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A NOTE ON HARDY'S INEQUALITY

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ABSTRACT. We prove a two-sided version of Hardy's inequality by methods arising from the proof of the Littlewood conjecture.

An inequality of G. H. Hardy states that for some $c > 0$,

$$(*) \qquad \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1,$$

for all $f \in L^1\big([0, 2\pi)\big)$ having one-sided Fourier series,

$$f(t) \sim \sum_{n=0}^{\infty} \hat{f}(n)e^{int}.$$

If we allow f to be an arbitrary L^1 function, the inequality $(*)$ fails. This can be seen by letting $f =$ Fejér kernel of order N for large N. However, it has been asked (for instance in [4]) whether $(*)$ generalizes as follows:

$$(**) \qquad \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1 + c\sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|}{n}, \quad f \in L^1.$$

This remains an open problem.

The purpose of this paper is to record the following alternative generalization of $(*)$. Its proof follows from a construction of L. Pigno and B. Smith [3], and from the methods for estimating such constructions [1], [2], [4].

THEOREM 1. *There is a constant $c > 0$ such that for any function $f \in L^1\big([0, 2\pi)\big)$,*

$$(1) \qquad \sum_{j=1}^{\infty}\Big(4^{-j}\sum_{4^{j-1}\leq n<4^j} |\hat{f}(n)|^2\Big)^{1/2} \leq c\|f\|_1 + c\sum_{j=1}^{\infty}\Big(4^{-j}\sum_{4^{j-1}\leq n<4^j} |\hat{f}(-n)|^2\Big)^{1/2}.$$

This is a generalization of $(*)$ because the left-hand side of $(*)$ is majorized by a constant times the left-hand side of (1), by the Cauchy-Schwarz inequality. The latter remark also means that (1) is stronger than $(*)$ already for the case of one-sided f. This was already "well-known", although I know of no precise reference: Roughly speaking,

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if one carries out a proof of the Littlewood conjecture starting with the "data" given by (2) below, one obtains (1) (when f is one-sided) without any additional work. In this paper, we are going to do some "additional work", and so obtain (1) for all f. Thus, Theorem 1 should be viewed as one more result from the "aftermath" of the proof of the Littlewood conjecture as given in [2]. We have however found it easier to use a version of a construction given by J. Fournier in [1]. This construction is also the one in [3], but there it had not been noticed that it gives the Littlewood conjecture. It has the specific advantage of producing trigonometric polynomials at every step. There are 3 other constructions given in [1], and we believe that with some modifications (such as convolving with a Fejér kernel at each step), each of them would also work here.

For completeness, we have repeated most of the known estimates. As the specialists will note, the main new observation is the estimate (4), which accounts for the right-hand side of (1). This observation was in a sense hinted at by B. Smith in [4], where it was remarked that one can show for $\mu \in M(T)$ that

$$\sum_{n>0} \frac{|\hat{\mu}(-n)|}{\sqrt{n}} < \infty \Rightarrow \sum_{n>0} \frac{|\hat{\mu}(n)|}{n} < \infty$$

using essentially the present construction; see [4, page 144, problem 1]. Smith's remark is equivalent to replacing our estimate (4) by the weaker statement

$$(4') \qquad |\hat{F}(-n)| < c/\sqrt{n}, \quad n > 0.$$

PROOF OF THEOREM 1. It suffices to consider the case when f is a trigonometric polynomial. Let us change notation to $a_n \equiv \hat{f}(n)$, $n \in \mathbb{Z}$. Thus $a_n = 0$ for $|n|$ sufficiently large.

Define trigonometric polynomials f_j, $j = 1, 2, \ldots$, by either $f_j = 0$, if $a_n = 0$ for all n in the interval $[4^{j-1}, 4^j)$, or otherwise by:

$$(2) \qquad f_j(t) = 4^{-j/2} \Big(\sum_{4^{j-1} \le k < 4^j} |a_k|^2 \Big)^{-1/2} \sum_{4^{j-1} \le n < 4^j} a_n e^{int}, \quad t \in [0, 2\pi).$$

The theorem will be derived from the following lemma by a standard duality argument.

LEMMA 1. *There exist absolute constants $c_1, c_2 > 0$ and a trigonometric polynomial F satisfying:*

$$(3) \qquad \|F\|_\infty \le c_1$$

$$(4) \qquad \Big(\sum_{4^{j-1} \le n < 4^j} |\hat{F}(-n)|^2 \Big)^{1/2} \le c_2 4^{-j/2}, \quad j \ge 1$$

$$(5) \qquad \Big(\sum_{4^{j-1} \le n < 4^j} |\hat{F}(n) - \hat{f}_j(n)|^2 \Big)^{1/2} \le \frac{1}{2} 4^{-j/2}, \quad j \ge 1.$$

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Assuming Lemma 1, we have

$$c_1\|f\|_1 \geq \left|\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{F(t)}\,dt\right| = \left|a_0\overline{\hat{F}(0)} + \sum_{n<0} a_n\overline{\hat{F}(n)} + \sum_{n>0} a_n\overline{\hat{F}(n)}\right|.$$

Therefore, by the triangle inequality,

$$\begin{aligned}\left|\sum_{n>0} a_n\overline{\hat{F}(n)}\right| &\leq c_1\|f\|_1 + |a_0\overline{\hat{F}(0)}| + \left|\sum_{n<0} a_n\overline{\hat{F}(n)}\right| \\ &\leq 2c_1\|f\|_1 + \sum_{n<0}|a_n|\,|\hat{F}(n)| \\ &= 2c_1\|f\|_1 + \sum_{j=1}^{\infty}\sum_{4^{j-1}\leq n<4^j}|a_{-n}|\,|\hat{F}(-n)| \\ &\leq 2c_1\|f\|_1 + c_2\sum_{j=1}^{\infty}\Big(4^{-j}\sum_{4^{j-1}\leq n<4^j}|a_{-n}|^2\Big)^{1/2},\end{aligned} \tag{6}$$

where we have used (4) and the Cauchy-Schwarz inequality on the last line.

On the left-hand side of (6),

$$\left|\sum_{n>0} a_n\overline{\hat{F}(n)}\right| \geq \sum_{j=1}^{\infty}\operatorname{Re}\Big(\sum_{4^{j-1}\leq n<4^j} a_n\overline{\hat{F}(n)}\Big).$$

Let $j \geq 1$. If $a_n = 0$ for all $n \in [4^{j-1}, 4^j)$, then

$$\operatorname{Re}\Big(\sum_{4^{j-1}\leq n<4^j} a_n\overline{\hat{F}(n)}\Big) = 0 = \Big(4^{-j}\sum_{4^{j-1}\leq n<4^j}|a_n|^2\Big)^{1/2}.$$

If the a_n, $4^{j-1} \leq n < 4^j$ are not all zero, (2) gives

$$\|f_j\|_2 = 4^{-j/2}. \tag{7}$$

This means that (5) may be re-stated in the form

$$\|A - B\|_2 \leq \frac{1}{2}\|B\|_2$$

where the vectors A and B are given by

$$A_n = \hat{F}(n), \quad B_n = \hat{f}_j(n), \quad 4^{j-1} \leq n < 4^j.$$

Letting $\langle\ ,\ \rangle$ denote the usual complex inner product, we have

$$\|A\|_2^2 - 2\operatorname{Re}\langle A, B\rangle + \|B\|_2^2 \leq \frac{1}{4}\|B\|_2^2,$$

whence

$$\begin{aligned}2\operatorname{Re}\langle A, B\rangle &\geq \frac{3}{4}\|B\|_2^2 + \|A\|_2^2 \\ &\geq \frac{3}{4}\|B\|_2^2 \\ &= \frac{3}{4}4^{-j}.\end{aligned}$$

It follows that

$$\operatorname{Re}\Big(\sum_{4^{j-1}\le n<4^j} a_n\overline{\hat{F}(n)}\Big) = 4^{j/2}\Big(\sum_{4^{j-1}\le n<4^j}|a_n|^2\Big)^{1/2}\operatorname{Re}\Big(\sum_{4^{j-1}\le n<4^j}\hat{f}_j(n)\overline{\hat{F}(n)}\Big)$$
$$\ge \frac{3}{8}\Big(4^{-j}\sum_{4^{j-1}\le n<4^j}|a_n|^2\Big)^{1/2}.$$

Combining the two sides of (6), we obtain (1) with $c = \frac{8}{3}\max(2c_1, c_2)$.

PROOF OF LEMMA 1. Let $0<\epsilon<1$ be a parameter to be specified later, and define trigonometric polynomials $F_j, j = 0,1,2,\ldots$ (depending on ϵ) by $F_0 = 0$ and

$$F_{j+1} = \frac{\epsilon}{2}f_{j+1} + (1-\epsilon^2|f_{j+1}|^2)F_j - \frac{\epsilon}{2}\bar{f}_{j+1}F_j^2. \tag{8}$$

Then define

$$F = \frac{2}{\epsilon}F_k,$$

where k is chosen so large that $F_k = F_{k+1} = F_{k+2} = \cdots$. This happens because $f_j = 0$ eventually, f being a trigonometric polynomial.

We will prove that F satisfies (3), (4) and (5), for appropriate absolute constants $\epsilon, c_1, c_2, > 0$.

First observe that

$$\|f_j\|_\infty \le \sum_{4^{j-1}\le n<4^j}|\hat{f}_j(n)| \le 4^{j/2}\|f_j\|_2 \le 1. \tag{9}$$

It follows [3], [4] (by a clever application of the maximum modulus principle, and induction on j) that

$$\|F_j\|_\infty \le 1, \quad j\ge 0, \tag{10}$$

so that we have (3) with $c_1 = 2/\epsilon$.

Next, we prove that

$$\Big(\sum_{|n|\ge 4^{j-1}}|\hat{F}_k(n)|^2\Big)^{1/2} \le 16\epsilon 4^{-j/2} \tag{12}$$

for all $j,k\ge 0$. This implies (4) with $c_2 = 32$, and it will be used again for the proof of (5). We first show that

$$\operatorname{spec}(F_j) \subset \big(-2\cdot 4^{j+1}, 4^j\big) \equiv \{n\in\mathbb{Z} : -2\cdot 4^{j+1} < n < 4^j\}. \tag{13}$$

Clearly $\operatorname{spec}(F_0) = \emptyset \subset (-2\cdot 4, 1)$. Suppose (13) is true for a fixed $j\ge 0$. We have

$$\operatorname{spec}(f_{j+1}) \subset [4^j, 4^{j+1}),$$
$$\operatorname{spec}(|f_{j+1}|^2F_j) \subset \big(-(4^{j+1}-4^j), 4^{j+1}-4^j\big) + (-2\cdot 4^{j+1}, 4^j)$$
$$\subset (-3\cdot 4^{j+1}, 4^{j+1}),$$
$$\operatorname{spec}(\bar{f}_{j+1}F_j^2) \subset (-4^{j+1}, -4^j] + (-4\cdot 4^{j+1}, 2\cdot 4^j)$$
$$\subset (-5\cdot 4^{j+1}, 4^j), \text{ and therefore}$$
$$\operatorname{spec}(F_{j+1}) \subset (-5\cdot 4^{j+1}, 4^{j+1}) \subset (-2\cdot 4^{j+2}, 4^{j+1}).$$

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So (13) is true for all $j \geq 0$ by induction. To begin the proof of (12), note that for $k \leq j-3$ we have

$$\begin{aligned}\operatorname{spec}(F_k) &\subset (-2\cdot 4^{k+1}, 4^k)\\ &\subset (-2\cdot 4^{j-2}, 4^{j-3}),\end{aligned}$$

so that the left-hand side of (12) is 0, and there is nothing to prove. Next suppose $k > j-3$ and $k, j \geq 0$. For convenience define $F_\ell = 0 = F_{\ell+1}$ for indices $\ell < 0$, and write

$$\begin{aligned}F_k &= F_{j-3} + \sum_{\ell=j-3}^{k-1} (F_{\ell+1} - F_\ell)\\ &= F_{j-3} + \sum_{\ell=j-3}^{k-1} \Big(\frac{\epsilon}{2} f_{\ell+1} - \epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2}\bar{f}_{\ell+1} F_\ell^2\Big). \qquad (14)\end{aligned}$$

Again note that $\hat{F}_{j-3}(n) = 0$ for $|n| \geq 4^{j-1}$, so

$$\hat{F}_k = \Big(\sum_{\ell=j-3}^{k-1} \frac{\epsilon}{2} f_{\ell+1} - \epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2}\bar{f}_{\ell+1} F_\ell^2\Big)^{\wedge} \qquad (15)$$

on the set of n with $|n| \geq 4^{j-1}$. Recalling that $|F_\ell|, |f_{\ell+1}| \leq 1$, $\|f_{\ell+1}\|_2 \leq 4^{-(\ell+1)/2}$, we also get

$$\begin{aligned}\|\, |f_{\ell+1}|^2 F_\ell\|_2 &\leq \|f_{\ell+1}\|_2 \leq 4^{-(\ell+1)/2}, \text{ and}\\ \|\bar{f}_{\ell+1} F_\ell^2\|_2 &\leq \|f_{\ell+1}\|_2 \leq 4^{-(\ell+1)/2}.\end{aligned}$$

Taking ℓ^2 norms on both sides of (15), for $|n| \geq 4^{j-1}$, and substituting the latter estimates (via Plancherel's theorem and the triangle inequality) we get

$$\begin{aligned}\Big(\sum_{|n|\geq 4^{j-1}} |\hat{F}_k(n)|^2\Big)^{1/2} &\leq \sum_{\ell=j-3}^{k-1} \Big(\frac{\epsilon}{2} + \epsilon^2 + \frac{\epsilon}{2}\Big) 4^{-(\ell+1)/2}\\ &\leq 16\epsilon 4^{-j/2}\end{aligned}$$

(since $\epsilon \leq 1$), which is (12).

To obtain (5), fix $j \geq 1$ and recall that $F = \frac{2}{\epsilon} F_k$ for some $k > j$. Write

$$F_k - \frac{\epsilon}{2} f_j = \Big(F_{j-1} + \sum_{\ell=j}^{k-1} \frac{\epsilon}{2} f_{\ell+1}\Big) + \Big(\sum_{\ell=j-1}^{k-1} -\epsilon^2 |f_{\ell+1}|^2 F_\ell - \frac{\epsilon}{2}\bar{f}_{\ell+1} F_\ell^2\Big). \qquad (16)$$

Note that

$$\Big(F_{j-1} + \sum_{\ell=j}^{k-1} \frac{\epsilon}{2} f_{\ell+1}\Big)^{\wedge}(n) = 0, \quad 4^{j-1} \leq n < 4^j, \qquad (17)$$

by (13) and (2). Observe [1] also that

$$\begin{aligned}\operatorname{spec}(\bar{f}_{\ell+1} F_\ell) &\subset (-4^{\ell+1}, -4^\ell] + (-2\cdot 4^{\ell+1}, 4^\ell)\\ &\subset (-\infty, 0).\end{aligned}$$

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Therefore, on the interval $4^{i-1} \leq n < 4^j$, we can write

$$(18) \qquad \begin{aligned}(\bar{f}_{\ell+1}F_\ell^2)^\wedge &= (\bar{f}_{\ell+1}F_\ell F_\ell)^\wedge \\ &= (\bar{f}_{\ell+1}F_\ell F_{\ell,j})^\wedge,\end{aligned}$$

where $F_{\ell,j}$ is the "truncation":

$$F_{\ell,j}(t) = \sum_{n \geq 4^{j-1}} \hat{F}_\ell(n)e^{int}.$$

Combining (16), (17), and (18), we get that

$$(19) \qquad \Big(F_k - \frac{\epsilon}{2}f_j\Big)^\wedge(n) = \Big(\sum_{\ell=j-1}^{k-1} -\epsilon^2|f_{\ell+1}|^2F_\ell - \frac{\epsilon}{2}\bar{f}_{\ell+1}F_\ell F_{\ell,j}\Big)^\wedge(n)$$

for $4^{j-1} \leq n < 4^j$. This time we will use the 1-norm estimates,

$$\begin{aligned}\| \, |f_{\ell+1}|^2F_\ell\|_1 &\leq \|f_{\ell+1}\|_2^2 \leq 4^{-(\ell+1)}, \text{ and} \\ \|\bar{f}_{\ell+1}F_\ell F_{\ell,j}\|_1 &\leq \| \, |f_{\ell+1}| \cdot |F_{\ell,j}| \, \|_1 \\ &\leq \|f_{\ell+1}\|_2\|F_{\ell,j}\|_2 \\ &\leq 4^{-(\ell+1)/2}\Big(\sum_{n \geq 4^{j-1}} |\hat{F}_\ell(n)|^2\Big)^{\frac{1}{2}} \\ &\leq 4^{-(\ell+1)/2} \cdot 16\epsilon 4^{-j/2}\end{aligned}$$

where we have used (12) on the last line. Therefore, for each $n \in [4^{j-1}, 4^j)$ we have

$$\begin{aligned}\Big|\hat{F}_k(n) - \frac{\epsilon}{2}\hat{f}_j(n)\Big| &\leq \Big\|\sum_{\ell=j-1}^{k-1} -\epsilon^2|f_{\ell+1}|^2F_\ell - \frac{\epsilon}{2}\bar{f}_{\ell+1}F_\ell F_{\ell,j}\Big\|_1 \\ &\leq \sum_{l=j-1}^{k-1}\Big(\epsilon^2 4^{-(\ell+1)} + \frac{\epsilon}{2}4^{-(\ell+1)/2} \cdot 16\epsilon 4^{-j/2}\Big) \\ &\leq 18\epsilon^2 4^{-j}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big(\sum_{4^{j-1} \leq n < 4^j}\Big|\hat{F}_k(n) - \frac{\epsilon}{2}\hat{f}_j(n)\Big|^2\Big)^{1/2} &\leq 4^{j/2} \cdot 18\epsilon^2 4^{-j} \\ &= (36\epsilon)\frac{\epsilon}{2}4^{-j/2}.\end{aligned}$$

Since $F = \frac{2}{\epsilon}F_k$, we get (5) by choosing $\epsilon = \frac{1}{72}$.

This completes the proof of Lemma 1.

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