha_k = 0$ ($k = 1, 2, 3$).

In this case, substituting $X = \xi_k$ and $Y = Z \in D$ in (3.1), we have

$$
-3\sum_{l=1}^{3} g((\widetilde{\nabla}_W \phi_l)\xi_k, Y)\phi_l Y + g(AY, Y)(\widetilde{\nabla}_W A)\xi_k - g((\widetilde{\nabla}_W A)\xi_k, Y)AY = 0.
$$

Using the analogous argument as in Case I, we have the assertion. □

LEMMA 3.2. *If M is naturally reductive homogeneous space, then $g(\widetilde{\nabla}_W \xi_k, \xi_k) = 0$ and $g(\widetilde{\nabla}_W \xi_k, X) = 0$ are satisfied for all $X \in D$ and $W \in TM$.*

PROOF. Since $g(\xi_k, \xi_k) = 1$, using the condition (i) of Theorem 1.2, we deduce $g(\widetilde{\nabla}_W \xi_k, \xi_k) = 0$. By Lemma 3.1, we arrive at

$$
(3.4) \qquad 0 = g((\widetilde{\nabla}_W A)\xi_k, X) = g(\widetilde{\nabla}_W \xi_k, (\alpha_k I - A)X).
$$

Choose a principal curvature vector $X \in D$ satisfying $AX = \lambda X$. Further, substituting this X in the right-hand side of (3.4), we are led to $(\alpha_k - \lambda)g(\widetilde{\nabla}_W \xi_k, X) = 0$. Since $\lambda \neq \alpha_k$, we get the assertion. □

Now, we define the following local 1-forms p_{k+1} and p_{k+2} by

$$p_{k+1}(X) := -g(\widetilde{\nabla}_X \xi_k, \xi_{k+2}), \quad p_{k+2}(X) := g(\widetilde{\nabla}_X \xi_k, \xi_{k+1}). \tag{3.5}$$

Then, by Lemma 3.2, we obtain $\widetilde{\nabla}_X \xi_k = p_{k+2}(X)\xi_{k+1} - p_{k+1}(X)\xi_{k+2}$.

LEMMA 3.3. *Let M be a naturally reductive homogeneous space. Then we have*

$$\begin{aligned}
(\widetilde{\nabla}_W \phi_k)\xi_k &= -p_{k+1}(W)\xi_{k+1} - p_{k+2}(W)\xi_{k+2},\\
(\widetilde{\nabla}_W \phi_{k+1})\xi_k &= p_k(W)\xi_{k+1},\\
(\widetilde{\nabla}_W \phi_{k+2})\xi_k &= p_k(W)\xi_{k+2}, \quad W \in TM.
\end{aligned}$$

PROOF. According to Lemma 3.2, we have

$$(\widetilde{\nabla}_W \phi_k)\xi_k = \widetilde{\nabla}_W(\phi_k \xi_k) - \phi_k \widetilde{\nabla}_W \xi_k = -\phi_k \widetilde{\nabla}_W \xi_k = -p_{k+2}(W)\xi_{k+2} - p_{k+1}(W)\xi_{k+1}.$$

This proves the first equation. The second and third equations can be proved analogously. □

LEMMA 3.4. *If M is naturally reductive homogeneous, then we have*

$$(\widetilde{\nabla}_W \phi_k)X = p_{k+2}(W)\phi_{k+1}X - p_{k+1}(W)\phi_{k+2}X, \quad \textit{for } X, W \in TM.$$

PROOF. By Lemma 3.3, we only need to prove the equation for $X \in D$. Substituting $X \in D$, $Y = \xi_k$ and $Z = \xi_{k+1}$ in (3.1) and using Lemma 3.1 and Lemma 3.3, we deduce

$$\begin{aligned}
0 &= g((\widetilde{\nabla}_W \phi_k)\xi_k, \xi_{k+1})\phi_k X + g((\widetilde{\nabla}_W \phi_{k+1})\xi_k, \xi_{k+1})\phi_{k+1}X\\
&\quad + g((\widetilde{\nabla}_W \phi_{k+2})\xi_k, \xi_{k+1})\phi_{k+2}X + g(\phi_{k+2}\xi_k, \xi_{k+1})(\widetilde{\nabla}_W \phi_{k+2})X\\
&= -p_{k+1}(W)\phi_k X + p_k(W)\phi_{k+1}X + (\widetilde{\nabla}_W \phi_{k+2})X.
\end{aligned}$$

This proves the lemma. □

Now we define local 1-forms r_{k+1} and r_{k+2} as follows:

$$r_{k+1}(X) := q_{k+1}(X) - p_{k+1}(X), \quad r_{k+2}(X) := q_{k+2}(X) - p_{k+2}(X).$$

Then, by (2.9) and (2.10), we get the following:

LEMMA 3.5. *If M is naturally reductive homogeneous, then we obtain*

$$\begin{aligned}
T_X \xi_k &= r_{k+2}(X)\xi_{k+1} - r_{k+1}(X)\xi_{k+2} + \phi_k A X,\\
(T_X \cdot \phi_k)Y &= r_{k+2}(X)\phi_{k+1}Y - r_{k+1}(X)\phi_{k+2}Y + \eta_k(Y)AX - g(AX, Y)\xi_k.
\end{aligned}$$

LEMMA 3.6. *If M is naturally reductive homogeneous. then the following relations hold*:

$$\phi_k AX = A\phi_k X, \quad X \in D,$$
$$r_{k+1}(\xi_k) = r_{k+2}(\xi_k) = 0 \quad (k = 1, 2, 3).$$

PROOF. By Lemma 3.5, we have

$$(T_{\xi_k} \cdot \phi_k)Y = r_{k+2}(\xi_k)\phi_{k+1}Y - r_{k+1}(\xi_k)\phi_{k+2}Y. \tag{3.6}$$

On the other hand, using the condition (iv) of Theorem 1.2, we obtain

$$\begin{aligned}(T_{\xi_k} \cdot \phi_k)Y &= T_{\xi_k}(\phi_k Y) - \phi_k(T_{\xi_k} Y) = -T_{\phi_k Y}\xi_k + \phi_k(T_Y\xi_k) \\ &= -\phi_k A\phi_k Y - AY + (r_{k+1}(Y) - r_{k+2}(\phi_k Y))\xi_{k+1} \\ &\quad + (r_{k+2}(Y) + r_{k+1}(\phi_k Y))\xi_{k+2} + \alpha_k\eta_k(Y)\xi_k.\end{aligned}$$

Combining this with (3.6), we have

$$\begin{aligned}\phi_k A\phi_k Y + AY = r_{k+1}(\xi_k)\phi_{k+2}Y - r_{k+2}(\xi_k)\phi_{k+1}Y + (r_{k+1}(Y) - r_{k+2}(\phi_k Y))\xi_{k+1} \\ + (r_{k+2}(Y) + r_{k+1}(\phi_k Y))\xi_{k+2} + \alpha_k\eta_k(Y)\xi_k.\end{aligned} \tag{3.7}$$

Substituting $Y = \phi_k X$ in (3.7), we arrive at

$$\begin{aligned}(A\phi_k - \phi_k A)X = {} & r_{k+1}(\xi_k)\phi_{k+1}X + r_{k+2}(\xi_k)\phi_{k+2}X \\ & + (r_{k+1}(\phi_k X) + r_{k+2}(X) - 2\eta_k(X)r_{k+2}(\xi_k))\xi_{k+1} \\ & + (r_{k+2}(\phi_k X) - r_{k+1}(X) + 2\eta_k(X)r_{k+1}(\xi_k))\xi_{k+2}.\end{aligned} \tag{3.8}$$

Further, taking a principal curvature vector $X \in D$ in (3.8), then the left-hand side of (3.8) belongs to span$\{\phi_k X\}$ and the right-hand side of (3.8) belongs to span$\{\phi_{k+1}X, \phi_{k+2}X, \xi_{k+1}, \xi_{k+2}\}$. So, we conclude that $(A\phi_k - \phi_k A)X = 0$, for $X \in D$ and $r_{k+1}(\xi_k) = r_{k+2}(\xi_k) = 0$. This proves Lemma 3.6. □

4. Proof of the theorem

We now prove our main theorem.

THEOREM 4.1. *Let M be a simply connected curvature-adapted real hypersurface in a quaternionic space form $\overline{M}^n(c)$ ($c \neq 0$, $n \geq 3$). In the case $c < 0$, we further assume that M has constant principal curvatures. Then M is naturally reductive homogeneous space if and only if M is Q-quasiumbilical.*

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PROOF. We only need to prove the only if part, since the if part is known by Theorem 1.4. According to Lemma 3.5 and Lemma 3.6, we get the following equation for X, $Y \in D$:

$$g(T_{\xi_{k+2}}(\phi_k Y), X) = -g(\phi_k Y, T_{\xi_{k+2}} X) = g(\phi_k Y, T_X \xi_{k+2}) = -g(\phi_{k+1} A Y, X)$$

and $g(\phi_k T_{\xi_{k+2}} Y, X) = g(\phi_{k+1} A Y, X)$. Therefore, we obtain

$$g((T_{\xi_{k+2}} \cdot \phi_k) Y, X) = -2g(\phi_{k+1} A Y, X). \tag{4.1}$$

On the other hand, we also have

$$g((T_{\xi_{k+2}} \cdot \phi_k) Y, X) = r_{k+2}(\xi_{k+2}) g(\phi_{k+1} Y, X). \tag{4.2}$$

So, from (4.1)–(4.2), when we define a local smooth function λ by $\lambda = -r_{k+2}(\xi_{k+2})/2$, we then deduce

$$\phi_{k+1} A Y = \lambda \phi_{k+1} Y, \quad \text{for } Y \in D. \tag{4.3}$$

Substituting $Y = \phi_{k+1} X$, $X \in D$ in both sides of (4.3), we arrive at

$$AX = \lambda X, \quad X \in D. \tag{4.4}$$

In our model spaces, only Q-quasiumbilical real hypersurfaces satisfy (4.4). This proves the theorem. □

REMARK. According to the argument in the proof of Theorem 4.1, we conclude that all the functions $r_k(\xi_k)$ coincide with the constant $\lambda = \cot r$ for $c > 0$ ($\lambda = \coth r$ for $c < 0$, respectively).

Concerning homogeneous structure tensors, we have the following:

THEOREM 4.2. *On the real hypersurfaces $P_1^l(r)$, $H_1^l(r)$ and H_3 in $\overline{M}^n(c)$, the following tensors T define homogeneous structures for all $\sigma \in \mathbb{R}$:*

$$\begin{aligned} T_X Y = &\sum_{k=1}^{3} \{\eta_k(Y)\phi_k A X + \sigma\, \eta_k(X)\phi_k Y - g(\phi_k A X, Y)\xi_k\} \\ &+ (\alpha + \sigma) \sum_{k=1}^{3} (\eta_{k+1}(X)\eta_{k+2}(Y) - \eta_{k+2}(X)\eta_{k+1}(Y))\xi_k, \end{aligned} \tag{4.5}$$

where $\alpha = 2\cot 2r$ for $P_1^k(r)$, $\alpha = 2\coth 2r$ for $H_1^k(r)$ and $\alpha = 2$ for H_3, respectively.

PROOF. We have to prove (i)–(iii) of Theorem 1.1. By a straightforward calculation, we get

$$\widetilde{\nabla} g = 0. \tag{4.6}$$

Using (2.5), Lemma 2.1 and Lemma 2.2, we obtain

$$\widetilde{\nabla} A = 0. \tag{4.7}$$

Further, by a straightforward calculation, we have

$$(\widetilde{\nabla}_X \phi_k) Y = \gamma_{k+2}(X)\phi_{k+1} Y - \gamma_{k+1}(X)\phi_{k+2} Y, \tag{4.8}$$

where $\gamma_{k+1}(X) = q_{k+1}(X) - 2\sigma\, \eta_{k+1}(X)$, $\gamma_{k+2}(X) = q_{k+2}(X) - 2\sigma\, \eta_{k+2}(X)$. Therefore, using (2.6), (4.6), (4.7) and (4.8), we are led to $\widetilde{\nabla} R = 0$.

Finally, we shall prove $\widetilde{\nabla} T = 0$. By a straightforward calculation, we get

$$\widetilde{\nabla}_X \xi_k = \gamma_{k+2}(X)\xi_{k+1} - \gamma_{k+1}(X)\xi_{k+2}, \tag{4.9}$$

$$\widetilde{\nabla}_X \eta_k = \gamma_{k+2}(X)\eta_{k+1} - \gamma_{k+1}(X)\eta_{k+2}. \tag{4.10}$$

Therefore, using (4.6), (4.7), (4.8), (4.9) and (4.10), we arrive at $(\widetilde{\nabla}_X T)_Y Z = 0$. The theorem is now proved by all the above arguments. □

REMARK. In the case of a Q-quasiumbilical real hypersurface the structure T of Theorem 4.2 reduces to (1.2) if we put $\sigma = -\lambda$.

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