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POINCARÉ DUALITY FOR K-THEORY OF EQUIVARIANT COMPLEX PROJECTIVE SPACES

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Abstract. We make explicit *Poincaré duality* for the equivariant K-theory of equivariant complex projective spaces. The case of the trivial group provides a new approach to the K-theory orientation [**3**].

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1. Introduction. In well behaved cases one expects the cohomology of a finite complex to be a contravariant functor of its homology. However, orientable manifolds have the special property that the cohomology is covariantly isomorphic to the homology, and hence in particular the cohomology ring is self-dual. More precisely, Poincaré duality states that taking the cap product with a fundamental class gives an isomorphism between homology and cohomology of a manifold.

Classically, an n-manifold M is a topological space locally modelled on $\mathbb{R}^n$, and the fundamental class of M is a homology class in $H_n(M)$. Equivariantly, it is much less clear how things should work. If we pick a point x of a smooth G-manifold, the tangent space V_x is a representation of the isotropy group G_x, and its G-orbit is locally modelled on $G \times_{G_x} V_x$; both G_x and V_x depend on the point x. It may happen that we have a W-manifold, in the sense that there is a single representation W so that V_x is the restriction of W to G_x for all x, but this is very restrictive. Even if there are fixed points x, the representations V_x at different points need not be equivalent. It is therefore not clear even in which dimension we should hope to find a fundamental class. In general one needs complicated apparatus to provide a suitable context [**6**], and ordinary cohomology is especially complicated. Fortunately, particular examples can be better behaved.

The purpose of the present paper is to look at the very concrete example of linear complex projective spaces: these are not usually W-manifolds for any W, but we observe that in equivariant K-theory there is a natural choice of fundamental class, and we make the resulting Poincaré duality isomorphism explicit. In the non-equivariant case this gives an elementary approach to the classical K-theory fundamental class [**3**].

2. Preliminaries.

2.1. Linear projective spaces. Let V be a unitary complex representation of a finite group G. We write $S(V)$ for the unit sphere, $D(V)$ for the unit disc in V, and S^V for the one-point compactification, $S^V = D(V)/S(V)$. We write T for the circle group $T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and z for the natural representation of T.

Definition 2.1. We write $\mathbb{C}P(V)$ for the G-space of complex lines in V, so that

$$\mathbb{C}P(V) \cong S(V \otimes z)/T.$$

2.2. Equivariant stable homotopy theory. Although our principal results are stated in terms of homology and cohomology, we often work in the equivariant stable homotopy category. We summarise some standard results (see [**1**], [**11**] or [**12**, XVI §5] for details). The relevance arises since equivariant homology and cohomology theories are represented by G-spectra in the sense that for based G-spaces X,

$$\widetilde{E}^*_G(X) = [X, E]^*_G \text{ and } \widetilde{E}^G_*(X) = [S^0, E \wedge X]^G_*,$$

where E is the representing G-spectrum of the theory.

Lemma 2.2 (Change of groups [**11**, II.4.3 and II.6.5]). *Let H be a subgroup of G, and suppose that A is an H-spectrum and B is a G-spectrum. Then there are natural isomorphisms*

$$\theta : [A, B]_H \xrightarrow{\cong} [G_+ \wedge_H A, B]_G \text{ and } \phi : [B, A]_H \xrightarrow{\cong} [B, G_+ \wedge_H A]_G. \qquad \square$$

Theorem 2.3 (Adams isomorphism [**11**, II.7.1]). *Suppose B is a T-free $(G \times T)$-spectrum. For any G-spectrum A there is a natural isomorphism*

$$[A, \Sigma B/T]_G \cong [A, B]_{G\times T},$$

induced by a suitable transfer map. $\square$

2.3. Spanier-Whitehead duality. Using function spectra we may define the functional duality functor $DX = F(X, S^0)$ on G-spectra X. When restricted to finite G-spectra, the natural map $X \longrightarrow D^2X$ is an equivalence, and one may give a more concrete description: if X is a based G-space which embeds in the sphere $S^{1\oplus V}$, we have

$$\Sigma^V DX \simeq S^{1\oplus V}\backslash X,$$

where we have supressed notation for the suspension spectrum. The formal properties of the category of G-spectra give a useful statement relating homology and cohomology.

Lemma 2.4 (Spanier-Whitehead duality [**11**, III.2.9]). *If X, Y are finite G-CW-spectra and E is a G-spectrum, then*

(i) *there is an isomorphism $SW : E^*_G(X) \xrightarrow{\cong} E^G_*(DX)$;*

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(ii) *a G-map $f : X \longrightarrow Y$ gives rise to a commutative diagram*

$$\begin{tikzcd} E_G^*(Y) \arrow[r, "f^*"] \arrow[d, "SW"', "\cong"] & E_G^*(X) \arrow[d, "\cong"', "SW"] \\ E_*^G(DY) \arrow[r, "(Df)_*"] & E_*^G(DX). \end{tikzcd}$$

□

2.4. Equivariant K-theory. We are concerned with the equivariant K-theory of Atiyah and Segal [**13**] of finite G-CW-complexes, so that $K_G^0(X)$ is the Grothendieck group of equivariant vector bundles over X, and K_G^* is $R(G)$ in even degrees and zero in odd degrees. We use the represented extension to arbitrary spectra: there is a G-spectrum K so that for a based G-space X we have

$$\widetilde{K}_G^0(X) = [X, K]_G \text{ and } \widetilde{K}_0^G(X) = [S^0, K \wedge X]_G.$$

Equivariant K-theory has its version of the *Thom isomorphism*: if E is a bundle over X then we have an isomorphism $\tau : \widetilde{K}_G^*(X) \xrightarrow{\cong} \widetilde{K}_G^*(X^E)$, where X^E denotes the Thom space of E. The isomorphism is made explicit in [**13**, §3], and this permits a definition of the *Euler class* $\chi(V) = i_V^*\tau(1) \in K_G^*$, where i_V is the inclusion $S^0 \hookrightarrow S^V$. In turn, this paves the way for the equivariant *Bott periodicity*.

Theorem 2.5 (Equivariant Bott periodicity [**13**]). *For a based G-space X, and a complex representation V of G, multiplying by the Bott class $\tau(1) \in \widetilde{K}_G^0(S^V)$ gives a natural isomorphism*

$$\widetilde{K}_G^0(X) \xrightarrow{\cong} \widetilde{K}_G^0(S^V \wedge X).$$

Moreover, if $\dim_{\mathbb{C}}(V) = n$ *then*

$$\chi(V) = 1 - \lambda V + \lambda^2 V - \cdots + (-1)^n \lambda^n V \in R(G),$$

where $\lambda^r V$ denotes the r^{th} exterior power of V.

2.4.1. *Restriction in equivariant K-theory.* For $H \leq G$, let $\pi : G/H \longrightarrow G/G$ denote projection. It is not hard to verify from the explicit form of the change of groups isomorphisms that the restriction maps in homology and cohomology are represented in the following sense.

Lemma 2.6. *There are commutative diagrams*

$$\begin{tikzcd} \widetilde{K}_G^0(X) \arrow[r, "\mathrm{Res}_H^G"] \arrow[d, equal] & \widetilde{K}_H^0(X) \arrow[d, "\cong"', "\theta"] \\ \widetilde{K}_G^0(G/G_+ \wedge X) \arrow[r, "(\pi\wedge 1)^*"] & \widetilde{K}_G^0(G/H_+ \wedge X) \end{tikzcd}$$

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and

$$\begin{tikzcd}
\widetilde{K}_0^G(X) \arrow[rr, "\mathrm{Res}_H^G"] \arrow[d, equal] & & \widetilde{K}_0^H(X) \arrow[d, "\phi", "\cong"'] \\
\widetilde{K}_0^G(G/G_+ \wedge X) \arrow[rr, "(D(\pi)\wedge 1)_*"] & & \widetilde{K}_0^G(G/H_+ \wedge X).
\end{tikzcd}$$

□

The restriction maps are not, in general, injective. However, one finds that

$$\mathrm{Res}_*^G : K_G^0(\mathbb{C}P(V)) \xrightarrow{\{\mathrm{Res}_H^G\}} \prod_{\substack{H\leq G \\ H \text{ cyclic}}} K_H^0(\mathbb{C}P(V)) \tag{2.7}$$

and the analogous map in homology *are* both injective. This is easily deduced from the corresponding statement about representation rings. For example, it follows from the calculations in Subsection 4.1 that $K_G^0(\mathbb{C}P(V))$ and $K_H^0(\mathbb{C}P(V))$ are both free modules on generators which map to each other under restriction. This is explained in more detail in [**14**].

3. Equivariant Poincaré duality.

3.1. Orientation of topological G-manifolds. We work with smooth G-manifolds M, for which the *Slice Theorem* [**4**, II Theorem 5.4] asserts that given $x \in M$ with isotropy $G_x \leq G$, there is a neighbourhood U of the orbit Gx, which is G-homeomorphic to $G \times_{G_x} V_x$, where V_x is the tangent space to M at x.

Lemma 3.1. *Using the notation of the Slice Theorem, for each i there are isomorphisms*

(i) $E_i^G(M, M\backslash Gx) \cong E_i^G(U, U\backslash Gx)$;
(ii) $E_i^G(U, U\backslash Gx) \cong E_i^G(G \times_{G_x} V_x, (G \times_{G_x} V_x)\backslash Gx)$;
(iii) $E_i^G(G \times_{G_x} V_x, (G \times_{G_x} V_x)\backslash Gx) \cong \widetilde{E}_i^G(G_+ \wedge_{G_x} S^{V_x})$.

Proof. For (i) and (ii), use excision. Part (iii) is equivalent to showing that

$$E_i^{G_x}(V_x, V_x \setminus \{0\}) \cong \widetilde{E}_i^{G_x}(S^{V_x}),$$

and this follows since $S^{V_x} \setminus \{0\} \cong_G V_x$, which is contractible. □

Composing the three isomorphisms of Lemma 3.1, the outcome is that

$$E_*^G(M, M\backslash Gx) \cong \widetilde{E}_*^{G_x}(S^{V_x}). \tag{3.2}$$

Provided we restrict to cohomology theories E_G^* and manifolds M so that the modules $\widetilde{E}_*^{G_x}(S^{V_x})$ that occur in this way are free on one generator, we may copy the classical definitions.

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DEFINITION 3.3 (Fundamental classes). (i) A cohomology theory $E_G^*(\cdot)$ is said to be *complex stable* if, for each complex representation V, there are classes $\sigma_V \in \widetilde{E}_G^{|V|}(S^V)$ giving isomorphisms

$$\widetilde{E}_G^*(S^{|V|} \wedge X) \xrightarrow{\cong} \widetilde{E}_G^*(S^V \wedge X)$$

for any G-spectrum X. Note in particular that this means $\widetilde{E}_G^*(S^V)$ is a free E_G^*-module on one generator.

(ii) Let M be a smooth G-manifold of dimension n, and let $E_G^*(\cdot)$ be a complex stable cohomology theory. Consider the composite ϕ_{Gx} below. The maps labelled (i), (ii), (iii) are the corresponding isomorphisms of Lemma 3.1, ϕ is the change of group isomorphism (Lemma 2.2) and

$$i_*^{Gx} : E_*^G(M) \cong E_*^G(M, \emptyset) \longrightarrow E_*^G(M, M\backslash Gx)$$

is the map induced by G-inclusion of the G-pairs $(M, \emptyset) \xhookrightarrow{i^{Gx}} (M, M\backslash Gx)$.

$$\begin{tikzcd}
E_*^G(M) \arrow[rr, "\phi_{Gx}"] \arrow[d, "i_*^{Gx}"'] & & \widetilde{E}_*^{G_x}(S^{V_x}) \\
E_*^G(M, M\backslash Gx) \arrow[d, "(i)"', "\cong"] & & \widetilde{E}_*^G(G_+ \wedge_{G_x} S^{V_x}) \arrow[u, "\cong"', "\phi^{-1}"] \\
E_*^G(U, U\backslash Gx) \arrow[rr, "\cong", "(ii)"'] & & E_*^G(G \times_{G_x} V_x, (G \times_{G_x} V_x)\backslash Gx). \arrow[u, "\cong"', "(iii)"]
\end{tikzcd} \tag{3.4}$$

An element $\xi \in E_n^G(M)$ is a *fundamental class* for M if the image $\phi_{Gx}(\xi)$ is an $\widetilde{E}_*^{G_x}$-module generator for $\widetilde{E}_*^{G_x}(S^{V_x})$ for all $x \in M$, in which case one writes $[M]$ for such a ξ.

3.2. Poincaré duality. Before we can state the Poincaré duality theorem we must first recall [**11**, III §3] how cap products work in the represented setting.

DEFINITION 3.5 (Cap products). Let E be a commutative ring G-spectrum with multiplicative structure μ, and let X be a G-CW-complex. The *cap product* $E_G^*(X) \otimes E_*^G(X) \longrightarrow E_*^G(X)$ is defined by setting $c \cap h$ to be the composite

$$S \xrightarrow{h} E \wedge X \xrightarrow{1\wedge\Delta} E \wedge X \wedge X \xrightarrow{1\wedge c\wedge 1} E \wedge E \wedge X \xrightarrow{\mu\wedge 1} E \wedge X.$$

THEOREM 3.6 (Poincaré duality). *Let $E_G^*(\cdot)$ be a complex stable cohomology theory. If M is a smooth G-manifold with E_G^*-fundamental class $[M]$ then there is an isomorphism*

$$E_G^*(M) \xrightarrow{\cong} E_*^G(M)$$

given by capping with the fundamental class, precisely $a \longmapsto a \cap [M]$ for $a \in E_G^*(M)$.

Proof. The classical proof (see, for example, [**8**, §26]) proceeds by showing that $(-) \cap [M]$ induces an isomorphism on larger and larger subsets of M, starting from a point, and using Mayer-Vietoris sequences and excision. The only difference in our

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case is that we must start with a G-point, in other words the orbit Gx for $x \in M$. By definition, the fundamental class provides exactly this input. □

4. Construction of the fundamental class.

4.1. Equivariant K-theory of $\mathbb{C}P(V)$. Our computation of $K_G^*(\mathbb{C}P(V))$ arises from the based cofibre sequence

$$S(V \otimes z)_+ \longrightarrow D(V \otimes z)_+ \longrightarrow D(V \otimes z)/S(V \otimes z) \cong S^{V \otimes z} \tag{4.1}$$

and the following fundamental result of Atiyah and Segal [**13**].

THEOREM 4.2. *Let G be a compact Lie group. Suppose N is a normal subgroup which acts freely on the G-CW-complex X. Then the quotient $X \longrightarrow X/N$ induces an isomorphism $K_{G/N}^*(X/N) \xrightarrow{\cong} K_G^*(X)$.* □

Applying $\widetilde{K}_{G\times T}^*(-)$ to (4.1) and appealing to Theorem 4.2 gives the long exact sequence

$$\cdots \longrightarrow \widetilde{K}_{G\times T}^0(S^{V\otimes z}) \longrightarrow \widetilde{K}_{G\times T}^0 \longrightarrow \widetilde{K}_G^0(\mathbb{C}P(V)_+) \longrightarrow \widetilde{K}_{G\times T}^1(S^{V\otimes z}) \longrightarrow \cdots .$$

PROPOSITION 4.3. *We have $K_G^0(\mathbb{C}P(V)) \cong \frac{R(G)[z]}{\chi(V\otimes z)}$.*

Proof. We claim that the long exact sequence above gives a short exact sequence

$$0 \longrightarrow R(G \times T) \xrightarrow{\psi} R(G \times T) \longrightarrow \widetilde{K}_G^0(\mathbb{C}P(V)_+) \longrightarrow 0. \tag{4.4}$$

Indeed, by equivariant Bott periodicity $\widetilde{K}_{G\times T}^1(S^{V\otimes z}) \cong \widetilde{K}_{G\times T}^1(S^0) = 0$. The Thom isomorphism tells us that $\widetilde{K}_{G\times T}^0(S^{V\otimes z}) \cong K_{G\times T}^0$ and, by definition of the Euler class, $\mathrm{Im}(\psi)$ is the ideal generated by $\chi(V \otimes z)$. The fact that multiplication by the Euler class is injective in (4.4) follows since $\widetilde{K}_{G\times T}^{-1}(S^{V\otimes z}) = 0$. The first isomorphism theorem now tells us that

$$\widetilde{K}_G^0(\mathbb{C}P(V)_+) \cong \frac{R(G \times T)}{\chi(V \otimes z)},$$

and we observe [**2**] that $R(G \times T) \cong R(G)[z, z^{-1}]$, from which the proposition follows. □

When we come to consider homology, the Adams isomorphism takes the role of Theorem 4.2 and we have a subtle dimension shift, viz

$$\widetilde{K}_0^G(\mathbb{C}P(V)_+) \cong \widetilde{K}_{-1}^{G\times T}(S(V \otimes z)_+).$$

Excepting this technical point, we find in a similar fashion a short exact sequence

$$0 \longrightarrow R(G \times T) \xrightarrow{\psi} R(G \times T) \longrightarrow \widetilde{K}_0^G(\mathbb{C}P(V)_+) \longrightarrow 0, \tag{4.5}$$

in which ψ is again multiplication by the Euler class.

We now choose a notation which will be convenient for comparing results for projective spaces of different representations in §5.

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PROPOSITION 4.6. *We have* $K_0^G(\mathbb{C}P(V)) \cong \frac{\frac{1}{\chi(V\otimes z)}R(G\times T)}{R(G\times T)}$, *where* $\frac{1}{\chi(V\otimes z)}R(G \times T)$ *is the* $R(G \times T)$*-submodule generated by* $\frac{1}{\chi(V\otimes z)}$ *in the total ring of fractions of* $R(G \times T)$.

Proof. Just replace the short exact sequence (4.5) with the isomorphic short exact sequence

$$0 \longrightarrow R(G \times T) \hookrightarrow \frac{1}{\chi(V \otimes z)} R(G \times T) \longrightarrow \widetilde{K}_0^G(\mathbb{C}P(V)_+) \longrightarrow 0. \tag{4.7}$$

□

4.2. Duality from the Universal Coefficient Theorem. It is convenient to record a simple case of the algebraic relation between homology and cohomology. For any ring G-spectrum E and any G-spectrum Y we have a natural map

$$p_Y : E_G^*(Y) \longrightarrow \mathrm{Hom}_{E_*^G}(E_*^G(Y), E_*^G).$$

A suitable Universal Coefficient Theorem (UCT) would state that p_Y is an isomorphism if $E_*^G(X)$ is projective as an E_*^G-module. In equivariant topology the existence of such a UCT is more than the formality it is non-equivariantly [7], for a variety of linked reasons. From one point of view, the issue is that on the one hand the usual building blocks of G-spaces are the orbits G/H, whilst on the other $E_G^*(G/H) \cong E_H^*$ is unlikely to be projective. For these reasons, the sort of UCT that exists for formal reasons [**10**, **9**] is based on Mackey functor valued homology and cohomology. Since this does not directly discuss p_Y, additional work is required, which relies upon special properties of the cohomology theory, or the group of equivariance, or the space. For K-theory, one does expect a UCT for general G-spaces, but for present purposes we will be content to prove the very special case that concerns us.

LEMMA 4.8. *If* $X = \mathbb{C}P(V)$ *then we have isomorphisms*

$$K_G^*(X) \xrightarrow{\cong} \mathrm{Hom}_{K_*^G}(K_*^G(X), K_*^G)$$

and

$$K_*^G(X) \xrightarrow{\cong} \mathrm{Hom}_{K_G^*}(K_G^*(X), K_G^*).$$

Proof. Taking $E = K$, p_X gives the first comparison map, and applying Spanier-Whitehead duality to p_{DX} gives the second.

First, we prove that if V is a sum of one dimensional representations the map p_X is an isomorphism. The same argument shows p_{DX} is an isomorphism. We argue by induction on the dimension of V. If V is one dimensional then $\mathbb{C}P(V)$ is a point and the conclusion is clear. Now suppose that $V = W \oplus \alpha$ with α one dimensional, and that $p_{\mathbb{C}P(W)}$ is known to be an isomorphism. There is a cofibre sequence

$$\mathbb{C}P(W) \longrightarrow \mathbb{C}P(V) \longrightarrow S^{W\otimes\alpha^{-1}},$$

which induces a short exact sequence of free K_G^*-modules in both homology and cohomology. Since $p_{S^{W\otimes\alpha^{-1}}}$ is an isomorphism, we conclude that $p_{\mathbb{C}P(V)}$ is an isomorphism as required.

This shows that $p_{\mathbb{C}P(V)}$ is an isomorphism for all V if G is abelian, and we now consider the general case. We have a commutative square

$$\begin{tikzcd}
K_G^*(X) \arrow[r, "p_X"] \arrow[d] & \mathrm{Hom}_{K_*^G}(K_*^G(X), K_*^G) \arrow[d] \\
\prod_{\substack{H\leq G\\ H \text{ cyclic}}} K_H^*(X) \arrow[r, "\cong"] & \prod_{\substack{H\leq G\\ H \text{ cyclic}}} \mathrm{Hom}_{K_*^H}(K_*^H(X), K_*^H).
\end{tikzcd}$$

Since the left hand vertical is the monomorphism (2.7), it follows that p_X is a monomorphism. The same applies to p_{DX}.

We also have a commutative square

$$\begin{tikzcd}
K_*^G(X) \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & K_*^G(D^2X) \arrow[d] \\
\mathrm{Hom}_{K_*^G}(\mathrm{Hom}_{K_*^G}(K_*^G(X), K_*^G), K_*^G) \arrow[rr, "(p_X)^*"] & & \mathrm{Hom}_{K_*^G}(K_G^*(X), K_*^G).
\end{tikzcd}$$

The top horizontal is an isomorphism because X is finite, so that the natural map $X \xrightarrow{\simeq} D^2X$ is an equivalence. The left hand vertical is an isomorphism because $K_*^G(X)$ is a finitely generated free module. The right hand vertical is p_{DX}, combined with Spanier-Whitehead duality, so that the composite obtained by travelling the square first horizontally, then vertically, is the second comparison map. This shows that the second comparison map is the algebraic dual of p_X. Since p_X is a monomorphism, duality shows that the second comparison map is an epimorphism, and hence an isomorphism. The first comparison map is dealt with similarly. $\square$

REMARK 4.9. There is an alternative approach to the duality statement which is perhaps more illuminating from the algebraic point of view. Writing $R = R(G)$ and $S = R(G \times T)$, and $\chi = \chi(V \otimes z)$ we calculated the homology

$$K_0^G(\Sigma^2\mathbb{C}P(V)) = S/\chi$$

from the short exact sequence arising from the sequence of $G \times T$-spaces $S^0 \longrightarrow S^{V\otimes z} \longrightarrow \Sigma S(V \otimes z)_+$, which we regard as a projective resolution over S. This means that the cohomology of $S(V \otimes z)_+ \longrightarrow S^0 \longrightarrow S^{V\otimes z}$ shows that

$$K_G^0(\mathbb{C}P(V)) = \mathrm{Ext}_S^1(S/\chi, S).$$

Thus the UCT duality statement is

$$\mathrm{Ext}_S^1(S/\chi, S) \cong \mathrm{Hom}_R(S/\chi, R),$$

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and one can write down the isomorphism explicitly in these terms. Furthermore, the short exact sequence

$$0 \longrightarrow S \xrightarrow{\chi} S \longrightarrow \mathrm{Ext}^1_S(S/\chi, S) \longrightarrow 0$$

can be viewed as an exact sequence of R-modules; since the R-modules are all free, applying $(\cdot)^* = \mathrm{Hom}_R(\cdot, R)$ we see the more elementary isomorphism

$$\mathrm{Ext}^1_S(S/\chi, S)^* \cong \mathrm{Hom}_R(S/\chi, R)$$

which corresponds to Poincaré duality. By contrast with topology, from the algebraic point of view, it is the UCT that is the more subtle statement, and Poincaré duality that is formal.

4.3. The fundamental class. The following identification of the fundamental class is the key result of the paper.

THEOREM 4.10. *Let G be a finite group and V a complex representation of G with $\dim_{\mathbb{C}}(V) = n$. Then $\frac{1}{\chi(V\otimes z)} \in K_0^G(\mathbb{C}P(V))$ is a fundamental class in equivariant K-theory for $\mathbb{C}P(V)$.*

We break our proof into convenient pieces as follows. For brevity we write Vz for $V \otimes z$, *etc.*

LEMMA 4.11. *The notation is compatible with restriction, in the sense that for any subgroup H of G, we have $\mathrm{Res}^G_H\left(\frac{1}{\chi(Vz)}\right) = \frac{1}{\chi(Vz)}$.*

Proof. If we use 4.5 to say $K_0^G(\mathbb{C}P(V)) = R(G\times T)/(\chi(Vz))$ the element $1/\chi(Vz)$ corresponds to the unit of $R(G\times T)$. The lemma simply states that the restriction of the unit in $R(G\times T)$ is the unit in $R(H\times T)$. □

LEMMA 4.12. *If $x \in \mathbb{C}P(V)$ is G-fixed, then $i_*^{Gx}(\frac{1}{\chi(Vz)})$ is an $R(G\times T)$-generator for $K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\backslash Gx)$.*

Proof. The point x represents a line in V, and since it is fixed, this is a 1-dimensional representation α of G, and we have $V \cong W \oplus \alpha$ for some W. Thus

$$\mathbb{C}P(V)\backslash Gx = \mathbb{C}P(V) \setminus \mathbb{C}P(\alpha) \simeq \mathbb{C}P(W),$$

so we are required to prove that $K_{-1}^{G\times T}(S(Vz), S(Wz)))$ is $R(G\times T)$-generated by $i_*^{Gx}(\frac{1}{\chi(Vz)})$.

We have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \widetilde{K}_0^{G\times T}(S^0) \arrow[r, "\chi(Wz)"] \arrow[d, equal] & \widetilde{K}_0^{G\times T}(S^{Wz}) \arrow[r] \arrow[d, "\chi(\alpha z)"] & \widetilde{K}_{-1}^{G\times T}(S(Wz)_+) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{K}_0^{G\times T}(S^0) \arrow[r, "\chi(Vz)"] & \widetilde{K}_0^{G\times T}(S^{Vz}) \arrow[r, "a"] \arrow[d, "b"'] & \widetilde{K}_{-1}^{G\times T}(S(Vz)_+) \arrow[r] \arrow[d, "c"] & 0 \\
 & & K_{-1}^{G\times T}(S(Vz), S(Wz)) \arrow[r, equal] \arrow[d] & K_{-1}^{G\times T}(S(Vz), S(Wz)) & \\
 & & 0 & &
\end{tikzcd}$$

in which the rows and columns are exact. (The rows are (4.5), the centre column is the homology sequence of the G-triple $(D(Vz), S(Vz), S(Wz))$ and the right-hand column comes from the G-pair $(S(Vz), S(Wz))$.) Writing β for the Bott class in $\widetilde{K}_0^{G\times T}(S^{Vz})$, we must show that $ca(\beta) = b(\beta)$ is an $R(G \times T)$-generator. This is clear, since β obviously $R(G \times T)$-generates $\widetilde{K}_0^{G\times T}(S^{Vz})$. □

LEMMA 4.13. *Under the hypothesis of Lemma* 4.12, $i_*^{Gx}(\frac{1}{\chi(Vz)})$ *is an* $R(G)$*-generator for* $K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\backslash Gx)$.

Proof. We work in cohomology, where the module structure is transparent, and the result in homology follows via duality. Our proof now amounts to showing that the action of $z \in R(G \times T)$ on $K^0_{G\times T}(S(Vz), S(Wz))$ is the same as that of $\alpha^{-1} \in R(G)$. We have an equivalence

$$S(Vz)/S(Wz) \simeq S(\alpha z)_+ \wedge S^{Wz},$$

and writing $\kappa = \ker(\alpha z)$ we have $S(\alpha z) \cong (G \times T)/\kappa$ so we may work in $K^0_\kappa(S^{Wz})$. Finally, we identify κ with G by the isomorphism $f : G \xrightarrow{\cong} \kappa$ defined by $f(g) = (g, \alpha(g)^{-1})$. Thus $f^*(Wz) = W\alpha^{-1}$. The result now follows by considering the commutative diagram

$$\begin{tikzcd}
\widetilde{K}^0_{G\times T} \times \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{Wz}) \arrow[r, "m"] \arrow[d, "\mathrm{Res}^{G\times T}_\kappa \times \theta^{-1}"'] & \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{Wz}) \arrow[d, "\theta^{-1}", "\cong"'] \\
\widetilde{K}^0_\kappa \times \widetilde{K}^0_\kappa(S^{Wz}) \arrow[r, "m"] \arrow[d, "f^*"', "\cong"] & \widetilde{K}^0_\kappa(S^{Wz}) \arrow[d, "f^*", "\cong"'] \\
\widetilde{K}^0_G \times \widetilde{K}^0_G(S^{W\alpha^{-1}}) \arrow[r, "m"] & \widetilde{K}^0_G(S^{W\alpha^{-1}}),
\end{tikzcd}$$

in which m is the module structure. □

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Lemma 4.14. *Suppose $x \in \mathbb{C}P(V)$ has isotropy $H = G_x < G$. Given $B \subseteq A \subseteq \mathbb{C}P(V)$, write i_B^A for the inclusion of G-pairs*

$$i_B^A : (\mathbb{C}P(V), \mathbb{C}P(V)\backslash A) \hookrightarrow (\mathbb{C}P(V), \mathbb{C}P(V)\backslash B).$$

Writing i^B for $i_B^{\mathbb{C}P(V)}$, we have a commutative diagram

$$\begin{tikzcd}
K_0^G(\mathbb{C}P(V)) \arrow[r, "i_*^{Gx}"] \arrow[d, "\mathrm{Res}_H^G"'] & K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\backslash Gx) \arrow[r, "\cong", "(3.2)"'] \arrow[d, "\mathrm{Res}_H^G"] & K_0^H \arrow[dd, equal] \\
K_0^H(\mathbb{C}P(V)) \arrow[r, "i_*^{Gx}"] \arrow[d, equal] & K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\backslash Gx) \arrow[d, "(i_{Hx}^{Gx})_*"] & \\
K_0^H(\mathbb{C}P(V)) \arrow[r, "i_*^{Hx}"] & K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\backslash Hx) \arrow[r, "\cong", "(3.2)"'] & K_0^H.
\end{tikzcd} \tag{4.15}$$

Proof. Commutativity of the left hand squares in the diagram is obvious by naturality. For the right hand square, use Lemma 3.1 to write out the isomorphism (3.2) in full. □

Proof of Theorem 4.10. Equivariant Bott periodicity means that we can work everywhere in degree zero. Let $x \in \mathbb{C}P(V)$. Suppose x has isotropy $H \leq G$. By Lemma 4.14 it suffices to show that $(i^{Hx})_* \mathrm{Res}_H^G\left(\frac{1}{\chi(Vz)}\right)$ is a generator. Now Lemma 4.11 allows us to use Lemma 4.13 to complete the proof. □

5. Calculations with the fundamental class.

5.1. The abelian world. For the time being, let us impose the restriction that G be a finite *abelian* group A. Given an n-dimensional complex representation V of A, we can write $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ for one dimensional summands α_i. Following **[5]** we *choose* a complete flag

$$\mathcal{F} = \left(0 \subset V^1 \subset V^2 \subset \cdots \subset V^n = V\right) \tag{5.1}$$

in which $V^i / V^{i-1} = \alpha_i$. This choice gives rise to an $R(A)$-basis

$$\{1, y^{V^1}, y^{V^2}, \ldots, y^{V^{n-1}}\}$$

for $K_A^0(\mathbb{C}P(V))$, in which

$$y^{V^i} = y^{\alpha_1} y^{\alpha_2} \cdots y^{\alpha_i} \text{ and } y^{\alpha_j} = 1 - \alpha_j z.$$

We write $\{\beta_0^{\mathcal{F}}, \ldots, \beta_{n-1}^{\mathcal{F}}\}$ for the dual $R(A)$-basis for

$$K_0^A(\mathbb{C}P(V)) \cong \mathrm{Hom}_{R(A)}(K_A^0(\mathbb{C}P(V)), R(A)),$$

so that

$$\beta_i^{\mathcal{F}}(y^{V^j}) = \delta_i^j.$$

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THEOREM 5.2. *The fundamental class is given by*

$$\frac{1}{\chi(V \otimes z)} = \beta_0^{\mathcal{F}} + \cdots + \beta_{n-1}^{\mathcal{F}}. \tag{5.3}$$

REMARKS 5.4. (i) Since the left hand side of (5.3) is a topological invariant of V, so too is the right hand side and we may abbreviate to $\beta_0 + \cdots + \beta_{n-1}$ without ambiguity. It is striking that although the individual $\beta_i^{\mathcal{F}}$ depend on the flag $\mathcal{F}$, this sum does not.

(ii) This generalises Adams's classical identification of the (non-equivariant) K-theory fundamental class [**3**, Theorem III.11.15], and provides a more elementary proof (Adams's alternating signs arise by choosing the opposite orientation).

(iii) One can give a direct, algebraic proof that $\beta_0 + \cdots + \beta_{n-1}$ is independent of flag, without relating it to $\frac{1}{\chi(V \otimes z)}$. We refer to [**14**, Proposition 3.5.13] for details. Furthermore, in [**14**] it is shown directly that taking the cap product with $\beta_0 + \cdots + \beta_{n-1}$ gives a duality isomorphism.

Proof of Theorem 5.2. It suffices to prove the result if A is the n-torus T^n and $V = z_1 \oplus \cdots \oplus z_n$, where $z_i(\lambda_1, \ldots, \lambda_n) = \lambda_i$. This is because the pullback of $z_1 \oplus \cdots \oplus z_n$ along the homomorphism $\boldsymbol{\alpha} : A \longrightarrow T^n$, in which $\boldsymbol{\alpha}(a) = (\alpha_1(a), \ldots, \alpha_n(a))$, is $\alpha_1 \oplus \cdots \oplus \alpha_n$.

The proof proceeds by induction on $n = \dim_{\mathbb{C}}(V)$. The initial step is obvious, so now suppose the theorem holds for representations of dimension smaller than $n > 1$. For $1 \leq i \leq n$, we have T^n-inclusions

$$j_i : \mathbb{C}P(z_1 \oplus \cdots \oplus z_{i-1} \oplus z_{i+1} \oplus \cdots \oplus z_n) \hookrightarrow \mathbb{C}P(V),$$

and we write

$$\begin{aligned} \iota_i = & \frac{1}{\chi((z_1 \oplus \cdots \oplus z_{i-1} \oplus z_{i+1} \oplus \cdots \oplus z_n) \otimes z)} \\ & \in K_0^{T^n}(\mathbb{C}P(z_1 \oplus \cdots \oplus z_{i-1} \oplus z_{i+1} \oplus \cdots \oplus z_n)). \end{aligned}$$

Writing $\langle -, - \rangle$ for the Kronecker pairing we have

$$\begin{aligned} \langle y^{z_1} \cdots y^{z_i}, (j_n)_*(\iota_n) \rangle &= \langle (j_n)^*(y^{z_1} \cdots y^{z_i}), \iota_n \rangle \\ &= \langle y^{z_1} \cdots y^{z_i}, \iota_n \rangle \\ &= \begin{cases} 1 & 0 \leq i \leq n-2 \\ 0 & i = n-1 \end{cases}, \end{aligned} \tag{5.5}$$

from which

$$(j_n)_*(\iota_n) = \beta_0^{\mathcal{F}} + \cdots + \beta_{n-2}^{\mathcal{F}}. \tag{5.6}$$

In the final step of (5.5), we use the inductive hypothesis for $0 \leq i \leq n-2$, and for $i = n-1$, the fact that $y^{z_1} y^{z_2} \cdots y^{z_{n-1}} = 0$. Similarly, one finds that

$$(j_{n-1})_*(\iota_{n-1}) = \beta_0^{\mathcal{F}} + \cdots + \beta_{n-2}^{\mathcal{F}} + (1 - z_{n-1} z_n^{-1}) \beta_{n-1}^{\mathcal{F}}. \tag{5.7}$$

Taking a linear combination of (5.6) and (5.7), we find that

$$(j_{n-1})_*(\iota_{n-1}) - z_{n-1}z_n^{-1}(j_n)_*(\iota_n) = \big(1 - z_{n-1}z_n^{-1}\big)\big(\beta_0^{\mathcal{F}} + \cdots + \beta_{n-1}^{\mathcal{F}}\big).$$

We now simplify the left hand side, using the fact that

$$(j_n)_*(\iota_n) = \frac{1}{\chi((V/z_n)\otimes z)} = \frac{\chi(z_n \otimes z)}{\chi(V \otimes z)}$$

and similarly for $(j_{n-1})_*(\iota_{n-1})$. Since

$$\frac{\chi(z_{n-1}\otimes z)}{\chi(V\otimes z)} - z_{n-1}z_n^{-1}\frac{\chi(z_n\otimes z)}{\chi(V\otimes z)} = (1 - z_{n-1}z_n^{-1})\frac{1}{\chi(V\otimes z)},$$

we obtain

$$\big(1 - z_{n-1}z_n^{-1}\big)\frac{1}{\chi(V\otimes z)} = (1 - z_{n-1}z_n^{-1})(\beta_0^{\mathcal{F}} + \cdots + \beta_{n-1}^{\mathcal{F}}).$$

The result follows, since $1 - z_{n-1}z_n^{-1}$ is not a zero divisor in $R(T^n \times T)$. □

5.2. The non-abelian world. The proofs of §5.1 break down in the non-abelian case because V may not have a decomposition into one-dimensional representations and we cannot choose a flag as in (5.1).

NOTATION 5.8. Recall that $K_G^0(\mathbb{C}P(V)) \cong R(G)[z]/(\chi(V\otimes z))$ (irrespective of whether G is abelian). Observe that

$$\mathcal{B} = \{(1-z)^i \mid 0 \leq i \leq n-1\}$$

is always a basis for $K_G^0(\mathbb{C}P(V))$. (Whereas the construction of § 5.1 gives a basis for any complex orientable theory, the fact that $\mathcal{B}$ gives a basis is a special feature of K-theory). We write $\{\beta_0^{\mathcal{B}}, \ldots, \beta_{n-1}^{\mathcal{B}}\}$ for the corresponding dual basis for $K_0^G(\mathbb{C}P(V))$.

Happily, it turns out that in the abelian case, the explicit proof mentioned in Remarks 5.4 (iii) gives

$$\frac{1}{\chi(V\otimes z)} = \sum_{i=0}^{n-1}\beta_i = \sum_{i=0}^{n-1}\beta_i^{\mathcal{B}}. \tag{5.9}$$

LEMMA 5.10. *If $H \leq G$ then we have $\mathrm{Res}_H^G(\beta_i^{\mathcal{B}}) = \beta_i^{\mathcal{B}}$ for $i = 0, \ldots, n-1$.* □

The proof involves considering the interaction of restriction with the Kronecker pairing. Details may be found in [**14**, §4.4].

THEOREM 5.11. *Let V be a complex representation, $\dim_{\mathbb{C}} V = n$, of the finite group G. Take $\mathcal{B} = \{(1-z)^i\}_{i=0}^{n-1}$ as a basis for $K_G^0(\mathbb{C}P(V))$ and let the dual basis for $K_0^G(\mathbb{C}P(V))$ be $\{\beta_i^{\mathcal{B}}\}_{i=0}^{n-1}$. Then*

$$\sum_{i=0}^{n-1}\beta_i^{\mathcal{B}} = \frac{1}{\chi(V\otimes z)}.$$

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Proof. We use Lemma 5.10 to see that $\mathrm{Res}^G_H(\sum_{i=0}^{n-1}\beta_i^{\mathcal{B}}) = \sum_{i=0}^{n-1}\beta_i^{\mathcal{B}}$ and Lemma 4.11 to see that $\mathrm{Res}^G_H(\frac{1}{\chi(V\otimes z)}) = \frac{1}{\chi(V\otimes z)}$ for each $H \leq G$. Taking the product over cyclic subgroups, and using (5.9),

$$\mathrm{Res}^G_*\left(\sum_{i=0}^{n-1}\beta_i^{\mathcal{B}}\right) = \mathrm{Res}^G_*\left(\frac{1}{\chi(V\otimes z)}\right).$$

The theorem now follows from the injectivity of Res^G_*. $\square$

5.3. Perfect pairings. Recall that if M, N are modules over the commutative ring R then a bilinear map $b : M\otimes N \longrightarrow R$ is a *perfect pairing* if

$$\begin{array}{rcl} M & \longrightarrow & \mathrm{Hom}_R(N, R) \\ m & \longmapsto & (n \longmapsto b(m\otimes n)) \end{array}$$

defines an isomorphism of R-modules $M \xrightarrow{\cong} \mathrm{Hom}_R(N, R)$.

NOTATION 5.12. We define a pairing $\lceil -, -\rceil : K^0_G(\mathbb{C}P(V))\otimes K^0_G(\mathbb{C}P(V)) \longrightarrow R(G)$ by $\lceil x, y\rceil = \langle xy, \frac{1}{\chi(V\otimes z)}\rangle$.

THEOREM 5.13. *The pairing*

$$\lceil -, -\rceil : K^0_G(\mathbb{C}P(V))\otimes K^0_G(\mathbb{C}P(V)) \longrightarrow R(G)$$

is perfect, and the corresponding isomorphism

$$K^0_G(\mathbb{C}P(V)) \xrightarrow{\cong} \mathrm{Hom}_{R(G)}\big(K^0_G(\mathbb{C}P(V)), R(G)\big) = K_0^G(\mathbb{C}P(V))$$

is a Poincaré duality isomorphism.

Proof. One can show directly that $\lceil -, -\rceil$ is perfect in the abelian case, but it is far more satisfactory (and general) to observe that the map

$$K^0_G(\mathbb{C}P(V)) \xrightarrow{\cap_\xi} \mathrm{Hom}_{R(G)}\big(K^0_G(\mathbb{C}P(V)), R(G)\big) = K_0^G(\mathbb{C}P(V)),$$

in which $x \overset{\cap_\xi}{\longmapsto} (y \longmapsto \langle xy, \xi\rangle)$, is capping with $\xi \in K_0^G(\mathbb{C}P(V))$ – in other words $\cap_\xi(x) = x\cap\xi$. This is easily verified, using Lemma 4.8 and the definition of the cap product. $\square$

6. Examples. We conclude by explaining how to compute the pairing $\lceil -, -\rceil$ of Notation 5.12 for any $\mathbb{C}P(V)$. We make the results explicit in dimensions ≤ 4.

As observed above, $K^0_G(\mathbb{C}P(V)) \cong R(G)[z]/\chi(Vz)$, and we use the basis $\{1, y, y^2, \ldots, y^{n-1}\}$ if V is of dimension n, where $y = 1 - z$. As described above $\lceil a, b\rceil = \varepsilon(ab)$ where

$$\varepsilon(a_0 + a_1 y + \cdots + a_{n-1}y^{n-1}) = a_0 + a_1 + \cdots + a_{n-1} \in R(G).$$

Given $s \geq 0$, we therefore need to find expressions for y^{n+s} in terms of the basis: in fact if

$$y^{n+s} = \sum_{j=0}^{n-1} \lambda_j^s y^j,$$

we will find recursive formulae for λ_j^s, and then

$$\lceil y^i, y^j \rceil = \varepsilon(y^{i+j}) = \lambda_0^s + \cdots + \lambda_{n-1}^s,$$

if $i + j = n + s$.

We first apply the splitting principle to obtain a formula for $\chi(Vz)$ in terms of y, and we use notation suggested by the theory of equivariant formal group laws. Indeed if α is one dimensional,

$$\chi(\alpha z) = 1 - \alpha z = e(\alpha) + \alpha y = \alpha(y - e(\alpha^{-1})),$$

where $e(\alpha) = 1 - \alpha$. Now, if $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ is a sum of one dimensional representations,

$$\det(V)^{-1}\chi(Vz) = \prod_{i=1}^{n}(y - e(\alpha_i^{-1})) = \sigma_n + \sigma_{n-1}y + \cdots + \sigma_1 y^{n-1} + y^n,$$

where we have used the elementary symmetric polynomials

$$\sigma_j = \sigma_j\big(-e\big(\alpha_1^{-1}\big), -e\big(\alpha_2^{-1}\big), \ldots, -e\big(\alpha_n^{-1}\big)\big).$$

Since the σ_j are symmetric, the coefficients can be expressed in terms of exterior powers. Explicitly, writing V^* for the dual representation of V, we have the formula

$$\sigma_m = \lambda^m(V^*) - \binom{n-m+1}{n-m}\lambda^{m-1}(V^*) + \binom{n-m+2}{n-m}\lambda^{m-2}(V^*) - \cdots$$
$$\cdots + (-1)^{m-1}\binom{n-1}{n-m}\lambda^1(V^*) + (-1)^m\binom{n}{n-m}.$$

Thus we have an equality

$$\det(V)^{-1}\chi(Vz) = \sigma_n + \sigma_{n-1}y + \cdots + \sigma_1 y^{n-1} + y^n,$$

between elements of $R(G \times T)$: we have verified it when V is a sum of one dimensional representations, and it therefore holds in general by the splitting principle.

Thus the condition $\chi(Vz) = 0$ is equivalent to

$$y^n = -(\sigma_n + \sigma_{n-1}y + \cdots + \sigma_1 y^{n-1}),$$

or $\lambda_j^0 = -\sigma_{n-j}$. Now

$$y^{n+s+1} = yy^{n+s} = \sum_{j=1}^{n-1} \lambda_{j-1}^s y^j - \lambda_{n-1}^s \sum_{j=0}^{n-1} \sigma_{n-j} y^j,$$

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or, interpreting λ^s_{-1} as zero,

$$\lambda_j^{s+1} = \lambda_{j-1}^s - \lambda_{n-1}^s \sigma_{n-j}.$$

When adding up, it is useful to note that $1 - e(\alpha) = \alpha$, so in particular

$$\det(V)^{-1} = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n.$$

Then we find

$$\varepsilon(y^n) = 1 - \det(V)^{-1}.$$

Similarly,

$$\varepsilon(y^{n+s+1}) = \varepsilon(y^{n+s}) - \lambda_{n-1}^s \det(V)^{-1},$$

and an inductive argument then shows

$$\varepsilon(y^{n+s}) = 1 - \frac{1}{\det(V)}\left(1 + \lambda_{n-1}^0 + \cdots + \lambda_{n-1}^{s-1}\right).$$

More explicitly, if we interpret σ_{n+s} as zero for $s > 0$,

$$\lambda_j^0 = -\sigma_{n-j},\ \lambda_j^1 = -\sigma_{n-j+1} + \sigma_1\sigma_{n-j},\ \lambda_j^2 = -\sigma_{n-j+2} + \sigma_1\sigma_{n-j+1} + \left(\sigma_2 - \sigma_1^2\right)\sigma_{n-j},$$

and so

$$\begin{aligned}
\varepsilon(y^n) &= 1 - \frac{1}{\det(V)},\\
\varepsilon(y^{n+1}) &= 1 - \frac{1}{\det(V)}(1 - \sigma_1),\\
\varepsilon(y^{n+2}) &= 1 - \frac{1}{\det(V)}\left(1 - (\sigma_1 + \sigma_2) + \sigma_1^2\right),\\
\varepsilon(y^{n+3}) &= 1 - \frac{1}{\det(V)}\left(1 - (\sigma_1 + \sigma_2 + \sigma_3) + \left(2\sigma_1\sigma_2 + \sigma_1^2\right) - \sigma_1^3\right).
\end{aligned}$$

Below are the results of the pairing $\lceil -, - \rceil$ for $\mathbb{C}P(V)$ when V is of small dimension. (For brevity, we write δ^* for $\det(V^*) = 1/\det(V)$).

	1	y
1	1	1
y	1	$1 - \delta^*$

Pairing for $\dim(V) = 2$

	1	y	y^2
1	1	1	1
y	1	1	$1 - \delta^*$
y^2	1	$1 - \delta^*$	$1 - \delta^*(4 - V^*)$

Pairing for $\dim V = 3$

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	1	y	y^2	y^3
1	1	1	1	1
y	1	1	1	$1-\delta^*$
y^2	1	1	$1-\delta^*$	$1-\delta^*(5-V^*)$
y^3	1	$1-\delta^*$	$1-\delta^*(5-V^*)$	$1-\delta^*(14-6V^*+(V^*)^2-\lambda^2(V^*))$

Pairing for dim $V = 4$

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