LINEAR TRANSFORMATIONS ON GRASSMANN SPACES

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1. Let U denote an n-dimensional vector space over a field F and let G_{nr} denote the set of non-zero decomposable r-vectors of the Grassmann product space $\wedge^r U$. Let T be a linear transformation of $\wedge^r U$ into itself which maps G_{nr} into itself. If F is algebraically closed, or if T is non-singular, then the structure of T is known. In this paper we show that if T is singular, then the image of $\wedge^r U$ has a very special form with dimension equal to the larger of the integers $r + 1$ and $n - r + 1$. We give an example to show that this can occur.

2. We adopt the notation of (**1**). We recall that if $z = x_1 \wedge \ldots \wedge x_r \in G_{nr}$, then $[\mathbf{z}] = \langle x_1, \ldots, x_r \rangle$ is a well-defined r-dimensional subspace of U. We say that z determines $[z]$. The two classes of maximal subspaces of $\wedge^r U$ whose non-zero elements belong to G_{nr} are denoted by A_r and B_r. The r-dimensional subspaces determined by the non-zero elements of an $X \in A_r$ contain a common $(r - 1)$-dimensional subspace which we will denote by $\xi(X)$. The r-dimensional subspaces determined by the non-zero elements of a $Y \in B_r$ are contained in an $(r + 1)$-dimensional subspace of U which we will denote by $\eta(Y)$.

For maps $f\colon S \to T$, where S and T are arbitrary, we adopt the following conventions. If $S_0 \subseteq S$, then $f(S_0)$ denotes $\{f(s)\colon s \in S_0\}$ and if $\mathscr{S}$ is a family of subsets of S, then $f(\mathscr{S})$ is the family $\{f(S_0)\colon S_0 \in \mathscr{S}\}$ of subsets of T.

The following elementary facts are used throughout the paper. Distinct elements of A_r or of B_r intersect in at most one dimension. On the other hand, if $X \in A_r$ and $Y \in B_r$, then $\dim(X \cap Y) = 0$ or 2 according as $\xi(X) \not\subseteq \eta(Y)$ or $\xi(X) \subseteq \eta(Y)$. The dimensions of the elements of A_r and B_r are $n - r + 1$ and $r + 1$, respectively. We note that these are equal only when $n = 2r$. Finally, since $T(G_{nr}) \subseteq G_{nr}$, T is one-to-one on each member of $A_r \cup B_r$.

Our main result is the following.

3. Theorem. *If $T\colon \wedge^r U \to \wedge^r U$ is a singular linear transformation such that $T(G_{nr}) \subseteq G_{nr}$, then $T(\wedge^r U) \in A_r \cup B_r$.*

Proof. We first consider the case when $T(B_r) \subseteq B_r$. Let k be the maximal integer such that the image of every $\wedge^r U_0$ with $\dim(U_0) = k$ is a $\wedge^r W$ with

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$\dim(W) = k$. Then $r < k < n$, where the latter inequality is strict since T is singular. If U_1 and U_2 are an adjacent pair of k-dimensional subspaces of U and $T(\wedge^r U_i) = \wedge^r W_i$, then, since $T(\wedge^r(U_1 \cap U_2)) \subseteq \wedge^r(W_1 \cap W_2)$ and T is one-to-one on $\wedge^r(U_1 \cap U_2)$, W_1 and W_2 are either adjacent or equal. If W_1 and W_2 are distinct, then T is one-to-one on $\wedge^r(U_1 + U_2)$ since its image, $\wedge^r(W_1 + W_2)$, then has dimension equal to $\dim(\wedge^r(U_1 + U_2))$. Therefore, by the maximality of k, there is a pair of adjacent k-dimensional subspaces U_1 and U_2 of U such that $T(\wedge^r(U_1 + U_2)) = \wedge^r W$, where $\dim(W) = k$. Suppose that $k > r + 1$. Since $T\colon \wedge^r U_1 \to \wedge^r W$ is one-to-one and maps B_r into B_r, it is induced by a linear transformation $A\colon U_1 \to W$. Let $X \in A_r$ with $\xi(X) = \langle x_1, \ldots, x_{r-1}\rangle \subseteq U_1$. Since $k > r + 1$, $\dim(X \cap \wedge^r U_1) = k - r + 1 > 2$. If $Y \in A_r$ with $\xi(Y) = \langle Ax_1, \ldots, Ax_{r-1}\rangle$, then $\dim(T(X) \cap Y) > 2$. Therefore, $T(X) = Y$. But then $T(X \cap \wedge^r(U_1 + U_2)) \subseteq Y \cap \wedge^r W$ which is impossible since we have $\dim(X \cap \wedge^r(U_1 + U_2)) = 1 + \dim(Y \cap \wedge^r W)$. Therefore, $k = r + 1$, and for every pair of adjacent $(r + 1)$-dimensional subspaces U_1 and U_2 of U we have that $T(\wedge^r U_1) = T(\wedge^r U_2)$. Then $T(\wedge^r U) \in B_r$, since for any pair $X, Y \in B_r$ there is a finite chain $X_1, \ldots, X_m$ of elements of B_r with X_i, X_{i+1} adjacent and $X = X_1$, $Y = X_m$.

Next, we suppose that $T(A_r) \subseteq A_r$ and there is a pair $X \in B_r$, $Y \in A_r$ for which $T(X) \subseteq Y$. If $Z \in A_r$ with $\xi(Z) \subseteq \eta(X)$, then $\dim(Z \cap X) = 2$. Therefore, $\dim(T(Z) \cap Y) \geqq 2$, and since $T(Z) \in A_r$ also, we have that $T(Z) = Y$. Let U_1 be a subspace of largest possible dimension such that for each $Z \in A_r$ with $\xi(Z) \subseteq U_1$ we have that $T(Z) = Y$. Then $\dim(U_1) > r$. Suppose that $U_1 \neq U$ and select $U_2 \supset U_1$ such that $\dim(U_2) = 1 + \dim(U_1)$. Let $Z \in A_r$ with $\xi(Z) \subseteq U_2$. Then $\dim(\xi(Z) \cap U_1) = r - 2$ or $\xi(Z) \subseteq U_1$. If the latter, then $T(Z) = Y$. Otherwise, let $\xi(Z) = \langle y_1, \ldots, y_{r-1}\rangle$, where $\langle y_1, \ldots, y_{r-2}\rangle \subseteq U_1$. For each $y \in U_1$, $y \notin \langle y_1, \ldots, y_{r-2}\rangle$ there is a $Z_y \in A_r$ with $\xi(Z_y) = \langle y_1, \ldots, y_{r-2}, y\rangle$. Now, $\dim(Z \cap Z_y) = 1$. Choose y and y' so that $\{y_1, \ldots, y_{r-1}, y, y'\}$ is independent and $y, y' \in U_1$. Then $Z \cap Z_y \neq Z \cap Z_{y'}$, and therefore, since $T(Z_y) = T(Z_{y'}) = Y$, we have that $\dim(T(Z) \cap Y) > 1$. It follows that $T(Z) = Y$. This contradicts the maximality of U_1, and thus $U_1 = U$. Then $T(\wedge^r U) = Y \in A_r$.

If $n < 2r$, then $T(B_r) \subseteq B_r$, since $X \in B_r$ and $T(X) \subseteq Y$ for some $Y \in A_r$ would imply that T is singular on X. If $n > 2r$, then $T(A_r) \subseteq A_r$ and for some $X \in B_r$, $T(X) \notin B_r$; for, $T(B_r) \subseteq B_r$ would imply that $T(\wedge^r U) \in B_r$, and consequently T would be singular on each member of A_r. Therefore, the above paragraphs prove the theorem for the case when $n \neq 2r$.

When $n = 2r$, we show that either $T(B_r) \subseteq B_r$ or $T(B_r) \subseteq A_r$. Suppose the contrary. Then we can select $X_1, X_2 \in B_r$ with $\eta(X_1)$ and $\eta(X_2)$ adjacent such that $T(X_1) = Y_1 \in A_r$ and $T(X_2) = Y_2 \in B_r$. Let $U_0 = \eta(X_1) \cap \eta(X_2)$ and let $\mathscr{Y} = \{Y \in A_r \colon \xi(Y) \subseteq U_0\}$. For each $Y \in \mathscr{Y}$, $\dim(Y \cap X_1) = \dim(Y \cap X_2) = 2$, and therefore both $\dim(T(Y) \cap Y_1)$ and $\dim(T(Y) \cap Y_2)$ are at least 2. Since Y_1 and Y_2 are of different types, it follows that $T(Y) = Y_1$

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or $T(Y) = Y_2$. Furthermore, since $\bigcup\{Y: Y \in \mathscr{Y}\} \supseteq X_1 \cup X_2$, not every $Y \in \mathscr{Y}$ is mapped into the same Y_i. We select $Y_i' \in \mathscr{Y}$ such that $T(Y_i') = Y_i$, $i = 1, 2$. Let $X \in B_r$ such that $U_0 \subseteq \eta(X) \subseteq \eta(X_1) + \eta(X_2)$ while $\eta(X)$ is distinct from both $\eta(X_1)$ and $\eta(X_2)$. Then $\dim(X \cap Y_i') = 2$, and therefore $T(X) = Y_1$ or Y_2. However, since $\eta(X) + \eta(X_i) = \eta(X_1) + \eta(X_2)$, we obtain $Y_1 + Y_2 = Y_1$ or Y_2, which is impossible.

To complete the proof of the theorem, we proceed as follows. Since we have already dealt with the possibility that $T(B_r) \subseteq B_r$, we may suppose that $T(B_r) \subseteq A_r$. Let T_0 denote a linear transformation of $\wedge^r U$ which is induced by a correlation of the r-dimensional subspaces of U. Then $T_0(A_r) = B_r$ and $T_0(B_r) = A_r$. Therefore, $T_0 T(B_r) \subseteq B_r$, and consequently $T_0 T(\wedge^r U) \in B_r$. Therefore, $T(\wedge^r U) \in A_r$, and the proof is complete.

4. When $\dim(U) = 4$ and $r = 2$, we can decide for which fields F a singular T exists. In fact, such a T exists if and only if there exist $a_i \in F$, $i = 1, \ldots, 6$, such that the only solution in F of

$$(*) \qquad a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz = 0$$

is trivial; that is, $x = y = z = 0$.

Suppose that there are elements $a_i \in F$ such that the only solution of $(*)$ is trivial. Let $\{u_1, u_2, u_3, u_4\}$ be a basis of U and define

$$z_1 = u_1 \wedge (a_5u_2 + a_1u_4) + u_2 \wedge u_3,$$

$$z_2 = u_1 \wedge (a_4u_4 - a_2u_3) + u_2 \wedge u_4,$$

$$z_3 = u_1 \wedge (a_3u_2 - a_6u_3) + u_3 \wedge u_4.$$

Let $V = \langle z_1, z_2, z_3\rangle$. Then V contains no non-zero decomposable vectors, since a linear combination $xz_1 + yz_2 + zz_3 = (a_5x + a_3z)u_1 \wedge u_2 + (-a_2y - a_6z)u_1 \wedge u_3 + (a_1x + a_4y)u_1 \wedge u_4 + xu_2 \wedge u_3 + yu_2 \wedge u_4 + zu_3 \wedge u_4$ is decomposable if and only if $(a_5x + a_3z)z - (-a_2y + a_6z)y + (a_1x + a_4y)x = 0$, that is, if and only if $x = y = z = 0$. Since $\dim(\wedge^2 U) = 6$, there exist $T: \wedge^2 U \to X$, where $X \in A_2$ or B_2, and the kernel of T is V. On the other hand, suppose that there is a $T: \wedge^2 U \to X$, where $X \in A_2 \cup B_2$. Then $\dim(\text{kernel } (T)) = 3$. If $\{z_1, z_2, z_3\}$ is a basis for this kernel, then we can write

$$z_i = \sum_{j<k} a_{ijk} u_j \wedge u_k.$$

Then

$$xz_1 + yz_2 + zz_3 = \sum_{j<k} f_{jk} u_j \wedge u_k,$$

where each f_{jk} is a linear form in x, y, z with coefficients in F. The quadratic p-relation $f_{12}f_{34} - f_{13}f_{24} + f_{14}f_{23}$ is a form $a_1x^2 + \ldots + a_6yz$ with the $a_i \in F$.

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Since the kernel has no decomposable vectors, this form is zero only when $x = y = z = 0$.

Reference

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