GROUPS OF BREADTH FOUR HAVE CLASS FIVE

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A conjecture of reputable vintage states that $c(G) \leq b(G)+1$ for a finite p-group G of class $c(G)$ and breadth $b(G)$. This result has been proved in a medley of special cases and in particular whenever $b(G) \leq 3$. We now prove it for $b(G)=4$.

1. Introduction. Let G be a finite p-group and let $C(x)$ denote the centraliser of the element x in G. The breadth $b(x)$ is defined by

$$p^{b(x)} = |G : C(x)|,$$

and the breadth $b = b(G)$ of G is defined by

$$b = b(G) = \max\{b(x) : x \in G\}.$$

Discussion of the conjecture that the class $c(G)$ of the finite p-group G is bounded by $b(G)+1$ may be found in the references, especially [**6**] and [**7**]. The result for $b(G)=1$ was proved by Burnside [**1**, pp. 125–6]. Knoche [**5**] proved that $c(G) \leq b(G)+1$ for $2 \leq b(G) \leq 3$. Other special cases have been studied. The general case may yet prove to be false. The results of [**7**] suggest that the cases with $b(G) \leq 6$ may well be decisive because counter-examples with $b(G)=6$ are there presented to settle certain closely-related conjectures, which appear as Problems 3.5 and 3.6 in [**6**].

In this note we show that well-tried methods suffice to settle the case $b(G)=4$. More precisely we prove:

THEOREM. *If G is a finite p-group with $b(G)=4$ then $c(G) \leq 5$.*

2. Definitions and preliminaries. To prove the theorem we need the apparatus of commutator manipulation. In this section we survey the necessary equipment and, to give a taste of the argument, apply the methods to the case $b(G)=3$.

Let $x_1, x_2, \ldots, x_n, \ldots$ be elements of the group G. Then

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$$

and if $n \geq 2$ then

$$[x_1, x_2, \ldots, x_{n+1}] = [[x_1, x_2, \ldots, x_n], x_{n+1}].$$

We use the "semicolon notation", according to which

$$[x_1, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}] = [c_p, c_q],$$
$$[x_1, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}; x_{p+q+1}, \ldots, x_{p+q+r}] = [c_p, c_q, c_r],$$

and so on, where

$$c_p=[x_1,\dots,x_p],\ c_q=[x_{p+1},\dots,x_{p+q}],\ c_r=[x_{p+q+1},\dots,x_{p+q+r}].$$

The terms $\gamma_i(G)$ of the lower central series of G are defined by putting $\gamma_1(G)=G$ and

$$\gamma_i(G)=\langle[x_1,\dots,x_i]:x_1\in G,\dots,x_i\in G\rangle$$

for $i>1$. The group G is said to have class $n=c(G)$ if $\gamma_{n+1}(G)=1$ but $\gamma_n(G)\neq 1$. The two-step centralisers C_i corresponding to the lower central series are defined for $1\le i<c$ by

$$C_i=\langle x\in G:\text{ if } y\in\gamma_i(G)\quad\text{then}\quad [x,y]\in\gamma_{i+2}(G)\rangle.$$

Notice that $C_i\neq G$, and that C_i is normal in G, for $1\le i<c$.

Next we summarise some useful results. The multilinearity property of commutators will be referred to (if at all) as (ML), at its frequent appearances; we are thinking of statements like

$$[y_1z_1,x_2,\dots,x_n]\equiv[y_1,x_2,\dots,x_n][z_1,x_2,\dots,x_n]\bmod\gamma_{n+1}.$$

(In fact what $\equiv$ really denotes in statements like $u\equiv v\bmod\gamma_{n+1}$ or $u\equiv v\bmod\gamma_{n+1}(G)$ is that the cosets $u\gamma_{n+1}(G)$ and $v\gamma_{n+1}(G)$ in $G/\gamma_{n+1}(G)$ are equal.)

We denote by (JW) the result of a standard identity, namely

$$[c_p,c_q,c_r][c_q,c_r,c_p][c_r,c_p,c_q]\equiv 1\bmod\gamma_{p+q+r+1}$$

or

$$[c_p,[c_q,c_r]]\equiv[c_p,c_q,c_r][c_p,c_r,c_q]^{-1}\bmod\gamma_{p+q+r+1}$$

where c_p, c_q, c_r are defined as above.

Two consequences of (JW) are helpful. The first, (LN), states that a commutator of the form $[x_1,\dots,x_n]$ is the product of commutators like $[x_n,y_1,\dots,y_{n-1}]$ and its inverse, mod γ_{n+1}, where $\{y_1,\dots,y_{n-1}\}=\{x_1,\dots,x_{n-1}\}$. The second is denoted by (AB):

$$[x_1,x_2,x_1,x_2]\equiv[x_1,x_2,x_2,x_1]\bmod\gamma_5.$$

Now consider a finite p-group G with breadth b and class $c>b+1$. We can make a reduction by replacing G with $G/\gamma_{b+3}(G)$, for an obvious inductive assumption allows us to suppose that the breadth of $G/\gamma_{b+3}(G)$ is not less than b. In other words, *in proving the theorem we may take* $c(G)=b(G)+2$.

Lemma 3.1 of **[6]** shows that $C_1\cup\dots\cup C_{c-1}\neq G$ implies that $c\le b+1$. But $C_1\le C_i$ for $1<i<c$, as (LN) clearly implies. We shall prove that if $3\le b\le 4$ then $c\le b+1$ by first establishing: *if* $3\le b\le 4$ *and* $c=b+2$ *then* $C_2\cup\dots\cup C_{b+1}\neq G$.

Let us consider the case of a finite p-group G with $b(G)=3$ and $c(G)=5$, and let us suppose that $G=C_2\cup C_3\cup C_4$. It is a known fact that if $D=C_2\cap C_3\cap C_4$ then G/D is non-cyclic of order 4, and though this is due to Scorza (1926), according to **[3]**, a more convenient reference is **[4]**.

Therefore $p=2$, $G=\langle a, b, D\rangle$, $C_2=\langle a, D\rangle$, $C_3=\langle b, D\rangle$, $C_4=\langle ab, D\rangle$. Let $x_1, \dots, x_5$ be elements of G for which $w=[x_1, \dots, x_5]\neq 1$. Clearly D does not contain x_3 or x_4 or x_5, and without losing generality we may take $x_3=b$, $x_4=a$. If we work mod $\gamma_5(G)$ then

$$[x_1, x_2, b, a]\equiv[x_1, x_2; b, a][x_1, x_2, a, b] \quad \text{(JW)}.$$

But $[x_1, x_2, a, b]\equiv 1$ since for instance $a\in C_2$. Further

$$[x_1, x_2; b, a]\equiv[a, b; x_1, x_2]$$

and another application of (JW), whose details we suppress, shows that if x_1 or x_2 lies in D, and so in both C_2 and C_3, then $[a, b; x_1, x_2]\equiv 1$. So we may assume that $x_1=a$, $x_2=b$, in which case by (AB),

$$[x_1, x_2, x_3, x_4]=[a, b, b, a]\equiv[a, b, a, b].$$

Since $a\in C_2$, it follows that $w=1$. This is a contradiction, and shows that $G\neq C_2\cup C_3\cup C_4$. The result that if $b=3$ then $c\leq 4$ follows as explained above.

3. Proof of the theorem: the redundant case. In order to prove the theorem we suppose that G is a finite p-group with $b(G)=4$, $c(G)=6$, and $G=C_2\cup C_3\cup C_4\cup C_5$. The calculations in this case will be presented in a more succinct form than above.

A complication immediately arises, for G may be the union of just three of the proper subgroups $C_i (2\leq i\leq 5)$, and disposing of this case is not trivial. Consideration of subcases is necessary. We always suppose that $w\neq 1$ where

$$w=[x_1, x_2, x_3, x_4, x_5, x_6].$$

(i) Suppose that $G=C_3\cup C_4\cup C_5$. Put $D=C_3\cap C_4\cap C_5$, $C_3=\langle a, D\rangle$, $C_4=\langle b, D\rangle$, $C_5=\langle ab, D\rangle$. We take $x_4=b$, $x_5=a$, $x_6=a$.

Our first aim is to show that no x_i lies in D. By (LN), if some $x_i\in D$ then we can take $i=1$. We have

$$[x_1, x_2, x_3, b, a]\equiv[x_1, x_2, x_3; b, a] \qquad \text{(JW; } a\in C_3)$$

and

$$\begin{aligned} w&=[x_1, x_2, x_3; b, a; a] \\ &=[a, b, a; x_1, x_2, x_3][x_1, x_2, x_3, a; b, a] \qquad \text{(JW)}. \end{aligned}$$

The former of these commutators is trivial because $x_1\in C_3\cap C_4\cap C_5$—note that (JW) is used here—and the latter because $a\in C_3$. So if $x_i\in D$ then $w=1$.

This means that we can take $x_1=a$, $x_2=b$. But if $x_3=a$ then (AB) gives $w=1$; so $x_3=b$. Then

$$\begin{aligned} w&=[a, b, b, b, a, a] \\ &=[a, b, b; a, b; a]^{-1} && \text{(JW; } a\in C_3) \\ &=[a, b, a; a, b, b][a, b, b, a; a, b]^{-1} && \text{(JW)} \\ &=[a, b, a; [a, b], b] && (a\in C_3) \\ &=[a, b, a; a, b; b][a, b, a, b; a, b]^{-1} && \text{(JW)} \\ &=1 && \text{(AB; } a\in C_3). \end{aligned}$$

(ii) Suppose that $G = C_2 \cup C_4 \cup C_5$. Put $D = C_2 \cap C_4 \cap C_5$, $C_2 = \langle a, D\rangle$, $C_4 = \langle b, D\rangle$, $C_5 = \langle ab, D\rangle$. We take $x_3 = b$, $x_5 = a$, $x_6 = a$.

Note that since

$$[x_1, x_2, b] \equiv [b, x_2, x_1][b, x_1, x_2]^{-1} \bmod \gamma_4$$

we have $w = 1$ if $x_1 \in D$ and $x_2 \in D$; or if $x_1 \in D$ and $x_2 = a$. Indeed we can suppose that $x_1 \in D$ or $x_1 = a$, and that $x_2 = b$.

Let us next consider x_4. If $x_4 \in D$ then

$$\begin{aligned} w &= [x_1, b, b, x_4, a, a] \\ &= [x_1, b, b; x_4, a; a][x_1, b, b, a, x_4, a] && \text{(JW)} \\ &= [x_1, b, b; x_4, a; a] && (x_4 \in C_4) \\ &= [x_4, a, a; x_1, b, b]^{-1}[x_1, b, b, a; x_4, a] && \text{(JW)} \\ &= 1 && (a \in C_2;\ x_4 \in C_4 \cap C_5). \end{aligned}$$

Next suppose that $x_4 = a$. If c and d are commutators of weight 2 then modulo γ_6 we have

$$[c, d, a] \equiv [d, a, c]^{-1}[c, a, d] \qquad \text{(JW)}$$

and in our case, with $a \in C_2$, we have $w = 1$. We conclude therefore that we can take $x_4 = b$ without losing any generality.

Finally we can take $x_1 = a$ by the following reasoning. If (LN) is applied to $[x_1, b, b, b, a]$ then this element becomes a product (modulo γ_6) of commutators of the form $[a, \ldots]^{\pm 1}$, all of which are trivial when both b and x_1 lie in C_4. Therefore $x_1 \notin D$, and as above we have $x_1 = a$. Then

$$\begin{aligned} w &= [a, b, b, b, a, a] \\ &= [a, b, b; a, b; a]^{-1}[a, b, b, a, b, a] && \text{(JW)} \\ &= [a, b, a; a, b, b][a, b, b, a; a, b]^{-1} && \text{(JW)} \\ &= 1 && (a \in C_2). \end{aligned}$$

Note that $[a, b, b, a] \equiv 1$ because $a \in C_2$.

(iii) The cases in which C_4 is redundant and C_5 is redundant may be combined. Make the obvious definitions of D and put $C_2 = \langle a, D\rangle$, $C_3 = \langle b, D\rangle$. We take $x_3 = b$, $x_4 = a$; and applying (JW) twice to

$$[x_1, x_2; a, b][a, b; x_1, x_2] \equiv 1 \bmod \gamma_5$$

we obtain

$$[x_1, x_2, a, b][x_1, x_2, b, a]^{-1}[a, b, x_1, x_2][a, b, x_2, x_1]^{-1} \equiv 1.$$

It is clear that if $x_1 \in D$ then $w = 1$. So by (LN), every entry of w may be chosen from $\{a, b\}$. But in that case $[a, b, b, a] \equiv [a, b, a, b] \equiv 1$, and we are finished.

4. Proof of the theorem: the irredundant case. In this section we suppose that G is

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a finite p-group with $b(G)=4$, $c(G)=6$, and $G=C_2\cup C_3\cup C_4\cup C_5$ as above, but G is not the union of any three of the C_i. Put $D=C_2\cap C_3\cap C_4\cap C_5$. We have to consider the possibilities for G/D and, just as important, for each C_i/D.

Fortunately the structure of a group covered by four proper subgroups has been given by Greco [**3**] and by Neumann [**8**]. For convenience we use the latter reference. Thus either $p=2$ and $|G:D|=8$, or $p=3$ and $|G:D|=9$.

Because the results of [**8**] do not give us the coverings of G/D in the various cases we shall need a little elaboration. Let H be a finite p-group which is irredundantly the union of its proper subgroups S_1, S_2, S_3, S_4 whose intersection is trivial. Suppose first that $p=2$ and that H is elementary abelian, $H=\langle a, b, c\rangle$. We may choose a, b, c so that $S_1=\langle a, b\rangle$ and $S_2=\langle a, c\rangle$ (see the table on p. 239 of [**8**]). Some juggling shows that in case (i) of that table we can choose a, b, c so that

$$S_1=\langle a, b\rangle,\qquad S_2=\langle a, c\rangle,\qquad S_3=\langle b, c\rangle,\qquad S_4=\langle abc\rangle. \tag{1}$$

Case (ii) which is simpler yields

$$S_1=\langle a, b\rangle,\qquad S_2=\langle a, c\rangle,\qquad S_3=\langle bc\rangle,\qquad S_4=\langle abc\rangle. \tag{2}$$

Another possibility is that H is abelian of order 8 and $H=\langle a, b\rangle$ with $a^4=b^2=1$. Though case (i) does not occur now, case (ii) gives

$$S_1=\langle a\rangle,\qquad S_2=\langle ab\rangle,\qquad S_3=\langle b\rangle,\qquad S_4=\langle a^2b\rangle. \tag{3}$$

Finally we may have $p=3$ and $H=\langle a, b\rangle$ of order 9 with

$$S_1=\langle a\rangle,\qquad S_2=\langle ab\rangle,\qquad S_3=\langle a^2b\rangle,\qquad S_4=\langle b\rangle. \tag{4}$$

Our next move is to cut down the number of possibilities embodied in (1)–(4) by borrowing some arguments of Gallian [**2**]. Suppose that x is an element of G such that $x\notin C_4\cup C_5$. Since $x\notin C_4$, $b(x\gamma_5)<b(x\gamma_6)$; since $x\notin C_5$, $b(x\gamma_6)<b(x)\leq 4$—here we are using that part of Lemma 2.1 of [**6**] stated as Lemma 2 of [**2**]. So $b(x\gamma_5)\leq 2$. It follows that $c(\langle x\gamma_5 : x\notin C_4\cup C_5\rangle)\leq 3$, by Theorem 2 of [**5**] also to be found as Lemma 1 of [**2**]. Because $K=\langle x\in G : x\notin C_4\cup C_5\rangle$ therefore has $c(K)\leq 5$, we see that $K\neq G$. (In rough terms we can say that "C_4 and C_5 must not be too small.")

This at once implies that (4) does not occur. Neither does (3), though this is less obvious. First we note that if $4\leq i\leq 5$ then C_i/D must have order 4. Next we lose no generality in taking $b\in C_2$, $a^2b\in C_3$, $a\in C_4$, $ab\in C_5$. As usual in these calculations we put

$$w=[x_1, x_2, x_3, x_4, x_5, x_6]\neq 1,$$

and we assume that $x_3=a$, $x_5=b$. Suppose every x_i lies in $\{a, b\}$ and take $x_1=a$, $x_2=b$. Since $ab\in C_5$, we lose no generality in putting $x_6=b$. Thus

$$w=[a, b, a, x_4, b, b].$$

If $x_4=b$ we find that $w=1$ by applying (AB) to $[a, b, a, x_4]$ and noting that $b\in C_2$. If

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$x_4 = a$ then

$$\begin{aligned} w &= [a, b, a; a, b; b][a, b, a, b, a, b] && \text{(JW)} \\ &= [a, b, a; a, b; b] && (\text{AB}; b \in C_2) \\ &= [a, b, b; a, b, a]^{-1}[a, b, a, b; a, b] && \text{(JW)} \\ &= 1 && (\text{AB}; b \in C_2). \end{aligned}$$

Thus the final stage of the argument requires $x_1 \in D$. If $x_4 = b$ then

$$[x_1, x_2, a, b][x_1, x_2, b, a]^{-1}[a, b, x_1, x_2][a, b, x_2, x_1]^{-1} \equiv 1 \bmod \gamma_5$$

gives $w = 1$, while if $x_4 = a$ then

$$\begin{aligned} w &= [x_1, x_2, a; a, b; a] && (\text{JW}; a \in C_4) \\ &= [a, b; x_1, x_2; a; a]^{-1}[a, b, a; x_1, x_2; a] && \text{(JW)} \\ &= 1 && (x_1 \in D). \end{aligned}$$

So (3) does not occur.

In cases (1) and (2) G has at least three generators, and we make a remark which will be important later: *we can assume that the case when G is generated by two elements has been disposed of.*

Gallian's argument shows that in both (1) and (2) we can take $C_4 = \langle a, b, D\rangle$, $C_5 = \langle a, c, D\rangle$. Correspondingly we can assume that $x_5 = c$, $x_6 = b$ in the usual way. In fact, we get more by applying (LN) to $[x_1, \ldots, x_4, c]$; we may suppose that *in addition* $x_1 = c$. So

$$w = [c, x_2, x_3, x_4, c, b].$$

We proceed to dispose of the subcase of (1) in which $C_2 = \langle b, c, D\rangle$, $C_3 = \langle abc, D\rangle$. We can suppose that $x_3 = a$. Since

$$[c, b, a][b, a, c][a, c, b] \equiv 1 \bmod \gamma_4,$$

the composition of C_2 shows that $[c, b, a] \equiv 1 \bmod \gamma_4$, and so if $x_2 = b$ then $w = 1$. So

$$w = [c, a, a, x_4, c, b].$$

If $x_4 = b$ then

$$\begin{aligned} [c, a, a, b, c] &\equiv [c, a; a; b, c] && (\text{JW}; b \in C_4) \\ &\equiv [b, c, a; c, a][b, c; c, a; a]^{-1} && \text{(JW)} \\ &\equiv 1 \end{aligned}$$

because $[b, c, a] \equiv 1$ and because $a \in C_4$. If $x_4 = c$ then $w = 1$ by (AB) and $c \in C_2$. Therefore the fact that $abc \in C_3$ shows that if $x_4 = a$ then $w = 1$.

The subcase does not occur then, and we recapitulate by rewriting (1) and (2) as

$$C_2 = \langle abc, D\rangle, \quad C_3 = \langle b, c, D\rangle, \quad C_4 = \langle a, b, D\rangle, \quad C_5 = \langle a, c, D\rangle, \qquad (1')$$

$$C_2 = \langle abc, D\rangle, \quad C_3 = \langle bc, D\rangle, \quad C_4 = \langle a, b, D\rangle, \quad C_5 = \langle a, c, D\rangle. \qquad (2')$$

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Note that interchanging C_2 and C_3 in (2′) gives nothing essentially new.

In (1′) we have $w=[c, x_2, x_3, a, c, b]$. First suppose that $x_2=a$. If $x_3=c$ then (AB) gives $w=1$. If $x_3=b$ then

$$\begin{aligned} w &= [c, a, b, a, c, b] \\ &= [c, a; b, a; c; b] && (\text{JW}; b \in C_3) \\ &= [b, a, c; c, a; b]^{-1}[c, a, c; b, a; b] && (\text{JW}) \\ &= [b, a, c, a, c, b] && (\text{JW}; a \in C_4; b \in C_4). \end{aligned}$$

However both $[b, a, a, a, c, b]$ and $[b, a, b, a, c, b]$ are trivial, because (LN) can be applied to $[b, a, x, a, c]$ and the composition of C_4 used. The fact that $abc \in C_2$ then shows that $w=1$. Next we suppose that $x_3=a$. In that case $abc \in C_2$ implies that $w=1$. So if $x_2=a$ then $w=1$.

If $x_2=b$ then we apply (LN) to $[c, b, x_3, a]$; when $x_3=b$ or c we have $[c, b, x_3, a]\equiv 1$ because of C_3, and so $w=1$. It follows that if $x_3=a$ then $w=1$ because $abc \in C_2$.

To complete the case (1′) we discuss the implications of taking $x_2 \in D$. Mod γ_5,

$$\begin{aligned} [c, x_2, x_3, a] &\equiv [c, x_2, a, x_3][c, x_2; x_3, a] && (\text{JW}) \\ &\equiv [c, x_2, a, x_3] \end{aligned}$$

because $x_2 \in \cap C_3$ gives $[x_3, a; c, x_2]\equiv 1$; so if $x_3=b$ or c then $w=1$ in view of C_3, and consequently if $x_3=a$ then $w=1$ because $abc \in C_2$. Thus if $x_2 \in D$ then $w=1$.

In case (2′) we have $w=[c, x_2, x_3, x_4, c, b]$. First suppose that $x_2=a$. It suffices to prove that $w=1$ whenever $x_3, x_4 \in \{a, c\}$ in view of C_2 and C_3. But this is obvious.

Secondly suppose that $x_2=b$. If $x_4=b$ then $c(G)<6$ by a remark above (note that we may assume $x_3=b$ or c), so we take $x_4=a$. Mod γ_6 we have

$$\begin{aligned} [c, b, x_3, a, c] &\equiv [c, b, x_3; a, c] && (\text{JW}; a \in C_4) \\ &\equiv [a, c, x_3; c, b][a, c; c, b; x_3]^{-1} && (\text{JW}) \\ &\equiv [a, c, x_3; c, b] && (x_3 \in C_4), \end{aligned}$$

provided we assume (as we may) that $x_3=a$ or b. So

$$w=[a, c, x_3, b, c, b]^{-1}$$

since $b \in C_4$, and $w=1$ follows because the case $x_2=a$ was dealt with in the previous paragraph.

Thirdly suppose that $x_2 \in D$. We may take

$$w=[c, x_2, x_3; x_4, c; b]$$

because $x_4=a$ or b in view of C_3, and so $x_4 \in C_4$. Next we apply (LN) to $(c, x_2, x_3, [x_4, c]]$. The fact that $x_2 \in D$ then shows that $w=1$.

This completes the case (2′) and with it the proof of the theorem.

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