EXTENSIONS OF LIE ALGEBRAS AND THE THIRD COHOMOLOGY GROUP

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Introduction. Cohomology theories of various algebraic structures have been investigated by several authors. The most noteworthy are due to Hochschild, MacLane and Eckmann, Chevalley and Eilenberg, who developed the theory of cohomology groups of associative algebras, abstract groups, and Lie algebras respectively. In this paper we are concerned primarily with a characterization of the third cohomology group of a Lie algebra by its extension properties.

In §1 necessary definitions from Chevalley and Eilenberg's theory are given [**2**]; §2 is concerned with a special type of extension. In §3 we define the invariant coboundary: a mapping of $H^q(L, P)$ into $H^{q+1}(L, Q)$ for any representation modules $\{V, P\}$ and $\{W, Q\}$ of L. In §4 we consider the special extension problem corresponding to the Teichmüller theory for simple (associative) algebras [**4**].

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1. Definition of the cohomology groups. Let L be a Lie algebra over a field F, and P a representation of L by means of linear endomorphisms of a vector space V of finite dimension over F. A q-linear alternating mapping of L into V will be called a q-dimensional V-cochain (or shorter: a q-V-cochain). The q-V-cochains form a space $C^q(L, V)$. By definition, $C^0(L, V) = V$. We define a linear mapping $f \to \delta f$ of $C^q(L, V)$ into $C^{q+1}(L, V)$ by the formula

$$(\delta f)(x_1, \ldots, x_{q+1}) = \sum_{k<l} (-1)^{k+l+1} f([x_k, x_l], x_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_l, \ldots, x_{q+1}) + \sum_{i=1}^{q+1} (-1)^{i+1} P(x_i) f(x_1, \ldots, \tilde{x}_i, \ldots, x_{q+1}),$$

where the tilde implies omission of the corresponding variable. If $q = 0$ then $f \in V$ and δf is defined by $(\delta f)(x) = P(x)f$. For any $f \in C^q(L, V)$ and all q, $\delta\delta f = 0$. A cochain f is a cocycle provided $\delta f = 0$. The cocycles of dimension q form a subspace $Z^q(L, P)$ of $C^q(L, V)$. A cochain $f \in C^q(L, V)$ is a coboundary if it is of the form δg for some $g \in C^{q-1}(L, V)$. The coboundaries of dimension

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q form a subspace $B^q(L, P)$ of $Z^q(L, P)$. By definition $B^0(L, P) = \{0\}$. The factor space

$$H^q(L, P) = Z^q(L, P)/B^q(L, P)$$

is called the qth cohomology group of L by P.

2. The extension $U = (L, V, W, \beta)$. Let S be an arbitrary set of elements. By an *S-module* on a field F we mean a pair $\{V, P\}$ formed by a vector space V of finite dimension over F and a mapping P which assigns to every element $x \in S$ a linear endomorphism $P(x)$ of V. In particular, let S be the set of elements of a Lie algebra L. An S-module $\{V, P\}$ is called a *representation module* of L if the following condition is satisfied:

$$P([x, y]) = P(y)\, P(x) - P(x)\, P(y)$$

for any elements x, y of L. In this case the mapping P is called a *representation* of L.

The group $H^2(L, P)$ was related by Chevalley and Eilenberg to the extension L by P as follows: We define an *extension* $L^+ = (L, V)$ of L by P to be a Lie algebra with the following properties:

(i) V is an ideal in L^+,

(ii) $[V, V] = 0$, that is, V is an abelian ideal,

(iii) $L^+/V \cong L$,

(iv) The *linear representatives* $\rho_x = (\rho(x)) \in L^+$ corresponding to $x \in L$ by the isomorphism (iii) satisfy[1] $P_x v = [v, \rho(x)]$.

The structure of L^+ is completely determined by

$$[\rho_x, \rho_y] = \rho_{[x,y]} + g(x, y), \qquad x, y \in L, \quad g(x, y) \in V,$$

where g satisfies the condition corresponding to

$$[[\rho_x, \rho_y], \rho_z] + [[\rho_y, \rho_z], \rho_x] + [[\rho_z, \rho_x], \rho_y] = 0,$$

that is,

$$g([x, y], z) + g([y, z], x) + g([z, x], y) + P_z g(x, y) + P_x g(y, z) + P_y g(z, x) = 0.$$

Hence g is a 2-P-cocycle. Conversely, for any given $g \in Z^2(L, P)$ there exists an extension L^+ with this g. We denote this extension by $L^+ = (L, V, g)$. If we choose another system of representatives

$$\overset{+}{\rho_x} = \rho_x + h(x) \qquad (h(x) \in V),$$

the corresponding g^+ is given by

$$g^+(x, y) = g(x, y) + \{P_y h(x) - P_x h(y) - h([x, y])\},$$

namely $g^+ \equiv g(\text{mod } B^2(L, P))$. Hence cohomologous g's generate isomorphic extensions. The extension L^+ is said to *split* if there is a subalgebra L' of L^+

[1]Let ϕ be a homomorphism of L^+ onto L. The representatives ρ_x (ρ is a linear function of x) are any fixed set of elements of L^+ satisfying $\phi\rho_x = x$ and $\rho_0 = 0$. Furthermore, V is the kernel of the homomorphism ϕ.

such that ϕ maps L' isomorphically onto L. Hence the vanishing of $H^2(L, P)$ for all P implies the splitting of all extensions $L^+ = (L, V)$.

Let the pair $\{U, R\}$ be a representation module of L with an L-invariant submodule W (with the operation of L on W denoted by Q). On the factor space U/W one has then an induced operation by L; if this is isomorphic to a module $\{V, P\}$, we call U an *extension of* V *by* W *with respect to* L.

Denote the elements of L by $x, y, \ldots$ and those of V by $v_1, v_2, \ldots$. For each element $v \in V$ we take a representative $\mu_v \in U$ from the residue class corresponding to $v \in V$ by the isomorphism $U/W \cong V$ such that μ_v depends linearly on v. Hence

$$U = (W + \mathrm{O}) \cup (W + \mu_{v_1}) \cup (W + \mu_{v_2}) \cup \ldots$$

where O is the zero representative and

2.1 $$R_x \mu_v = \mu_{P_x v} + \beta(x, v), \qquad \beta(x, v) \in W.$$

It follows from equation 2.1 that β is a bilinear function of $x \in L$ and $v \in V$. Now, since R is a representation on U,

$$R_y\, R_x\, \mu_v - R_x\, R_y\, \mu_v = R_{[x,y]}\mu_v$$

for all $x, y \in L$ and $v \in V$. Hence

2.2 $$\beta(x, P_y v) - \beta(y, P_x v) + \beta([x, y], v) + Q_x\, \beta(y, v) - Q_y\, \beta(x, v) = 0$$

for all $x, y \in L$ and $v \in V$. If we choose another set of linear representatives

$$\mu_v^+ = \mu_v + K_v, \qquad v \in V, \quad K_v \in W,$$

we have

2.3 $$\beta^+(x, v) = \beta(x, v) + \{Q_x\, K_v - K_{P_x v}\}.$$

We call β satisfying 2.2 a factor system and denote it by $\{\beta\}$. Two factor systems $\{\beta\}$ and $\{\beta^+\}$ satisfying the relation 2.3 are said to be *associated*. The structure of an extension U is completely determined by the factor system $\{\beta\}$. Hence we write $U = (L, V, W, \beta)$. Conversely, for any factor system $\{\beta\}$ there exists an extension $U = (L, V, W, \beta)$ satisfying 2.1 Two extensions $U_i = (L, V, W, \beta_i)$ $(i = 1, 2)$ are isomorphic (as L-modules, each element of $W < U_i$ $(i = 1, 2)$ corresponding to itself) if and only if $\{\beta_1\}$ and $\{\beta_2\}$ are associated. In this case we identify U_1 with U_2.

We define $\{\beta_1 + \beta_2\} = \{\beta_1\} + \{\beta_2\}$. Then all the factor systems form a module $\Phi(V, W)$. In a splitting factor system there is a set of representatives μ_v such that $\beta(x, v) = 0$. Then

$$\beta^+(x, v) = Q_x K_v - K_{P_x v}.$$

The splitting factor systems form a subspace $\Sigma(V, W)$ of $\Phi(V, W)$. Hence we have

THEOREM 2.4. *The elements of* $\Phi(V, W)/\Sigma(V, W)$ *correspond in a one-to-one manner with the extensions* U *of* V *by* W *with respect to* L.

3. The invariant coboundary. This section is merely an adaptation of the well-known relative cohomology sequence for coefficients to Lie algebras. Given the extension $U = (L, V, W, \beta)$, it is known that there exists a relative cohomology sequence[2] of homomorphisms,

3.1
$$\ldots \to H^q(L, V) \xrightarrow{\Lambda} H^{q+1}(L, W) \xrightarrow{\mathrm{M}} H^{q+1}(L, U) \xrightarrow{\mathrm{N}} H^{q+1}(L, V) \to H^{q+2}(L, W) \to \ldots,$$

and that this sequence is *exact*. In this sequence the mapping M is the obvious one: regard a cochain with values in W as if it has values in the larger module U; N is also obvious: take a cochain with values in U and reduce the values modulo W to obtain one with values in V. The mapping Λ is usually called the *invariant coboundary* and is described normally as follows: Let ψ: $U \to V$ be the given homomorphism of U upon its quotient $V \cong U/W$, and let g be any cocycle in $C^q(L, V)$. Pick representatives $\bar{g}(x_1, \ldots, x_q)$ at random so that $\bar{g}$ is multilinear and

$$\psi\bar{g}(x_1, \ldots, x_q) = g(x_1, \ldots, x_q).$$

Then $f = \delta\bar{g}$ actually has values in W and the mapping Λ is the one obtained by sending the cohomology class of g in $H^q(L, V)$ into that of f in $H^{q+1}(L, W)$.

We define a linear mapping $F = F_\beta$ of $C^q(L, V)$ into $C^{q+1}(L, W)$ $(q \geqslant 0)$ as follows:

$$F_\beta(g) = f \in C^{q+1}(L, W), \qquad g \in C^q(L, V),$$

where

$$f(x_1, \ldots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1}\beta(x_i, g(x_1, \ldots, \tilde{x}_i, \ldots, x_{q+1})), \; x_1, \ldots, x_{q+1} \in L.$$

A minor computation shows that the mapping Λ is essentially the same as the mapping F_β when applied to cocycles g. When applied to cochains g, the two maps differ by

$$\mu_{(\delta g)(x_1, \ldots, x_{q+1})}.$$

The advantage of the invariant coboundary is that it avoids some of the long computations necessary when employing the mapping F_β. For example, it is not necessary to prove that the mappings F_β and δ commute. Also the proof that the sequence 3.1 is exact in the usual sense is entirely straightforward. The map Λ is defined for the extension (an easy argument shows that the choice of representatives does not matter)[3] and not from the factor sets β or β^+.

If the factor system $\{\beta\}$ splits, then $\beta = 0$, and so also $F_\beta = 0$. Hence:

THEOREM 3.2 *If the factor system* $\{\beta\}$ *splits, then* Λ *maps* $H^q(L, V)$ *into the zero cohomology class of* $H^{q+1}(L, W)$ $(q \geqslant 1)$.

4. The group $H^3(L, W)$**.** An interpretation of the third cohomology group in relation and analogous to the Teichmüller theory of factor systems of

[2]We use U, V, W here instead of R, P and Q respectively. The cohomology groups have the same meaning as before.

[3]If the factor systems $\{\beta\}$ and $\{\beta^+\}$ are associated, then F_β and F_{β^+} induce the same mapping of $H^q(L, V)$ into $H^{q+1}(L, W)$.

higher degree is now given. Let $L^+ = (L, V, g)$ be an extension of L with factor set g and $U = (L, V, W, \beta)$ an extension of V by W with respect to L. The extension U implies the existence of representation modules $\{U, R\}$ and $\{V, P\}$ of L satisfying

4.1 $$R_x\mu_v = \mu_{P_x v} + \beta(x, v), \qquad x \in L, \ v \in V, \ \mu_v \in U, \ \beta(x, v) \in W$$

(cf. equation 2.1) where the elements μ_v are the representatives corresponding to the isomorphism $U/W \cong V$. We consider the following problem.

To construct[4] *an extension* $L^{++} = (L, U)$ *of* L *by* R *satisfying* $L^{++}/W \cong L^+$. Suppose that we have such an extension. We then have the following lattice:

$$\begin{array}{ccccc} & \overset{\tau}{\longleftarrow} & & \overset{\rho}{\longleftarrow} & \\ L^{++} & \longrightarrow & L^+ & \longrightarrow & L \\ | & \overset{\mu}{\longleftarrow} & | & & | \\ U & \longrightarrow & V & \longrightarrow & 0 \\ | & & | & & \\ W & \longrightarrow & 0 & & \end{array}$$

From $L^{++}/W \cong L^+$ choose linear representatives

$$\tau_{x^+} \in L^{++} \qquad (x^+ \in L^+)$$

such that

$$\tau_{x^+} \equiv \mu_{x^+}$$

on V. Hence

$$L^{++} = (W + \tau_{x^+}) \cup (W + \tau_{y^+}) \cup \ldots, \qquad x^+, y^+, \ldots \in L^+; \quad \tau_{x^+}, \tau_{y^+}, \ldots \in L^{++}.$$

Then

4.2 $$[\tau_{x^+}, \tau_{y^+}] = \tau_{[x^+, y^+]} + l(x^+, y^+), \qquad l(x^+, y^+) \in W.$$

In particular,

$$[\tau_{\rho_x}, \tau_{\rho_y}] = \tau_{[\rho_x, \rho_y]} + l(\rho_x, \rho_y)$$

4.3 $$= \tau_{\rho[x,y]+g(x,y)} + l(\rho_x, \rho_y)$$

$$= \tau_{\rho[x,y]} + \mu_{g(x,y)} + l(\rho_x, \rho_y), \qquad l(\rho_x, \rho_y) \in W$$

where the representatives $\rho_x \in L^+$ are selected in such a way that

$$[\rho_x, \rho_y] = \rho_{[x,y]} + g(x, y).$$

Now, from the isomorphism $L^{++}/U \cong L$ choose linear representatives $\sigma_x \in L^{++}$ so that $\sigma_x = \tau(\rho_x)$. Then

$$L^{++} = (U + \sigma_x) \cup (U + \sigma_y) \cup \ldots, \qquad x, y, \ldots \in L; \quad \sigma_x, \sigma_y, \ldots \in L^{++}.$$

Therefore we have the following multiplication of representatives

4.4 $$[\sigma_x, \sigma_y] = \sigma_{[x,y]} + \mu_{h(x,y)} + \alpha(x, y), \qquad \alpha(x, y) \in W, \quad h \in C^2(L, V).$$

[4] L^{++}/W may then be regarded as an extension of L.

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Comparing equations 4.3 and 4.4 we observe that $\mu_{h(x,y)} = \mu_{g(x,y)}$, so that

4.4′ $$[\sigma_x, \sigma_y] = \sigma_{[x,y]} + \mu_{g(x,y)} + \alpha(x, y), \qquad \alpha(x, y) \in W.$$

Since $g \in Z^2(L, P)$ it is easy to see that the function α is a 2-W-cochain.

The extension L^{++} of L by R implies that $R_x u = [u, \sigma_x]$ $(u \in U)$, and so, by 4.1,

4.5 $$[\mu_v, \sigma_x] = \mu_{P_x v} + \beta(x, v), \qquad \beta(x, v) \in W.$$

From equations 4.4 and 4.5,

4.6 $$\begin{aligned}[\sigma_x, [\sigma_y, \sigma_z]] &= [\sigma_x, \sigma_{[y,z]} + \mu_{g(y,z)} + \alpha(y, z)] \\ &= \sigma_{[x,[y,z]]} + \mu_{g(x,[y,z])} + \alpha(x, [y\ z]) \\ &\quad - \mu_{P_x(g(y,z))} - \beta(x, g(y, z)) - Q_x\alpha(y, z)\end{aligned}$$

since $[w, \sigma_x] = R_x w \equiv Q_x w$ $(w \in W)$. The Jacobi identity for the representatives σ_x yields symbolically

4.7 $$\Lambda(g) + \delta\alpha = 0.$$

Denote by $k(\in Z^2(L, U))$ the factor set belonging to L^{++}. Then

$$k(x, y) = \mu_{g(x,y)} + \alpha(x, y) = \bar{g}(x, y) + \alpha(x, y),$$

and so $\delta\alpha = -\delta\bar{g}$. Hence $\Lambda(g) = \delta\bar{g}$, and in addition $k(x, y) \equiv \bar{g}(x, y) \bmod W$.

Conversely, if we have $\alpha(x, y) \in W$ satisfying equation 4.7 then we can construct the extension L^{++} as follows: To each $x \in L$ assign a symbol σ_x. The algebra L^{++} is to consist of all the elements of all the cosets $U + \sigma_x$. Multiplication of two σ_x's will be defined by 4.4′ and the multiplication of a σ_x and a μ_v by the equation 4.5. Multiplication of σ_x and w is defined by $[w, \sigma_x] = Q_x w$. Since Q_x is a linear endomorphism, W is an ideal in L^{++}. Further, for an arbitrary representative $\mu_v \in U$ and $w \in W$, $[\mu_v, w] = 0$ since U is abelian. There remains the verification of the Jacobi identity for the σ_x and this is equivalent to 4.7. We must also verify the Jacobi identities for mixed multiplications of σ_x's and μ_v's:

LEMMA 4.8.

(i) $$[\sigma_x, [\mu_{v_1}, \mu_{v_2}]] + [\mu_{v_1}, [\mu_{v_2}, \sigma_x]] + [\mu_{v_2}, [\sigma_x, \mu_{v_1}]] = 0,$$

(ii) $$[\sigma_x, [\mu_v, \sigma_y]] + [\mu_v, [\sigma_y, \sigma_x]] + [\sigma_y, [\sigma_x, \mu_v]] = 0.$$

The proof of (i) is obvious since U is abelian.

Proof of (ii). The expression on the left is equal to

$$\begin{aligned}&[\sigma_x, \mu_{P_y v} + \beta(y, v)] + [\mu_v, \sigma_{[y,x]} + \mu_{g(y,x)} + \alpha(y, x)] + [\sigma_y, -\mu_{P_x v} - \beta(x, v)] \\ &= -\mu_{P_x(P_y v)} - \beta(x, P_y v) - Q_x\beta(y, v) + \mu_{P[y,x]v} + \beta([y, x], v) + \mu_{P_y(P_x v)} \\ &\qquad + \beta(y, P_x v) + Q_y\beta(x, v) \\ &= -\beta(x, P_y v) + \beta(y, P_x v) + \beta([y, x], v) - Q_x\beta(y, v) + Q_y\beta(x, v) = 0,\end{aligned}$$

by the relation 2.2. Hence we have

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THEOREM 4.9. *Let L be a Lie algebra over a field F and $\{U, R\}$ a representation module of L where $U = (L, V, W, \beta)$ is an extension of V by W with respect to L. Then for a given extension $L^+ = (L, V, g)$ a necessary and sufficient condition for the existence of another extension $L^{++} = (L, U)$ of L by R such that $L^{++}/W \cong L^+$ is that the 3-Q-cocycle $\Lambda(g)$ is a coboundary.*

COROLLARY 4.91. *If $H^3(L, Q) = \{0\}$ then there is always such an extension.*

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