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RESEARCH ARTICLE

An orthogonality relation for GL$(4, \mathbb{R})$ (with an appendix by Bingrong Huang)

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Abstract
Orthogonality is a fundamental theme in representation theory and Fourier analysis. An orthogonality relation for characters of finite abelian groups (now recognized as an orthogonality relation on GL(1)) was used by Dirichlet to prove infinitely many primes in arithmetic progressions. Orthogonality relations for GL(2) and GL(3) have been worked on by many researchers with a broad range of applications to number theory. We present here, for the first time, very explicit orthogonality relations for the real group GL$(4, \mathbb{R})$ with a power savings error term. The proof requires novel techniques in the computation of the geometric side of the Kuznetsov trace formula.

1. Introduction and main theorem

Let $q > 1$ be an integer, and let $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character (mod q). The classical orthogonality relation for Dirichlet characters states that for integers m, n coprime to q,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = \begin{cases} 1 & \text{if } m \equiv n \pmod q, \\ 0 & \text{otherwise.} \end{cases}$$

This orthogonality relation is the basis for Dirichlet's proof that there are infinitely many primes $p \equiv a \pmod q$ if $(a, q) = 1$. It has played an essential role in the modern development of analytic number theory.

When they are lifted to the adele ring $\mathbb{A}$ over $\mathbb{Q}$, Dirichlet characters can be realized as automorphic representations of GL(1) (see chapter 2 in [GH11]). It is then very natural to try to generalize the above orthogonality relation to representations of higher rank reductive groups. When trying to do this, however, there is an immediate obstacle. In the case of GL(1), there are only finitely many characters (mod q) for any fixed $q > 1$. In higher rank, on the other hand, there will be infinitely many automorphic representations. It then becomes necessary to introduce a test function with rapid decay and define the orthogonality relation as an absolutely convergent integral over the automorphic representations.

The first successful attempt at obtaining an orthogonality relation for GL(2) was made by R. Bruggeman in 1978 (see [Bru78]) who considered the orthonormal basis $\{\phi_j\}_{j=1,2,\ldots}$ of Maass cusp forms for SL$(2, \mathbb{Z})$, where

$$\phi_j(z) = \sum_{n \neq 0} a_j(n) \sqrt{2\pi y}\, K_{it_j}(2\pi |n| y) \cdot e^{2\pi i n x}, \qquad (z = x + iy \in \text{upper-half plane}),$$

and K_{it_j} is the modified K-Bessel function of the second kind while $a_j(n) \in \mathbb{C}$ are the Fourier coefficients of ϕ_j. The Maass cusp form ϕ_j has Laplace eigenvalue $\lambda_j = 1/4 + t_j^2$ and is a Hecke eigenform. Each such Maass cusp form is associated with a unique irreducible cuspidal automorphic representation of $\mathrm{GL}(2)$. Then Bruggeman proved the following orthogonality relation for non-zero integers m, n:

$$\lim_{T\to\infty} \frac{4\pi^2}{T} \sum_{j=1}^{\infty} \frac{a_j(m)\,\overline{a_j(n)}}{\cosh(\pi t_j)} \cdot e^{-\lambda_j/T} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Other versions of GL(2) type orthogonality relations were later obtained by P. Sarnak [Sar87], and, for the case of holomorphic Hecke modular forms, by Conrey-Duke-Farmer [CDF97] and J.P. Serre [Ser97].

An orthogonality relation for Maass cusp forms on $\mathrm{GL}(3,\mathbb{R})$ was first proved independently by Goldfeld–Kontorovich [GK13] and Blomer [Blo13] in 2013. Further results on orthogonality relations for $\mathrm{GL}(3,\mathbb{R})$ were obtained by Blomer-Buttcane-Raulf [BBR14] and Guerreiro [Gue15]. In his 2013 thesis (see [Zho13], [Zho14]) Fan Zhou conjectured a very general orthogonality relation for $\mathrm{GL}(n)$ for $n \geq 2$. We now describe Zhou's conjecture.

Fix $n \geq 2$. A Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$ is a smooth function $\phi : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{C}$ that satisfies $\phi(gkr) = \phi(g)$ for all $g \in \mathrm{GL}(n,\mathbb{R})$, $k \in K = \mathrm{O}(n,\mathbb{R})$, and $r \in \mathbb{R}^\times$. In addition, ϕ is square integrable and is an eigenfunction of the Laplacian. If λ denotes the Laplace eigenvalue of ϕ, then λ can be expressed in terms of Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ of ϕ, where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$. The precise relation is given (see Section 6 in [Mil02]) by

$$\lambda = \left(\frac{n^3 - n}{24} - \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}{2} \right).$$

The Maass cusp form ϕ is said to be tempered at ∞ if the Langlands parameters $\alpha_1, \ldots, \alpha_n$ are all pure imaginary.

Let $\{\phi_j\}_{j=1,2,\ldots}$ denote an orthogonal basis of Maass cusp forms for $\mathrm{SL}(n,\mathbb{Z})$ with associated Langlands parameters $\alpha^{(j)} = \left(\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}\right)$ and M^{th} Fourier coefficient $A_j(M)$, where $M = (m_1, m_2, \ldots, m_{n-1})$ with $m_1 m_2 \cdots m_{n-1} \neq 0$. We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1, 1, \ldots, 1) = 1$. Let

$$\mathcal{L}_j := \operatorname*{Res}_{s=1} L(s, \phi_j \times \overline{\phi}_j)$$

be the residue, at the edge of the critical strip, of the Rankin-Selberg L-function attached to $\phi_j \times \overline{\phi}_j$ that is the value at $s = 1$ of the adjoint L-function $L(s, \mathrm{Ad}\ \phi_j)$.

For $T \to \infty$, and Langlands parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ of ϕ, let $h_T(\alpha)$ denote a good test function with exponential decay as $\sum_{k=1}^n |\alpha_k|^2 \to \infty$. Here 'good' means h_T is smooth, holomorphic in a region $-\eta < \mathrm{Re}(\alpha_i) < \eta$ for some $\eta > 0$, invariant under permutation of the Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$, real valued and positive, and is essentially supported on the Laplace eigenvalues of ϕ that are less than T^2.

Conjecture 1.0.1 (Orthogonality relation for $\mathrm{GL}(n,\mathbb{R})$**).** *Let* $\{\phi_j\}_{j=1,2,\ldots}$ *denote an orthogonal basis of Maass cusp forms for* $\mathrm{SL}(n,\mathbb{Z})$ *as above. Set* $M = (m_1, \ldots, m_{n-1})$ *and* $M' = (m'_1, \ldots, m'_{n-1}) \in \mathbb{Z}_+^{n-1}$. *Let* h_T *denote a good test function as above. Then*

$$\lim_{T\to\infty} \frac{\sum\limits_{j=1}^{\infty} A_j(M)\overline{A_j(M')}\, \frac{h_T\left(\alpha^{(j)}\right)}{\mathcal{L}_j}}{\sum\limits_{j=1}^{\infty} \frac{h_T\left(\alpha^{(j)}\right)}{\mathcal{L}_j}} = \begin{cases} 1 & \text{if } M = M', \\ 0 & \text{otherwise.} \end{cases}$$

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For applications, it is important to determine the rate of convergence as $T \to \infty$ in the above asymptotic relation. With this in mind, we reformulate Conjecture 1.0.1 with an error term.[1] In this case, the orthogonality relation is expected to take the form

Conjecture 1.0.2. *For some constant $\theta < 1$, we have*

$$\sum_{j=1}^{\infty} A_j(M)\overline{A_j(M')}\,\frac{h_T\left(\alpha^{(j)}\right)}{\mathcal{L}_j} = \delta_{M,M'} \sum_{j=1}^{\infty} \frac{h_T\left(\alpha^{(j)}\right)}{\mathcal{L}_j} + \mathcal{O}_{M,M'}\left(\sum_{j=1}^{\infty} \frac{h_T\left(\alpha^{(j)}\right)}{\mathcal{L}_j}\right)^{\theta}.$$

Here $\delta_{M,M'}$ is 1 or 0 depending on whether $M = M'$ or not.

In the above, since $\theta < 1$, the error term gives a power savings in the main term. In the case $n = 3$, this conjecture was proved in [GK13] with Langlands parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ and the following choice of test function:

$$h_{T,R}(\alpha) := e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{T^2}} \cdot \frac{\prod\limits_{1\le j\ne k\le 3} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)^2}{\prod\limits_{1\le j\ne k\le 3} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}.$$

More precisely, it was shown in [GK13] that $\sum_{j=1}^{\infty} \frac{h_{T,R}(\alpha^j)}{\mathcal{L}_j} \sim c\,T^{5+3R}$ and $\theta = \frac{3+3R+\varepsilon}{5+3R}$, for some constant $c > 0$, and any fixed $\varepsilon > 0$ as $T \to \infty$. Similar results were independently obtained by Blomer [Blo13] and improved later in [BBR14] and more recently in [BZ20], where an interesting technique is developed to remove the arithmetic weight $\mathcal{L}_j$.

Conjecture 1.0.2 has many important applications to low-lying zeros, Katz–Sarnak conjectures on symmetry types of families of automorphic L-functions, Sato–Tate conjectures, etc. Such applications, for the special case of $GL(3,\mathbb{R})$, constitute a major main theme in [Blo13], [BBR14], [BZ20], [GK13], [Gue15], [Zho13] and [Zho14]. See also [ST16], where results for these problems are obtained for general families of cohomological automorphic representations of reductive groups over number fields. In this paper, we focus only on the orthogonality conjecture, as the techniques to obtain the above type applications from Conjecture 1.0.2 are very well established.

Shin–Templier [ST16] obtain their results by an application of the Arthur-Selberg trace formula, with polynomial dependence on the Hecke eigenvalue and a power saving on the error term. Matz-Templier [MT15] establish the analogous results for the family $\{\phi_j\}$ of Maass cusp forms for $SL(n,\mathbb{Z})$, and this is strengthened and generalized in Finis-Matz [FM19]. A variant of Theorem 1.1.1 is obtained in [MT15], [FM19], without the arithmetic weight $\mathcal{L}_j$, without the polynomial weight of size T^{8R}, and with different test functions that are indicator functions of $\alpha^{(j)} \in T\Omega$, and where the error term would be $O(|\ell m|^{\frac{1}{2}} \cdot T^8)$. For comparison, note that in Theorem 1.1.1 (if the polynomial weights are removed) we obtain an error term of the form $\mathcal{O}\left(|\ell m|^{\frac{2}{5}+\varepsilon} \cdot T^{6+\varepsilon}\right)$. Also note that our main term is of a different form than that of [MT15], [FM19], in that ours entails some high-order asymptotics (the terms $\mathfrak{c}_2 T^{8+8R}$ and $\mathfrak{c}_3 T^{7+8R}$).

Shortly after our paper first appeared, Subhajit Jana [Jan20] obtained a proof of the conjecture for compactly supported functions (on the geometric side) for automorphic forms for $PGL_r(\mathbb{Z})$ with $r \ge 2$. Although a power savings error term is not given in Jana's paper, the author has informed us that the method can give a power savings error term that is very far from optimal (even for GL(2)).

1.1. Main Theorem

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4$, and let S_4 denote the symmetric group on a set of size four. The main result of this paper is a proof of Conjecture 1.0.2 for $GL(4,\mathbb{R})$ for the test function $h_{T,R}(\alpha)$ given by

[1]We adopt the standard convention that the constant implied by $\mathcal{O}_{M,M'}$ depends at most on M and M'.

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$$e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{T^2}} \prod_{1\le j\ne k\le 4} \frac{\left(\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)\right)^2}{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)} \prod_{\sigma\in S_4}\Big(1+\alpha_{\sigma(1)}-\alpha_{\sigma(2)}-\alpha_{\sigma(3)}+\alpha_{\sigma(4)}\Big)^{\frac{R}{12}},$$

where T is a large positive number and R (sufficiently large) is a fixed positive integer.

Theorem 1.1.1 (Main Theorem). *Let $\{\phi_j\}_{j=1,2,\dots}$ denote an orthogonal basis of even Maass cusp forms for $\mathrm{SL}(4,\mathbb{Z})$ (assumed to be tempered at ∞) with associated Langlands parameters*

$$\alpha^{(j)} = \big(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_4^{(j)}\big) \in (i\mathbb{R})^4$$

and L^{th} Fourier coefficient $A_j(L)$ (as in (2.8.1)), where $L=(\ell_1,\ell_2,\ell_3)\in\mathbb{Z}^3$. Let $\mathcal{L}_j = L(1,\mathrm{Ad}\,\phi_j)$. We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1,1,1)=1$. Let $\ell, m\in\mathbb{Z}$ with $\ell m\neq 0$. Then, for $T\to\infty$,

$$\sum_{j=1}^{\infty} A_j(\ell,1,1)\,\overline{A_j(m,1,1)}\,\frac{h_{T,R}\left(\alpha^{(j)}\right)}{\mathcal{L}_j} = \delta_{\ell,m}\cdot\Big(\mathfrak{c}_1T^{9+8R}+\mathfrak{c}_2T^{8+8R}+\mathfrak{c}_3T^{7+8R}\Big)$$
$$+\,\mathcal{O}_{\varepsilon,R}\left(|\ell m|^{\frac{2}{5}+\varepsilon}\cdot T^{6+8R+\varepsilon}+|\ell m|^{\frac{7}{32}+\varepsilon}\cdot T^{5+8R+\varepsilon}+|\ell m|^{\frac{15}{2}}\cdot T^{4+8R+\varepsilon}\right),$$

where $\delta_{\ell,m}$ is the Kronecker symbol and $\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{c}_3>0$ are absolute constants that depend at most on R. Note that $h_{T,R}$ is of size T^{8R} on the relevant support.

Remark 1.1.2. The polynomial $\prod\limits_{\sigma\in S_4}\Big(1+\alpha_{\sigma(1)}-\alpha_{\sigma(2)}-\alpha_{\sigma(3)}+\alpha_{\sigma(4)}\Big)^{\frac{R}{12}}$ is a new feature. It has not appeared in previous versions of this method for $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GL}(3,\mathbb{R})$, but we have included it because its inclusion improves the error terms.

Remark 1.1.3. For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>5/2$, the L-function associated with ϕ_j is given by

$$L(s,\phi_j)=\sum_{m=1}^{\infty}A_j(m,1,1)m^{-s}=\prod_p\left(1-\tfrac{A_j(p,1,1)}{p^s}+\tfrac{A_j(1,p,1)}{p^{2s}}-\tfrac{A_j(1,1,p)}{p^{3s}}+\tfrac{1}{p^{4s}}\right)^{-1}.$$

This shows that Theorem 1.1.1 gives the orthogonality relation on $\mathrm{GL}(4,\mathbb{R})$ for coefficients of cuspidal L-functions. It is possible, using the Hecke relations, to obtain a more general version of Theorem 1.1.1 involving A_L, A_M for arbitrary L, M, where $\prod\limits_{i=1}^{3}\ell_i m_i\neq 0$, but the formulas get quite complex and messy and so are omitted.

Remark 1.1.4. For a tempered Maass cusp form with Langlands parameters $\alpha\in(i\mathbb{R})^4$, note that

$$h_{T,R}(\alpha)>0$$

and is essentially supported on Laplace eigenvalues $\lambda<T^2$. It is not necessary to assume all Maass cusp forms for $\mathrm{SL}(4,\mathbb{Z})$ are tempered. A weaker version of Theorem 1.1.1 can be proved that assumes that almost all Maass cusp forms (except for a set of zero density) are tempered.

The proof of Conjecture 1.0.2 for $\mathrm{GL}(4,\mathbb{R})$ (with a strong power savings error term) has resisted all attempts up to now. Theorem 1.1.1 is the first orthogonality relation for $\mathrm{GL}(4,\mathbb{R})$ obtained that has a strong power savings error term. Many of the techniques used in the proof of the $\mathrm{GL}(3,\mathbb{R})$ conjecture do not generalize in an obvious way and new difficulties arise for the first time. We now point out the obstacles that we faced in the last seven years of work on this paper with some indications of how we overcame them.

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- *In the methods developed in [GK13], the Whittaker transform of a test function is estimated by first taking the Mellin transform of the Whittaker function and then taking the inverse Mellin transform to go back. This leads to multiple integrals involving ratios of Gamma functions that can be estimated by Stirling's asymptotic formula. When moving to* $\mathrm{GL}(4,\mathbb{R})$*, however, the Mellin transform of the Whittaker function is much more complex and does not satisfy a simple recurrence relation as on* $\mathrm{GL}(3,\mathbb{R})$*. The polynomials that appear in the recurrence formula in [FG93] are of large degree, and it did not seem possible to get good estimates for Mellin transforms of Whittaker functions via recurrence relations.*
- *Recent results (see Section 5.2) give precise control of the polynomials that appear in the recurrence formulae for Mellin transforms of shifted Whittaker functions, allowing us to overcome the problem discussed in the previous bullet.*
- *The classical Perron's formula allows one to obtain asymptotic formulae for the sum of coefficients of an L-function by computing a certain integral transform of that L-function and then evaluating the integral transform by shifting the contour of integration. An important tool in the proof of Theorem 1.1.1 is a novel higher-dimensional version of Perron's formula that gives asymptotic formulae for sums of terms arising in the cuspidal contribution to the trace formula. In the case of* $\mathrm{GL}(4,\mathbb{R})$*, the Perron type formula we develop involves a triple integral that requires shifting contours in three directions. It was necessary to generalize the method of Goldfeld-Kontorovich for finding the 'exponential zero set' that gets repeatedly used for each shifted term. We also introduce a very precise bound for elementary integrals (see Appendix A) that turns out to be critical for accurately estimating the integrals over the shifted contours.*
- *Another difficulty is that the Langlands spectral decomposition is much more complex on* $\mathrm{GL}(4,\mathbb{R})$ *with many more types of Langlands L-functions involving twists by Maass cusp forms of lower rank in the Levi components of the relevant parabolic subgroups. In order to obtain precise power savings error terms in the contribution of the continuous spectrum to the trace formula, it is necessary to have very explicit forms of the Fourier coefficients of the Eisenstein series. Although the Fourier coefficients are known in great generality (see, for example, Shahidi's book [Sha10]), the archimedean factors do not seem to have been worked out explicitly in the published literature. In Section 3.2, we review [GMW], where Borel Eisenstein series are used as a template to explicitly determine the non-constant Fourier coefficients of general Langlands Eisenstein series on* $\mathrm{GL}(4,\mathbb{R})$.

1.2. Roadmap for the proof of the Main Theorem

The proof of Theorem 1.1.1 is based on the Kuznetsov trace formula for $\mathrm{GL}(4,\mathbb{R})$ that is worked out in Section 3. The trace formula is the identity $\mathcal{C} = \mathcal{M} + \mathcal{K} - \mathcal{E}$, where

$$\mathcal{C} = \sum_{j=1}^{\infty} A_j(\ell,1,1)\,\overline{A_j(m,1,1)}\,\frac{h_{T,R}\left(\alpha^{(j)}\right)}{\mathcal{L}_j}$$

is the cuspidal contribution. The main term $\mathcal{M}$ is computed in Proposition 3.5.1 and is given by

$$\mathcal{M} = \delta_{L,M}\cdot\left(\mathfrak{c}_1 T^{9+8R} + \mathfrak{c}_2 T^{8+8R} + \mathfrak{c}_3 T^{7+8R} + \mathcal{O}\left(T^{6+8R}\right)\right).$$

The bound for the Kloosterman contribution $\mathcal{K}$ is worked in Proposition 4.0.4 (with $r = 4$), while the bound for the continuous spectrum $\mathcal{E}$ is given in Theorem 7.0.7. Combining these bounds with the main term $\mathcal{M}$ completes the proof. □

2. Whittaker functions, Maass cusp forms and Poincaré series for $\mathrm{SL}(4,\mathbb{Z})$

We review basic notation and the definitions of Whittaker functions, Maass cusp forms and Poincaré series following [Gol06].

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2.1. Iwasawa decomposition

Fix $n \geq 2$, and let $g \in \mathrm{GL}(n, \mathbb{R})$. We have the Iwasawa decomposition

$$g = utkr \tag{2.1.1}$$

where $u \in U_n(\mathbb{R})$ and $k \in K = \mathrm{O}(n, \mathbb{R})$ and $r \in \mathbb{R}^\times$ and $t \in T$, the subgroup of diagonal matrices with positive entries. Then $t = t(g)$ can be uniquely chosen to take the form

$$t = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & \ddots & & & \\ & & y_1 y_2 & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \tag{2.1.2}$$

for some $y = (y_1, y_2, \ldots, y_{n-1})$ with $y_i > 0$ $(i = 1, 2, \ldots, n-1)$.

2.2. Spectral and Langlands parameters

In the context of $\mathrm{SL}(n, \mathbb{R})$, we associate with a vector of complex numbers $s = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$, the spectral parameters

$$v := (v_1, v_2, \ldots, v_{n-1}), \qquad v_j := s_j - \frac{1}{n}, \qquad j = 1, \ldots, n-1,$$

and Langlands parameters

$$\alpha_i := \begin{cases} B_{n-1}(v) & \text{if } i = 1, \\ B_{n-i}(v) - B_{n-i+1}(v) & \text{if } 1 < i < n, \\ -B_1(v) & \text{if } i = n, \end{cases}$$

where

$$B_j(v) = \sum_{i=1}^{n-1} b_{i,j} v_i \qquad \text{with} \qquad b_{i,j} = \begin{cases} ij & \text{if } i + j \leq n, \\ (n-i)(n-j) & \text{if } i + j \geq n. \end{cases}$$

Note that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0. \tag{2.2.1}$$

On the other hand, given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ satisfying (2.2.1), it is straightforward to see that the Langlands parameters associated with

$$s_i = \frac{\alpha_i - \alpha_{i+1} + 1}{n} \tag{2.2.2}$$

are given by α.

To be completely explicit, in the special case of $\mathrm{SL}(4, \mathbb{Z})$, the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ associated with $s = \frac{1}{4} + (v_1, v_2, v_3)$ are given by

$$\alpha_1 = 3v_1 + 2v_2 + v_3, \ \alpha_2 = -v_1 + 2v_2 + v_3, \ \alpha_3 = -v_1 - 2v_2 + v_3, \ \alpha_4 = -v_1 - 2v_2 - 3v_3;$$

$$v_1 = \frac{\alpha_1 - \alpha_2}{4}, \qquad v_2 = \frac{\alpha_2 - \alpha_3}{4}, \qquad v_3 = \frac{\alpha_3 - \alpha_4}{4}.$$

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2.3. The I_s-function

Let $g \in \mathrm{GL}(n, \mathbb{R})$ (with toric element given by (2.1.2)). We define a power function in terms of either the spectral or Langlands parameters associated with $s = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$ via

$$I_s(g) = I_s(utkr) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j} s_j},$$

or, equivalently,

$$I_s(g) = I_s(utkr) := \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2} + \alpha_1 + \cdots + \alpha_{j-1}}.$$

It is easy to see that if s and α are associated with each other as in Section 2.2, then these two definitions are equivalent. For example, when $n = 4$ and $s = (1/4 + \nu_1, 1/4 + \nu_2, 1/4 + \nu_3) \in \mathbb{C}^3$, we have

$$\begin{aligned} I_s(g) &= y_1^{s_1+2s_2+3s_3} y_2^{2s_1+4s_2+2s_3} y_3^{3s_1+2s_2+s_3} \\ &= y_1^{\frac{3}{2}+\nu_1+2\nu_2+3\nu_3} y_2^{2+2\nu_1+4\nu_2+2\nu_3} y_3^{\frac{3}{2}+3\nu_1+2\nu_2+\nu_3} = y_1^{\frac{3}{2}+\alpha_1+\alpha_2+\alpha_3} y_2^{2+\alpha_1+\alpha_2} y_3^{\frac{3}{2}+\alpha_1}. \end{aligned}$$

2.4. Additive character of $U_n(\mathbb{R})$

Assume $n \geq 2$. Fix

$$M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}.$$

Let $g \in \mathrm{GL}(n, \mathbb{R})$ with Iwasawa decomposition $g = utkr$, where

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}.$$

Then associated with the vector M, we have an additive character $\psi_M : U_n(\mathbb{R}) \to \mathbb{C}$ defined by

$$\psi_M(g) := \psi_M(u) := e^{2\pi i \left(m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n}\right)}. \tag{2.4.1}$$

2.5. Jacquet's Whittaker function

Assume $n \geq 2$. Given Langlands parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, let $s = s(\alpha)$ be defined as in Section 2.2.

For $\mathrm{Re}(\nu_i) > 0$ $(i = 1, \ldots, n-1)$ and $w_{\text{long}} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, we define the completed Whittaker function $W_\alpha^{\pm} : \mathrm{GL}(n, \mathbb{R}) \big/ \left(\mathrm{O}(n, \mathbb{R}) \cdot \mathbb{R}^{\times}\right) \to \mathbb{C}$ by the absolutely convergent integral

$$W_\alpha^{\pm}(g) := \prod_{1 \leq j < k \leq n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \cdot \int\limits_{U_4(\mathbb{R})} I_s(w_{\text{long}} u g)\, \overline{\psi_{1,\ldots,1,\pm 1}(u)}\, du,$$

where du is the Haar measure on $U_n(\mathbb{R})$. The product of Gamma factors is added so that $W_\alpha^{\pm}$ is invariant under all permutations of the Langlands parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Remark 2.5.1. If g is a diagonal matrix in $\mathrm{GL}(n, \mathbb{R})$, then the value of $W_\alpha^{\pm}(g)$ is independent of sign, so we drop the $\pm$. We also drop the $\pm$ if the sign is $+1$.

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Let $\mathcal{D}^n$ denote the algebra of $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators on

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R}) \big/ \big(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^\times\big).$$

It is well known that $I_s(g)$ is an eigenfunction of all $\delta\in\mathcal{D}^n$. In particular, for the case of $\delta=\Delta$, the Laplacian, then $\Delta I_s = \lambda_\Delta(\alpha)\cdot I_s$, where

$$\lambda_\Delta(\alpha) = \left(\frac{n^3-n}{24} - \frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2}\right).$$

Define $\lambda_\delta(\alpha)\in\mathbb{C}$ by the eigenfunction equation

$$\delta I_s(g) = \lambda_\delta(\alpha)\cdot I_s(g), \qquad (\delta\in\mathcal{D}^n,\ g\in\mathrm{GL}(n,\mathbb{R})).$$

Jacquet's Whittaker function for $\mathrm{GL}(n,\mathbb{R})$ is characterized (up to scalars) by the following properties:

- $\delta W_\alpha^\pm(g) = \lambda_\delta(\alpha)\cdot W_\alpha^\pm(g), \qquad$ (for all $\delta\in\mathcal{D}^n$, $g\in\mathrm{GL}(n,\mathbb{R})$), (2.5.2)
- $W_\alpha^\pm(ug) = \psi_{1,\ldots,1,\pm1}(u)\cdot W_\alpha^\pm(g), \qquad$ (for all $u\in U_n(\mathbb{R})$, $g\in\mathrm{GL}(n,\mathbb{R})$),
- $W_\alpha^\pm(\mathrm{diag}(y_1y_2\cdots y_{n-1},\ \ldots\ ,y_1,1))$ has exponential decay as $y_i\to\infty$,
- $W_\alpha^\pm$ has holomorphic continuation to all $\alpha\in\mathbb{C}^n$, for all $g\in\mathrm{GL}(n,\mathbb{R})$,
- $W_\alpha^\pm = W_{\alpha'}^\pm$ where α' is any permutation of $\alpha=(\alpha_1,\ldots,\alpha_n)$.

2.6. Whittaker transform

Assume $n\geq 2$. Let

$$v = \frac{1}{n} + (v_1,v_2,\ldots,v_{n-1})\in\mathbb{C}^{n-1}$$

with the associated Langlands parameters $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$. Set

$$y := (y_1,y_2,\ldots y_{n-1}), \qquad t(y) := \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1y_2 & & \\ & & & y_1 & \\ & & & & 1\end{pmatrix}.$$

Let $f:\mathbb{R}_+^{n-1}\to\mathbb{C}$ be an integrable function. Then we define the Whittaker transform $f^\#:\mathbb{R}_+^n\to\mathbb{C}$ by

$$f^\#(\alpha) := \int_{y_1=0}^\infty\cdots\int_{y_{n-1}=0}^\infty f(y)\,W_\alpha\big(t(y)\big)\prod_{k=1}^{n-1}\frac{dy_k}{y_k^{k(n-k)+1}}, \tag{2.6.1}$$

provided the above integral converges absolutely and uniformly on compact subsets of $\mathbb{R}_+^{n-1}$. Assume that α is tempered: that is, $v_1,v_2,\ldots,v_{n-1}$ are all pure imaginary. The Whittaker transform was studied in [GK12], and the following explicit inverse Whittaker transform was obtained:

$$f(y) = \frac{1}{\pi^{n-1}}\int_{\mathrm{Re}(v_1)=0}\cdots\int_{\mathrm{Re}(v_{n-1})=0} f^\#(\alpha)W_{-\alpha}\big(t(y)\big)\frac{dv_1dv_2\cdots dv_{n-1}}{\prod\limits_{1\le k\ne\ell\le n}\Gamma\left(\frac{\alpha_k-\alpha_\ell}{2}\right)},$$

provided the above integral converges absolutely and uniformly on compact subsets of $(i\mathbb{R})^n$.

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2.7. The inner product of two Whittaker functions

Assume $n \geq 2$. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are Langlands parameters for which $\mathrm{Re}(\alpha_j) = \mathrm{Re}(\beta_k) = 0$ $(1 \leq j, k \leq n)$. Then

$$\int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} W_\alpha(y)\, \overline{W_\beta(y)} \prod_{j=1}^{n-1} y_j^{(n-j)(s-j)} \frac{dy_j}{y_j} = \frac{\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{n} \Gamma\left(\frac{s+\alpha_j-\beta_k}{2}\right)}{2\pi^{s\frac{n(n-1)}{2}}\,\Gamma\left(\frac{ns}{2}\right)}, \tag{2.7.1}$$

where the left side converges absolutely for $\mathrm{Re}(s)$ sufficiently large. This is given in [Sta02].

2.8. Fourier-Whittaker expansion of Maass cusp forms

Assume $n \geq 2$. Fix Langlands parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$. Let $\phi : \mathfrak{h}^n \to \mathbb{C}$ be a Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ that satisfies $\delta\phi(g) = \lambda_\alpha(\delta) \cdot \phi(g)$ for all $\delta \in \mathcal{D}^n$ and $g \in \mathrm{GL}(n, \mathbb{R})$, as in (2.5.2). Then for $g \in \mathrm{GL}(n, \mathbb{R})$, we have the following Fourier-Whittaker expansion:

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A_\phi(M)}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha^{\mathrm{sgn}(m_{n-1})}\left(t(M) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right), \tag{2.8.1}$$

where $M = (m_1, m_2, \ \ldots, \ m_{N-1})$ and $A_\phi(M)$ is the M^{th} Fourier coefficient of ϕ. This is proved in Section 9.1 of [Gol15].

2.9. First Fourier-Whittaker coefficient of a Maass cusp form

For $n \geq 2$, consider a Maass cusp form ϕ for $\mathrm{SL}(n, \mathbb{Z})$ with Fourier Whittaker expansion given by 2.8.1. Assume ϕ is a Hecke eigenform. Let $A_\phi(1) := A_\phi(1, 1, \ldots, 1)$ denote the first Fourier-Whittaker coefficient of ϕ. Then we have

$$A_\phi(M) = A_\phi(1) \cdot \lambda_\phi(M)$$

where $\lambda_\phi(M)$ is the Hecke eigenvalue (see Section 9.3 in [Gol15]), and $\lambda_\phi(1) = 1$.

Recall also the definition of the adjoint L-function:

$$L(s, \mathrm{Ad}\ \phi) := \frac{L(s, \phi \times \overline{\phi})}{\zeta(s)}$$

where $L(s, \phi \times \overline{\phi})$ is the Rankin-Selberg convolution L-function as in Section 12.1 of [Gol15].

Proposition 2.9.1. *Assume $n \geq 2$. Let ϕ be a Maass cusp form for* $\mathrm{SL}(n, \mathbb{Z})$ *with Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the first coefficient $A_\phi(1)$ is given by*

$$|A_\phi(1)|^2 = \frac{\mathfrak{c}_n \cdot \langle \phi, \phi \rangle}{L(1, \mathrm{Ad}\ \phi) \prod\limits_{1 \leq j \neq k \leq n} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}$$

where $\mathfrak{c}_n \neq 0$ is a constant depending on n only.

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Proof. We follow the Rankin-Selberg computations in Section 12.1 of [Gol15].

$$\langle \phi, \phi \rangle = \int\limits_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} |\phi(g)|^2\, d^*g$$

$$= \mathrm{vol}\,(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n) \cdot \underset{s=1}{\mathrm{Res}} \int\limits_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} \phi(g)\overline{\phi(g)}E(g,s)\, d^*g$$

where $E(g,s)$ is the maximal parabolic Eisenstein series. After unfolding and replacing ϕ with its Fourier-Whittaker expansion, we obtain

$$\int\limits_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} \phi(g)\overline{\phi(g)}E(g,s)\, d^*g = \frac{|A_\phi(1)|^2 L(s,\phi\times\overline{\phi})}{\zeta(ns)} \int\limits_{\mathbb{R}_+^{n-1}} W_\alpha(y)\overline{W_\alpha(y)} \prod_{j=1}^{n-1} \frac{dy_k}{y_k^{(k-s)(n-k)+1}}.$$

The proposition follows immediately from the formula (2.7.1), since

$$L(1, \mathrm{Ad}\ \phi) = \underset{s=1}{\mathrm{Res}}\ L(s, \phi\times\overline{\phi}).$$

□

2.10. *Vector or matrix notation depending on context*

Given a vector

$$a = (a_1, a_2, a_3) \in \mathbb{R}^3,$$

we shall define the toric element $t(a) := \mathrm{diag}(a_1a_2a_3, a_1a_2, a_1, 1)$.

Given a function $f : \mathbb{R}^3 \to \mathbb{C}$, we define $f(a) := f(a_1, a_2, a_3)$. On the other hand, if $f : \mathrm{GL}(4,\mathbb{R}) \to \mathbb{C}$ is a function defined on the group, then we let $f(a) := f(t(a))$, and more generally, for any $g_1, g_2 \in \mathrm{GL}(4,\mathbb{R})$, we define $f(g_1 a g_2) := f(g_1 t(a) g_2)$. In other words, we may consider a as a vector or a diagonal matrix depending on the context.

2.11. *Poincaré series for* SL$(4,\mathbb{Z})$

Let $H : \mathfrak{h}^4 \to \mathbb{C}$ be a smooth test function satisfying $H(utkr) = H(t)$ (see (2.1.1)). We assume that H has sufficient decay properties so that the series defining the Poincaré series (given below) converges absolutely. For $g \in \mathrm{GL}(4,\mathbb{R})$, $M = (m_1, m_2, m_3) \in \mathbb{Z}_+^3$, and $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, with $\mathrm{Re}(s_j)$ sufficiently large, the $\mathrm{SL}(4,\mathbb{Z})$ Poincaré series is defined by

$$P^M(g,s) := \sum_{\gamma\in U_4(\mathbb{Z})\backslash\,\mathrm{SL}(4,\mathbb{Z})} \psi_M(\gamma g)\, H\big(M\gamma g\big)\, I_s(\gamma g). \tag{2.11.1}$$

Remark: Following Section 2.10, for ψ_M, we take $M = (m_1, m_2, m_3)$. On the other hand for $H(M\gamma g)$, we take M to be the diagonal matrix $t(M)$.

2.12. *Inner product of the Poincaré series with a Maass cusp form*

Let ϕ be a Maass cusp form for $\mathrm{SL}(4,\mathbb{Z})$ with Fourier expansion (2.8.1). Let P^M denote the Poincaré series (2.11.1). The inner product is defined by

$$\left\langle P^M(*,s),\ \phi\right\rangle := \int\limits_{\mathrm{SL}(4,\mathbb{Z})\backslash\mathfrak{h}^4} P^M(g,s)\,\overline{\phi(g)}\, dg.$$

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It follows that

$$\left\langle P^M(*,s),\ \phi\right\rangle = \int\limits_{U_4(\mathbb{Z})\backslash\mathfrak{h}^4} \psi_M(x)H\big(My\big)I_s(y)\overline{\phi(xy)}\left(\prod_{1\le j<k\le 4} dx_{j,k}\right)\frac{dy_1dy_2dy_3}{y_1^4y_2^5y_3^4}$$

$$= \frac{\overline{A_\phi(M)}}{m_1^{\frac{3}{2}}m_2^2m_3^{\frac{3}{2}}}\int\limits_{y_1=0}^{\infty}\int\limits_{y_2=0}^{\infty}\int\limits_{y_3=0}^{\infty} H\big(My\big)\,I_s(y)\cdot\overline{W_\alpha(My)}\,\frac{dy_1dy_2dy_3}{y_1^4y_2^5y_3^4}.$$

We see that the above inner product picks out the M^{th} Fourier coefficient of ϕ multiplied by a certain Whittaker transform of $H\big(My\big)\cdot I_s(y)$. Letting $s\to 0$, it follows from (2.6.1) that

$$\lim_{s\to 0}\left\langle P^M(*,s),\ \phi\right\rangle = \frac{\overline{A_\phi(M)}}{m_1^{\frac{3}{2}}m_2^2m_3^{\frac{3}{2}}}\int\limits_{y_1=0}^{\infty}\int\limits_{y_2=0}^{\infty}\int\limits_{y_3=0}^{\infty} H\big(My\big)\cdot\overline{W_\alpha\big(My\big)}\,\frac{dy_1dy_2dy_3}{y_1^4y_2^5y_3^4}$$

$$= m_1^{\frac{3}{2}}m_2^2m_3^{\frac{3}{2}}\cdot\overline{A_\phi(M)}\cdot H^{\#}(\overline{\alpha}). \tag{2.12.1}$$

2.13. Fourier-Whittaker expansion of the Poincaré series

Let $W_4\cong S_4$ denote the Weyl group of $\mathrm{GL}(4,\mathbb{R})$. For $w\in W_4$, we define

$$\Gamma_w := \left(w^{-1}\cdot {}^tU_4(\mathbb{Z})\cdot w\right)\cap U_4(\mathbb{Z}),$$

where ${}^tU_4$ denotes the transpose of U_4.

We have the Bruhat decomposition

$$\mathrm{GL}(4,\mathbb{R}) = \bigcup_{w\in W_4} G_w, \qquad \Big(G_w = U_4(\mathbb{R})\cdot w\cdot T_4(\mathbb{R})U_4(\mathbb{R})\Big),$$

where $T_4(\mathbb{R})$ is the subgroup of diagonal matrices in $\mathrm{GL}(4,\mathbb{R})$.

Definition 2.13.1 (Twisted Character). Let

$$V_4 := \left\{ v = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & v_3 & \\ & & & v_4\end{pmatrix} \,\middle|\, v_1,v_2,v_3,v_4\in\{\pm1\},\ v_1v_2v_3v_4=1\right\}.$$

Let $M=(m_1,m_2,m_3)\in\mathbb{Z}^3$, and consider ψ_M an additive character of U_4. Then for $v\in V_4$, we define the twisted character $\psi_M^v : U_4(\mathbb{R})\to\mathbb{C}$ by

$$\psi_M^v(g) := \psi_M\left(v^{-1}gv\right).$$

Definition 2.13.2 (Kloosterman Sum). Fix

$$L=(\ell_1,\ell_2,\ell_3),\ M=(m_1,m_2,m_3)\in\mathbb{Z}^3.$$

Let ψ_L,ψ_M be characters of $U_4(\mathbb{R})$. Let $w\in W_4$, where W_4 is the Weyl group of $\mathrm{GL}(4)$. Let $c=\begin{pmatrix} 1/c_3 & & & \\ & c_3/c_2 & & \\ & & c_2/c_1 & \\ & & & c_1\end{pmatrix}$ with $c_i\in\mathbb{N}$. Then the Kloosterman sum is defined as

$$S_w(\psi_L,\psi_M,c) := \sum_{\substack{\gamma\in U_4(\mathbb{Z})\backslash\Gamma\cap G_w/\Gamma_w\\ \gamma=\beta_1cw\beta_2}} \psi_L(\beta_1)\,\psi_M(\beta_2),$$

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with notation as in Definition 11.2.2 of [Gol06]. The Kloosterman sum $S_w(\psi, \psi', c)$ is well defined and not equal to zero if and only if it satisfies the compatibility condition $\psi(cwuw^{-1}) = \psi'(u)$.

It follows from Theorem 11.5.4 of [Gol06] that the M^{th} Fourier coefficient of the Poincaré series $P^L(g, s)$ is given by

$$\int\limits_{U_4(\mathbb{Z})\backslash U_4(\mathbb{R})} P^L(ug, s) \cdot \overline{\psi_M(u)}\, d^*u$$
$$= \sum_{w \in W_4} \sum_{v \in V_4} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{\infty} \sum_{c_3=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c) J_w(g; s, \psi_L, \psi_M^v, c)}{c_1^{4s_3} c_2^{4s_2} c_3^{4s_1}}, \tag{2.13.3}$$

where

$$J_w(g; s, \psi_L, \psi_M^v, c) = \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\overline{U}_w(\mathbb{R})} \psi_L(wug)\, H\big(Lcwug\big)\, I_s(wug)\, \overline{\psi_M^v(u)}\, d^*u,$$

$$U_w(\mathbb{R}) = \Big(w^{-1} \cdot U_4(\mathbb{R}) \cdot w\Big) \cap U_4(\mathbb{R}), \qquad \overline{U}_w(\mathbb{R}) = \Big(w^{-1} \cdot {}^tU_4(\mathbb{R}) \cdot w\Big) \cap U_4(\mathbb{R}),$$

and ${}^tm$ denotes the transpose of a matrix m.

3. Kuznetsov trace formula for SL$(4, \mathbb{Z})$

3.1. Choice of test function

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (i\mathbb{R})^4$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Let $T > 1$ with $T \to \infty$ and $R \geq 14$ with R fixed. We consider the test function

$$p^\sharp_{T,R}(\alpha) \;:=\; e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}}\, \mathcal{F}_R(\alpha) \prod_{1\leq j\neq k\leq 4} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right), \tag{3.1.1}$$

where

$$\mathcal{F}_R(\alpha) := \left(\prod_{\sigma \in S_4} \Big(1 + \alpha_{\sigma(1)} - \alpha_{\sigma(2)} - \alpha_{\sigma(3)} + \alpha_{\sigma(4)}\Big)\right)^{\frac{R}{24}}. \tag{3.1.2}$$

The function $p^\sharp_{T,R}(\alpha)$ defined in (3.1.1) generalizes the similar function defined in [GK13]. As before, the choice is motivated by the fact that we need $p^\sharp_{T,R}$ to be invariant under the Weyl group, and have meromorphic continuation in $\alpha \in \mathbb{C}^4$, while also requiring it to have enough exponential decay to kill the exponential growth of certain Gamma factors appearing in the denominator of the Kuznetsov trace formula. The new feature is the introduction of the polynomial $\mathcal{F}_R(\alpha)$.

By the inverse Lebedev-Whittaker transform (see [GK12]), we see that $p_{T,R}$ is given by

$$p_{T,R}(y) = p_{T,R}(y_1, y_2, y_3) = \frac{1}{\pi^3} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} p^\#_{T,R}(\alpha)\, W_\alpha(y) \frac{d\alpha_1\, d\alpha_2\, d\alpha_3}{\prod\limits_{1\leq j\neq k\leq 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}. \tag{3.1.3}$$

3.2. Setting up the trace formula

Set $L = (\ell_1, \ell_2, \ell_3)$, $M = (m_1, m_2, m_3) \in \mathbb{Z}^3$, where we assume $\prod_{i=1}^{3} \ell_i m_i \neq 0$. Consider the Poincaré series P^L, P^M, as defined in (2.11.1) with the test function $H = p_{T,R}$.

Definition 3.2.1 (Normalization factor $C_{L,M}$). Let $\mathfrak{c}_4$ be as Proposition 2.9.1. We define $C_{L,M} := \mathfrak{c}_4 \cdot (\ell_1 m_1)^3 (\ell_2 m_2)^4 (\ell_3 m_3)^3$.

With the normalization factor $C_{L,M}$ defined above, the Kuznetsov trace formula is obtained by evaluating the inner product

$$C_{L,M}^{-1} \cdot \lim_{s \to 0} \left\langle P^L(*, s),\ P^M(*, s) \right\rangle \ = \ C_{L,M}^{-1} \cdot \lim_{s \to 0} \int\limits_{\mathrm{SL}(4,\mathbb{Z})\backslash \mathfrak{h}^4} P^L(g, s)\, \overline{P^M(g, s)}\, dg$$

in two different ways. The first approach is to use spectral theory while the second uses geometry. For the spectral theory approach, we will need the following definition.

Definition 3.2.2 (Generic and non-generic automorphic forms). Let $n \geq 2$. An automorphic form for $SL(n, \mathbb{Z})$ is said to be *generic* if there exists some nondegenerate character of $U_n(\mathbb{R})$ for which the form has a nonzero Fourier coefficient. Otherwise, the automorphic form is said to be *non-generic*. Note that a character $\psi_{m_1, m_2, \ldots, m_{n-1}}$ of $U_n(\mathbb{R})$ is nondegenerate if all $m_1, m_2, \ldots, m_{n-1}$ are all nonzero.

In order to prove the Kuznetsov trace formula for $\mathrm{GL}(4, \mathbb{R})$, we require the spectral expansion of the Poincaré series P^M and P^L. It will be shown (see the proof of Theorem 3.7.3) that these Poincaré series are orthogonal to the non-generic spectrum. Therefore, the following theorem suffices for our needs.

Theorem 3.2.3 (Langlands spectral decomposition for $\mathrm{SL}(4, \mathbb{Z})$). *Assume that $\phi_1, \phi_2, \phi_3, \ldots$ is an orthogonal basis of Maass cusp forms for $\mathrm{SL}(4, \mathbb{Z})$. Assume that $F, G \in \mathcal{L}^2(\mathrm{SL}(4, \mathbb{Z})\backslash \mathfrak{h}^4)$ are orthogonal to the non-generic spectrum. Then for $g \in \mathrm{GL}(4, \mathbb{R})$, we have*

$$F(g) \ = \ \sum_{j=1}^{\infty} \langle F, \phi_j \rangle \frac{\phi_j(g)}{\langle \phi_j, \phi_j \rangle} + \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \int\limits_{\mathrm{Re}(s_1)=0} \cdots \int\limits_{\mathrm{Re}(s_{r-1})=0} \Big\langle F, E_{\mathcal{P},\Phi}(*\,, s) \Big\rangle E_{\mathcal{P},\Phi}(g\,, s) \;\; ds_1 \cdots ds_{r-1};$$

$$\begin{aligned} \langle F, G \rangle = & \sum_{j=1}^{\infty} \frac{\langle F, \phi_j \rangle \, \langle \phi_j, G \rangle}{\langle \phi_j, \phi_j \rangle} \\ & + \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \int\limits_{\mathrm{Re}(s_1)=0} \cdots \int\limits_{\mathrm{Re}(s_{r-1})=0} \Big\langle F,\ E_{\mathcal{P},\Phi}(*\,, s) \Big\rangle \Big\langle E_{\mathcal{P},\Phi}(*\,, s),\ G \Big\rangle \, ds_1 \cdots ds_{r-1}; \end{aligned}$$

where the sum over $\mathcal{P}$ ranges over proper parabolic subgroups associated with partitions $4 = \sum_{k=1}^{r} n_k$ ($r > 1$), and the sum over Φ (see Definition 3.7.1) ranges over an orthonormal basis of Maass cusp forms associated with $\mathcal{P}$. Here $s = (s_1, \ldots, s_r)$, where $\sum_{k=1}^{r} n_k s_k = 0$ for the partition $4 = \sum_{k=1}^{r} n_k$. Furthermore, $c_{\mathcal{P}}$ is a fixed non-zero constant for each parabolic subgroup $\mathcal{P}$.

Proof. For proofs of the spectral expansion for arbitrary reductive groups, see [Art79], [Lan76], and [MW95].

It has long been known that the residual spectrum of $\mathrm{GL}(n, \mathbb{R})$ is non-generic (see, for example, [LM18]). According to the results of Kostant, Casselman-Zuckerman and Vogan (mentioned at the end of Vogan's paper [Vog78]), generic representations have Gelfand-Kirillov dimension as large as possible. It follows that Eisenstein series induced from the residual spectrum can never be generic. Therefore, the spectral expansion here only involves terms coming from cusp forms, as claimed. □

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In particular since $P^L, P^M \in \mathcal{L}^2\left(\mathrm{SL}(4,\mathbb{Z})\backslash\mathfrak{h}^4\right)$, the inner product can be computed with the spectral expansion of the Poincaré series. The geometric approach utilizes the Fourier Whittaker expansion of the Poincaré series that involve Kloosterman sums.

The trace formula takes the following form.

$$\boxed{\underbrace{\mathcal{C} + \mathcal{E}}_{\text{spectral side}} = \underbrace{\mathcal{M} + \mathcal{K}}_{\text{geometric side}}.} \tag{3.2.4}$$

Here $\mathcal{C}$ is the cuspidal contribution, and $\mathcal{M}$ is the main term coming from the identity element. Further, $\mathcal{E}$ = Eisenstein contribution, and $\mathcal{K}$ = Kloosterman sum contribution. These will be small with the special choice of the test function $p_{T,R}$. In the subsections that follow, we explicitly evaluate $\mathcal{C}, \mathcal{E}, \mathcal{M}$ and $\mathcal{K}$.

3.3. *Cuspidal contribution $\mathcal{C}$ to the Kuznetsov trace formula*

Proposition 3.3.1 (Cuspidal contribution to the trace formula). *Suppose that $L = (\ell_1, \ell_2, \ell_3)$ and $M = (m_1, m_2, m_3)$, where $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$ are fixed non-zero rational integers. Let $\phi_1, \phi_2, \ldots$ denote an orthogonal basis of Maass cusp forms for* $\mathrm{SL}(4,\mathbb{Z})$ *with spectral parameters $\alpha^{(1)}, \alpha^{(2)}, \ldots$, respectively, ordered by Laplace eigenvalue. Let $A_j(L)$ and $A_j(M)$ denote the L^{th} and M^{th} Fourier coefficients of ϕ_j, and assume that $A_j(1,1,1) = 1$ for all $j = 1, 2, \ldots$. Let $\mathcal{L}_j = L(1, \mathrm{Ad}\ \phi_j)$. Then the cuspidal contribution to the trace formula (3.2.4) is given by*

$$\mathcal{C} = \sum_{j=1}^{\infty} \frac{A_j(L)\overline{A_j(M)} \cdot \left| p^{\#}_{T,R}\left(\overline{\alpha^{(j)}}\right)\right|^2}{\mathcal{L}_j \prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}$$

where $p^{\#}_{T,R}$ is given by (3.1.1).

Proof. It follows from Theorem 3.2.3 that for $g \in \mathrm{GL}(4,\mathbb{R})$, the cuspidal contribution to the trace formula (3.2.4) is given by

$$\sum_{j=1}^{\infty} \left\langle P^L(*,0),\ \phi_j\right\rangle \cdot \frac{\phi_j(g)}{\langle \phi_j, \phi_j\rangle}.$$

Now, since $A_j(1,1,1) = 1$, Proposition 2.9.1 implies that

$$\left\langle \phi_j, \phi_j\right\rangle = \mathfrak{c}_4^{-1} \cdot \mathcal{L}_j \prod_{1\le j\ne k\le 4} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right).$$

The cuspidal contribution to the trace formula is given by

$$\mathcal{C} \ := \ C_{L,M}^{-1} \cdot \sum_{j=1}^{\infty} \frac{\left\langle P^L(*,0),\ \phi_j\right\rangle \cdot \overline{\left\langle P^M(*,0),\ \phi_j\right\rangle}}{\langle \phi_j, \phi_j\rangle}.$$

The proposition immediately follows from the inner product formula (2.12.1). □

3.4. *Geometric side of the Kuznetsov trace formula*

Next we consider the geometric side of the trace formula (3.2.4). This is computed with the Fourier-Whittaker expansion of the Poincaré series given in (2.13.3).

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Proposition 3.4.1 (Geometric side of the trace formula). *Fix $L = (\ell_1, \ell_2, \ell_3)$ and $M = (m_1, m_2, m_3)$ with $C_{L,M} \neq 0$. Then*

$$\lim_{s\to 0} \left\langle P^L(*, s),\ P^M(*, s)\right\rangle = \sum_{w\in W_4} \mathcal{I}_w,$$

where we have

$$\mathcal{I}_w := \sum_{v\in V_4}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\sum_{c_3=1}^{\infty} S_w(\psi_L, \psi_M^v, c) \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty}\ \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int_{\overline{U}_w(\mathbb{R})} \cdot \psi_L(wuy)\overline{\psi_M^v(u)}\, p_{T,R}(Lcwuy)\,\overline{p_{T,R}(My)}\, d^*u\, \frac{dy_1\, dy_2\, dy_3}{y_1^4\, y_2^5\, y_3^4}. \tag{3.4.2}$$

Proof. We compute the inner product

$$\lim_{s\to 0} \left\langle P^L(*, s),\ P^M(*, s)\right\rangle = \lim_{s\to 0} \int_{\mathrm{SL}(4,\mathbb{Z})\backslash\mathfrak{h}^4} P^L(g, s)\cdot \overline{P^M(g, s)}\, dg$$

$$= \lim_{s\to 0} \int_{U_4(\mathbb{Z})\backslash\mathfrak{h}^4} P^L(g, s)\cdot \overline{\psi_M(g)\, p_{T,R}(Mg)\, I_s(g)}\, dg$$

$$= \lim_{s\to 0} \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty} \left(\int_{U_4(\mathbb{Z})\backslash U_4(\mathbb{R})} P^L(uy, s)\cdot \overline{\psi_M(u)}\, du\right)\cdot \overline{p_{T,R}(My)\, I_s(y)}\ \frac{dy_1\, dy_2\, dy_3}{y_1^4\, y_2^5\, y_3^4}.$$

It follows from (2.13.3) that

$$\lim_{s\to 0} \left\langle P^L(*, s),\ P^M(*, s)\right\rangle$$

$$= \lim_{s\to 0} \sum_{w\in W_4}\sum_{v\in V_4}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\sum_{c_3=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c)}{c_1^{4s_1} c_2^{4s_2} c_3^{4s_3}} \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty}\ \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int_{\overline{U}_w(\mathbb{R})} \cdot \psi_L(wuy)\overline{\psi_M^v(u)}\, p_{T,R}(Lcwuy)\,\overline{p_{T,R}(My)}\, I_s(wuy)\,\overline{I_s(y)}\, d^*u\, \frac{dy_1\, dy_2\, dy_3}{y_1^4\, y_2^5\, y_3^4}$$

$$= \sum_{v\in V_4}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\sum_{c_3=1}^{\infty} S_w(\psi_L, \psi_M^v, c) \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty}\ \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int_{\overline{U}_w(\mathbb{R})} \cdot \psi_L(wuy)\overline{\psi_M^v(u)}\, p_{T,R}(Lcwuy)\,\overline{p_{T,R}(My)}\, du\, \frac{dy_1\, dy_2\, dy_3}{y_1^4\, y_2^5\, y_3^4},$$

which equals $\sum\limits_{w\in W_4} \mathcal{I}_w$, as claimed. □

3.5. *Main term $\mathcal{M}$ in the Kuznetsov trace formula*

Let w_1 denote the 4×4 identity matrix. The main term $\mathcal{M} = I_{w_1}$ in the trace formula (3.2.4) can now be easily computed.

Proposition 3.5.1 (Main term in the trace formula). *There exist fixed constants $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3 > 0$ (depending only on R) such that the main term $\mathcal{M}$ in the trace formula (3.2.4) is given by*

$$\mathcal{M} = \delta_{L,M}\cdot\left(\mathfrak{c}_1 T^{9+8R} + \mathfrak{c}_2 T^{8+8R} + \mathfrak{c}_3 T^{7+8R} + \mathcal{O}\left(T^{6+8R}\right)\right).$$

Proof. The Kloosterman sum in Definition 2.13.2 for the special case of the trivial Weyl group element w_1 is identically zero unless $c = (1, 1, 1)$, in which case $S_{w_1}\left(\psi_M, \psi_L^v, (1, 1, 1)\right) = 1$. It follows from

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(3.4.2) and the normalization (by $C_{L,M}$) of the cuspidal contribution $\mathcal{C}$ that

$$\mathcal{M} = C_{L,M}^{-1} \cdot \mathcal{I}_{w_1}$$
$$= C_{L,M}^{-1} \cdot \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty} \left(\int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \psi_L(u)\overline{\psi_M(u)}\, d^*u \right) p_{T,R}(Ly)\, \overline{p_{T,R}(My)}\, \frac{dy_1\, dy_2\, dy_3}{y_1^4 y_2^5 y_3^4}.$$

Next

$$\mathcal{M} = \delta_{L,M} \cdot \mathfrak{c}_4 \int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty}\int_{y_3=0}^{\infty} |p_{T,R}(y)|^2 \frac{dy_1\, dy_2\, dy_3}{y_1^4 y_2^5 y_3^4} = \langle p_{T,R}, p_{T,R}\rangle$$
$$= \delta_{L,M} \cdot \mathfrak{c}_4 \iiint_{\mathrm{Re}(\alpha_j)=0} \frac{\left|p^{\#}_{T,R}(\alpha)\right|^2 d\alpha_1\, d\alpha_2\, d\alpha_3}{\prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}$$
$$= \delta_{L,M}\, \mathfrak{c}_4 \cdot \langle p^{\#}_{T,R},\ p^{\#}_{T,R}\rangle,$$

where the second representation of $\mathcal{M}$ in terms of the norm of $p^{\#}_{T,R}$ follows from the Plancherel formula in Corollary 1.9 of [GK12].

It now follows from (3.1.1) that

$$\mathcal{M} = \delta_{L,M} \cdot \mathfrak{c}_4 \iiint_{\mathrm{Re}(\alpha_j)=0} \frac{\left| e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}} \mathcal{F}_R(\alpha) \prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)\right|^2}{\prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\, d\alpha_1 d\alpha_2 d\alpha_3.$$

Let $\alpha_j = i\tau_j$ with $\tau_j \in \mathbb{R}$, where $\tau_4 = -\tau_1 - \tau_2 - \tau_3$. Hence, from Stirling's asymptotic formula $|\Gamma(\sigma+it)|^2 \sim 2\pi |t|^{2\sigma-1} e^{-\pi|t|}$, it follows that

$$\mathcal{M} \sim \delta_{L,M} \cdot \mathfrak{c}_4 \cdot \iiint_{\mathbb{R}^3} e^{\frac{-\tau_1^2-\tau_2^2-\tau_3^2-\tau_4^2}{2T^2}} \Big((1+|\tau_1+\tau_2-\tau_3-\tau_4|^2)(1+|\tau_1-\tau_2+\tau_3-\tau_4|^2)$$
$$\cdot (1+|\tau_1-\tau_2-\tau_3+\tau_4|^2)\Big)^{\frac{R}{3}} \Big((1+|\tau_1-\tau_2|)(1+|\tau_1-\tau_3|)(1+|\tau_2-\tau_3|)$$
$$\cdot (1+|2\tau_1+\tau_2+\tau_3|)(1+|\tau_1+2\tau_2+\tau_3|)(1+|\tau_1+\tau_2+2\tau_3|)\Big)^{1+R}\, d\tau_1 d\tau_2 d\tau_3.$$

Next, make the change of variables

$$\tau_1 \longrightarrow \tau_1 T, \quad \tau_2 \longrightarrow \tau_2 T, \quad \tau_3 \longrightarrow \tau_3 T.$$

It follows that as $T \longrightarrow \infty$, we have $\mathcal{M} \sim \mathfrak{c}_1 \cdot \delta_{L,M}\, T^{9+8R}$, where

$$\mathfrak{c}_1 = \mathfrak{c}_4 \iiint_{\mathbb{R}^3} e^{\frac{-\tau_1^2-\tau_2^2-\tau_3^2-\tau_4^2}{2}} \Big((1+|\tau_1+\tau_2-\tau_3-\tau_4|^2)(1+|\tau_1-\tau_2+\tau_3-\tau_4|^2)$$
$$(1+|\tau_1-\tau_2-\tau_3+\tau_4|^2)\Big)^{\frac{R}{3}} \cdot \Big((1+|\tau_1-\tau_2|)(1+|\tau_1-\tau_3|)(1+|\tau_2-\tau_3|)$$
$$\cdot (1+|2\tau_1+\tau_2+\tau_3|)(1+|\tau_1+2\tau_2+\tau_3|)(1+|\tau_1+\tau_2+2\tau_3|)\Big)^{1+R}\, d\tau_1 d\tau_2 d\tau_3.$$

This method of proof can be extended by using additional terms in Stirling's asymptotic expansion for the Gamma function to obtain additional terms in the asymptotic expansion of $\mathcal{M}$. □

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3.6. *Kloosterman term $\mathcal{K}$ in the Kuznetsov trace formula*

It immediately follows from Proposition 3.4.1 that

$$\mathcal{K} = C_{L,M}^{-1} \cdot \sum_{\substack{w \in W_4 \\ w \neq w_1}} \mathcal{J}_w.$$

3.7. *Eisenstein contribution $\mathcal{E}$ to the Kuznetsov trace formula*

This section is based on [GMW]. There are four standard non-associate parabolic subgroups on GL(4) corresponding to the partitions

$$4 \;=\; 3+1 \;=\; 2+2 \;=\; 2+1+1 \;=\; 1+1+1+1.$$

Consider the minimal parabolic subgroup

$$\mathcal{P}_{\text{Min}} := \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \subset \text{GL}(4,\mathbb{R}) \right\}$$

associated with the partition 4 = 1+1+1+1. Let $s = \frac{1}{4} + (s_1, s_2, s_3)$ with $s \in \mathbb{C}^3$. The minimal parabolic Eisenstein series for $\Gamma = \text{SL}(4,\mathbb{Z})$ is defined by

$$E_{\mathcal{P}_{\text{Min}}}(g,s) := \sum_{\gamma \in (\mathcal{P}_{\text{Min}} \cap \Gamma)\backslash\Gamma} I_s(\gamma g), \qquad (g \in \text{GL}(4,\mathbb{R}),\;\; \text{Re}(s) \gg 1).$$

For the other three partitions $4 = 3+1 = 2+2 = 2+1+1$, consider the parabolic subgroups

$$\mathcal{P}_{3,1} := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & * \end{pmatrix} \right\}, \quad \mathcal{P}_{2,2} := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\}, \quad \mathcal{P}_{2,1,1} := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\},$$

respectively. For each of these parabolic subgroups (denoted $\mathcal{P}$), associated with a partition with $4 = n_1 + n_2 + \cdots + n_r$, we may define an infinite family of Eisenstein series $E_{\mathcal{P},\Phi}$ (as in [GMW]) given by

$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma \in (\mathcal{P}\cap\Gamma)\backslash\Gamma} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s}, \qquad (g \in \text{GL}(4,\mathbb{R}),\;\; \text{Re}(s) \gg 1).$$

Here Φ runs over cusp forms on the Levi components $m = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_r \end{pmatrix}$ of g (with $m_i \in \text{GL}(n_i,\mathbb{R})$) and $|g|_{\mathcal{P}}^{s} = \prod_{j=1}^{r} |\det(m_j)|^{s_j}$ is a toric character as in [GMW].

Definition 3.7.1. The cusp forms Φ and complex s values associated with Eisenstein series $E_{\mathcal{P},\Phi}(g,s)$ for $\text{SL}(4,\mathbb{Z})$ are given as follows.

- Let $\mathcal{P} = \mathcal{P}_{3,1}$; then Φ runs over Maass cusp forms for $\text{SL}(3,\mathbb{Z})$, and $s = (s_1, s_2)$ with $3s_1 + s_2 = 0$.
- Let $\mathcal{P} = \mathcal{P}_{2,2}$; then Φ runs over pairs $\Phi = (\phi_1, \phi_2)$, where ϕ_1, ϕ_2 are Maass cusp forms for $\text{SL}(2,\mathbb{Z})$, and $s = (s_1, s_2)$ with $2s_1 + 2s_2 = 0$.
- Let $\mathcal{P} = \mathcal{P}_{2,1,1}$; then Φ runs over Maass cusp forms for $\text{SL}(2,\mathbb{Z})$, and $s = (s_1, s_2, s_3)$ with $2s_1 + s_2 + s_3 = 0$.

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The Eisenstein series $E_{\mathcal{P},\Phi}$ has a Fourier-Whittaker expansion similar to (2.8.1). If $M = (m_1, m_2, m_3)$ with m_1, m_2 and m_3 positive integers, then the M^{th} Fourier-Whittaker coefficient of $E_{\mathcal{P},\Phi}(*, s)$ is given by

$$\int\limits_{U_4(\mathbb{Z})\backslash U_4(\mathbb{R})} E_{\mathcal{P}}(uy, s)\,\overline{\psi_M(u)}\, du \;=\; \frac{A_{E_{\mathcal{P},\Phi}}(M, s)}{|m_1|^{\frac{3}{2}}|m_2|^2|m_3|^{\frac{3}{2}}}\; W_\alpha(My),$$

where α denotes the Langlands parameter of $E_{\mathcal{P},\Phi}(g, s)$. Here

$$A_{E_{\mathcal{P},\Phi}}(M, s) = A_{E_{\mathcal{P},\Phi}}\big((1, 1, 1), s\big) \cdot \lambda_{E_{\mathcal{P},\Phi}}(M, s)$$

and $\lambda_{E_{\mathcal{P},\Phi}}(M, s)$ is the M^{th} Hecke eigenvalue of $E_{\mathcal{P},\Phi}$.

Proposition 3.7.2 (Inner product of the Poincaré series P^M with $E_{\mathcal{P},\Phi}$). *Consider an Eisenstein series $E_{\mathcal{P},\Phi}$ for* $\mathrm{SL}(4, \mathbb{Z})$. *Fix $M = (m_1, m_2, m_3) \in \mathbb{Z}_+^3$. Let P^M be the Poincaré series defined in* (2.11.1) *with test function $p_{T,R} : \mathfrak{h}^4 \to \mathbb{C}$ (as in* (3.1.1)*). Then*

$$\lim_{\delta\to 0} \Big\langle P^M(*, \delta),\; E_{\mathcal{P},\Phi}(*,\; s)\Big\rangle = m_1^{\frac{3}{2}} m_2^2 m_3^{\frac{3}{2}} \cdot \overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot p^{\#}_{T,R}(\overline{\alpha}).$$

Proof. The proof is similar to the proof of (2.12.1). One begins by taking $\mathrm{Re}(s)$ very large so that there is no problem with convergence and the result follows by analytic continuations in s. □

Theorem 3.7.3 (Spectral decomposition for the inner product of Poincaré series). *Fix $L = (\ell_1, \ell_2, \ell_3)$, $M = (m_1, m_2, m_3) \in \mathbb{Z}^3$.*

Let $\phi_1, \phi_2, \phi_3, \ldots$ denote a basis of Maass cusp forms for $\mathrm{SL}(4, \mathbb{Z})$ *with spectral parameters $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$, …, respectively, ordered by Laplace eigenvalue and normalized so that the first Fourier coefficient $A_j(1, 1, 1) = 1$ for all $j = 1, 2, \ldots$. Set $\mathcal{L}_j = L(1, \mathrm{Ad}\,\phi_j)$.*

Let $\mathcal{P}$ range over parabolics associated with partitions $4 = n_1 + \cdots + n_r$, and Φ range over an orthonormal basis of Maass cusp forms associated with $\mathcal{P}$. Let $E_{\mathcal{P},\Phi}(s)$ denote the Langlands Eisenstein series for $\mathrm{SL}(4, \mathbb{Z})$ *with Langlands parameter $\alpha_{\mathcal{P},\Phi}(s)$ and L^{th}, M^{th} Fourier coefficients $A_{E_{\mathcal{P},\Phi}}(L, s)$, $A_{E_{\mathcal{P},\Phi}}(M, s)$, respectively. Then*

$$C^{-1}_{L,M} \cdot \lim_{\delta\to 0} \Big\langle P^L(*, \delta),\; P^M(*, \delta)\Big\rangle = \sum_{j=1}^{\infty} \frac{A_j(L)\overline{A_j(M)} \cdot \Big|p^{\#}_{T,R}\left(\alpha^{(j)}\right)\Big|^2}{\mathcal{L}_j \prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}$$
$$+ \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int\limits_{\mathrm{Re}(s_1)=0} \cdots \int\limits_{\mathrm{Re}(s_{r-1})=0} A_{E_{\mathcal{P},\Phi}}(L, s)\,\overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot \Big|p^{\#}_{T,R}\big(\alpha_{(\mathcal{P},\Phi)}(s)\big)\Big|^2\, ds_1 \cdots ds_{r-1},$$

for constants $c_{\mathcal{P}} > 0$.

Proof. This follows immediately from Proposition 3.3.1, Theorem 3.2.3, Proposition 3.7.2, and the rapid decay of the $p_{T,R}$ function. Note that the Poincaré Series P^L, P^M are orthogonal to the non-generic spectrum. This can be seen as follows:

$$\int\limits_{\mathrm{SL}(4,\mathbb{Z})\backslash\mathfrak{h}^4} P^M(g)\overline{\phi(g)}\, dg \;=\; \int\limits_{U_4(\mathbb{Z})\backslash\mathfrak{h}^4} \psi(g)H(Mg)I_s(g)\overline{\phi(g)}\, dg$$
$$= \int\limits_{U_4(\mathbb{R})\backslash\mathfrak{h}^4} H(Mg)I_s(g) \int\limits_{U_4(\mathbb{Z})\backslash U_4(\mathbb{R})} \psi(n)\,\overline{\phi(ng)}\, dn\, dg.$$

The last integral is zero if ϕ is non-generic. □

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Proposition 3.7.4 (Eisenstein term $\mathcal{E}$ in the Kuznetsov trace formula). *With the notation of Theorem 3.7.3, the Eisenstein term $\mathcal{E}$ in the Kuznetsov trace formula is given by*

$$\mathcal{E} = \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \int_{\mathrm{Re}(s_1)=0} \cdots \int_{\mathrm{Re}(s_{r-1})=0} A_{E_{\mathcal{P},\Phi}}(L,s)\, \overline{A_{E_{\mathcal{P},\Phi}}(M,s)} \cdot \left| p^{\#}_{T,R}\big(\alpha_{(\mathcal{P},\Phi)}(s)\big)\right|^2 ds_1 \cdots ds_{r-1}.$$

Proof. The proof follows from Proposition 3.7.3. □

4. Bounding the geometric side

Recall from (3.2.4) and (3.4.2) that the Kloosterman contribution to the Kuznetsov trace formula is given by

$$\mathcal{K} = C_{L,M}^{-1} \sum_{w \in W_4} \mathcal{I}_w$$

where $C_{L,M} = \mathfrak{c}_4 \cdot (\ell_1 m_1)^3 (\ell_2 m_2)^4 (\ell_3 m_3)^3$, and

$$\mathcal{I}_w = \sum_{v \in V_4} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{\infty} \sum_{c_3=1}^{\infty} S_w(\psi_L, \psi_M^v, c) \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} \int_{y_3=0}^{\infty} \int_{\mathrm{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}} \int_{\overline{U}_w(\mathbb{R})} \cdot\, \psi_L(wuy)\, \overline{\psi_M^v(u)}\, p_{T,R}(Lcwuy)\, \overline{p_{T,R}(My)}\, d^*u\, \frac{dy_1\, dy_2\, dy_3}{y_1^4 y_2^5 y_3^4}, \qquad (4.0.1)$$

and $c = \begin{pmatrix} 1/c_3 & & & \\ & c_3/c_2 & & \\ & & c_2/c_1 & \\ & & & c_1 \end{pmatrix}$.

The main term in the Kuznetsov trace formula is given in Proposition 3.5.1 and consists of the first term (corresponding to the identity Weyl element $w = w_1$) on the geometric side of the trace formula. In this section we bound I_w in each of the remaining cases.

We remark first, by Friedberg [Fri87], that $\mathcal{I}_w = 0$ unless w is *relevant*. That is w must be of the form $w = \begin{pmatrix} & & I_{n_1} \\ & \iddots & \\ I_{n_k} & & \end{pmatrix}$, where I_{n_i} is the identity matrix of size $n_i \times n_i$ and $4 = \sum_{i=1}^{k} n_i$ with $1 \le k \le 4$ and $n_i \in \mathbb{Z}_{\ge 0}$. The relevant Weyl group elements are therefore,

$$w_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}, \quad w_3 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}, \quad w_4 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix},$$

$$w_5 = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}, \quad w_6 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad w_7 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ 1 & & & \\ & 1 & & \end{pmatrix}, \quad w_8 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

To motivate the strategy for bounding $\mathcal{I}_w$, we consider (4.0.1) (choosing $L = M = 1$) and then take absolute values to obtain

$$|I_w| \ll \sum_{c \in \mathbb{Z}_+^3} \int_{\mathbb{R}_+^3} \left|p_{T,R}(y)\right| \frac{\delta_w(y)}{\delta(y)} \int_{U_w(\mathbb{R})} \left|p_{T,R}(cwyu)\right| du\, d^{\times}y \qquad (4.0.2)$$

($y = \mathrm{diag}(y_1y_2y_3, y_1y_2, y_1, 1)$) for each Weyl group element w, where $\delta(y) = y_1^3 y_2^4 y_3^3$ is the modular character and $\delta_w(y)$ is the Jacobian of $U_w(\mathbb{R}) \ni u \mapsto yuy^{-1}$. It can be checked that $\frac{\delta_w(y)}{\delta(y)} = \delta^{-\frac{1}{2}}(y) \cdot \delta^{-\frac{1}{2}}(wyw^{-1})$. Theorem 4.0.3 is required to prove the absolute convergence of (4.0.2). From the trivial bound of the Kloosterman sum given by $S(1,1,c) \ll \delta^{-\frac{1}{2}}(c) \ll c_1c_2c_3$, one obtains the absolute

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convergence of the sum over c provided that $p_{T,R}(y) \ll \delta^{1+\varepsilon}(y)$. Fortunately, then one checks that the u-integral is also absolutely convergent and

$$\int\limits_{U_w(\mathbb{R})} \left|p_{T,R}(cwyu)\right| \, du \ll \delta^{1+\varepsilon}(c) \cdot \delta^{1+\varepsilon}(wyw^{-1}).$$

Then the remaining y integral becomes

$$\int\limits_{\mathbb{R}_+^3} \left|p_{T,R}(y)\right| \cdot \delta^{\frac{1}{2}+\varepsilon}(wyw^{-1}) \cdot \delta^{-\frac{1}{2}}(y) \, d^\times y.$$

So, to make the y integral convergent, one also needs

$$p_{T,R}(y) \ll \delta^{\frac{1}{2}}(y) \cdot \delta^{-\frac{1}{2}}\left(wyw^{-1}\right) \min(y^{-\varepsilon}, y^{\varepsilon})$$

for all Weyl elements w. Note that $\min(y^{-\varepsilon}, y^{\varepsilon})$ is needed as buffer as in the proof of Theorem 4.0.13 and that for $w = w_{\text{long}}$ this bound coincides with $\delta^{1+\varepsilon}(y)$. One also notes that $\delta^{\frac{1}{2}}(y)$ is the trivial bound of $p_{T,R}(y)$.

The main technical tool for bounding the geometric side, the proof of which occupies all of Section 6, is the following theorem.

Theorem 4.0.3. *Let* $0 < \varepsilon < 1$ *and* $r \in \mathbb{Z}_{\geq 0}$. *There exists* R *sufficiently large such that the following is true. Suppose* a_1, a_2, a_3 *satisfy the conditions*

$$\boxed{\varepsilon \leq |2r_j - a_j| \leq 1 - \varepsilon \quad (1 \leq j \leq 3)}$$

if $(r_1, r_2, r_3) \in \left\{(0, r, 0), (r, r, 0), (0, r, r), (r, r, r)\right\}$, *or*

$$\boxed{\varepsilon \leq |2r - a_1| \leq 1 - \varepsilon, \quad \varepsilon \leq |a_2| \leq 1 - \varepsilon, \quad -1 + \varepsilon \leq a_3 < -\varepsilon}$$

if $(r_1, r_2, r_3) = (r, 0, 0)$, *or*

$$\boxed{-1 + \varepsilon \leq a_1 < -\varepsilon, \quad \varepsilon \leq |a_2| \leq 1 - \varepsilon, \quad \varepsilon \leq |2r - a_3| \leq 1 - \varepsilon}$$

if $(r_1, r_2, r_3) = (0, 0, r)$. *Then we have the bound*

$$\boxed{p_{T,R}(y) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+9+\sum_{j=1}^{3}(\delta_{0,r_j}-r_j)}}.$$

Here, δ_{0,r_j} *is equal to* 1 *if* $r_j = 0$ *and is zero otherwise. The implicit constant depends on* ε *and* R.

Proof. We will show in Section 6 that $p_{T,R}(y)$ can be expressed as a sum of a 'shifted $p_{T,R}$ term,' 'single residue terms,' 'double residue terms' and 'triple residue terms' (see (6.1.8)). A bound for the shifted $p_{T,R}$ term is given in Proposition 6.3.4. Note that in the proof of Proposition 6.3.4 the first set of inequalities for a_j $(j = 1, 2, 3)$ is shown to hold for any (r_1, r_2, r_3) with $r_j \geq 0$. Bounds for the single residue terms are given in Proposition 6.4.2 and Proposition 6.4.18. See the proof of Lemma 6.4.13 for details of why the further inequalities given above are required. Bounds for the double residue terms are given in Proposition 6.5.4 and Proposition 6.5.25. Finally, bounds for the triple residue terms are given in Proposition 6.6.2. Combining these bounds gives the desired result. ◻

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Proposition 4.0.4. *Let $\mathcal{I}_w$ be as above. Let $M = (m_1, m_2, m_3)$, $L = (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$, where $C_{L,M} \neq 0$ is the normalization factor given in Definition 3.2.1. Let $r \geq 1$ be an integer. Then for R sufficiently large and any $\varepsilon > 0$, we have*

$$C_{L,M}^{-1}\left|\mathcal{I}_{w_j}\right| \ll_{\varepsilon,R} (\ell_1 m_1)^{2r-1/2}(\ell_2 m_2)^{2r-1}(\ell_3 m_3)^{2r-1/2} B_j(T),$$

where

$$B_j(T) = \begin{cases} T^{\varepsilon+8R+20-4r} & \text{if } j = 2,3,4, \\ T^{\varepsilon+8R+19-5r} & \text{if } j = 6,7, \\ T^{\varepsilon+8R+18-6r} & \text{if } j = 5,8. \end{cases}$$

Remark 4.0.5. Given Proposition 3.5.1 (the asymptotic formula for the main term with asymptotic error terms), we need $20 - 4r < 6$ in order for the contribution from $\mathcal{I}_w$ to be smaller than the main term $\mathcal{M}$. In other words, we will need to take $r \geq 4$. In order to minimize the contribution of L and M in the Main Theorem 1.1.1, we take $r = 4$.

Proof. The main idea of the proof of Proposition 4.0.4 is to apply Theorem 4.0.3 to each of the two instances of $p_{T,R}$ that appear on the right-hand side of (4.0.1). Before doing so, we make the change of variables $u \mapsto yuy^{-1}$. Note that, by definition, $d(yuy^{-1}) = \delta_w(y)\, du$ for any $u \in \overline{U}_w$.

$$|\mathcal{I}_w| \ll \sum_{v\in V_4} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{\infty} \sum_{c_3=1}^{\infty} |S_w(\psi_L, \psi_M^v, c)| \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} \int_{y_3=0}^{\infty} \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int_{\overline{U}_w(\mathbb{R})} \cdot\, \delta_w(y) \cdot |p_{T,R}(Lcwyu)| \cdot |p_{T,R}(My)|\, d^*u\, \frac{dy_1\, dy_2\, dy_3}{y_1^4 y_2^5 y_3^4} \tag{4.0.6}$$

For the term $p_{T,R}(Lcwyu)$, we need the Iwasawa form: $Lcwyu =: u'tk$ so that we can apply Theorem 4.0.3 for a particular choice of integers r_1, r_2, r_3 and parameters $2r_j - 1 < a_j < 2r_j$ for each $j = 1, 2, 3$.

For notational purposes, we write

$$t = t(Lcwuy) =: Y = (Y_1, Y_2, Y_3).$$

It is easy to see that Y_i factors as

$$Y_i =: Y_i(w,c)Y_i(w,L)Y_i(w,y)Y_i(w,u)$$

with each factor being an expression in the entries of c, L, y or u, respectively. (Note that we are following the notation of (2.1.2) for the element Y.) By inspection, we see that

$$Y_1(w,L) = \ell_1^{\frac{3}{2}+a_1}, \qquad Y_2(w,L) = \ell_2^{2+a_2}, \qquad Y_3(w,L) = \ell_3^{\frac{3}{2}+a_3} \tag{4.0.7}$$

for all w. Moreover,

$$Y_1(w,u) = \frac{\sqrt{\xi_2(w,u)}}{\xi_1(w,u)},\ Y_2(w,u) = \frac{\sqrt{\xi_1(w,u)\cdot\xi_3(w,u)}}{\xi_2(w,u)},\ Y_3(w,u) = \frac{\sqrt{\xi_2(w,u)}}{\xi_3(w,u)}, \tag{4.0.8}$$

where each function $\xi_j(w,u)$ is strictly positive for any $u \in U(\mathbb{R})$.

For example, in the case of $w = w_8$, the long element, we find that if

$$u = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad y = (y_1, y_2, y_3) = \begin{pmatrix} y_1y_2y_3 & 0 & 0 & 0 \\ 0 & y_1y_2 & 0 & 0 \\ 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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then

$$Y_1(w_8, y) = \frac{1}{y_3}, \qquad Y_2(w_8, y) = \frac{1}{y_2}, \qquad Y_3(w_8, y) = \frac{1}{y_1}, \tag{4.0.9}$$

and

$$\begin{aligned}
\xi_1(w_8, u) &= 1 + x_{12}^2 + x_{13}^2 + x_{14}^2, \\
\xi_2(w_8, u) &= 1 + x_{23}^2 + x_{24}^2 + \left(x_{12}x_{24} - x_{14}\right)^2 + \left(x_{12}x_{23} - x_{13}\right)^2 + \left(x_{13}x_{24} - x_{14}x_{23}\right)^2, \\
\xi_3(w_8, u) &= 1 + x_{34}^2 + \left(x_{23}x_{34} - x_{24}\right)^2 + \left(x_{12}x_{23}x_{34} - x_{13}x_{34} - x_{12}x_{24} + x_{14}\right)^2.
\end{aligned}$$

Returning now to (4.0.6), we apply Theorem 4.0.3 for (r, r, r) and yet to be determined values a_1, a_2, a_3 within the permissible bounds ($\varepsilon \le |2r - a_j| \le 1 - \varepsilon$) to the term $p_{T,R}(u'tk)$. To the other term we apply the theorem for a choice (r_1', r_2', r_3') and values b_1, b_2, b_3. As we will see, values of (r_1', r_2', r_3') and b_j' will be forced upon us in order to guarantee the convergence of the sum over c, the integral over (y_1, y_2, y_3) and the integral over u. In fact, it will become evident that these values are determined by w and a_1, a_2, a_3.

Independent of the choice of w, we define

$$I_0 := (0, 1], \qquad I_1 = (1, \infty),$$

hence

$$\int\limits_{y_1=0}^{\infty} \int\limits_{y_2=0}^{\infty} \int\limits_{y_3=0}^{\infty} = \sum_{i,j,k \in \{0,1\}} \int_{I_i} \int_{I_j} \int_{I_k}.$$

Using the bounds for $p_{T,R}$ (once with parameters a_1, a_2, a_3 and once with parameters b_1, b_2, b_3 as described above), it follows that

$$\begin{aligned}
|\mathfrak{I}_w| \ll \; & T^{\varepsilon + 8R + 18 - 3r - r_1' - r_2' - r_3' + \sum\limits_{i=1}^{3} \delta_{r_i',0}} (\ell_1 m_1)^{\frac{3}{2}} (\ell_2 m_2)^2 (\ell_3 m_3)^{\frac{3}{2}} \ell_1^{a_1} \ell_2^{a_2} \ell_3^{a_3} m_1^{b_1} m_2^{b_2} m_3^{b_3} \\
& \cdot \sum_{v \in V_4} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{\infty} \sum_{c_3=1}^{\infty} \frac{\left|S_w(\psi_M, \psi_L^v, c)\right|}{c_1^{1+2a_1-a_2} c_2^{1-a_1+2a_2-a_3} c_3^{1-a_2+2a_3}} \\
& \cdot \sum_{i,j,k \in \{0,1\}} \int_{I_i} \int_{I_j} \int_{I_k} Y_1(w, y)^{\frac{3}{2}+a_1} Y_2(w, y)^{2+a_2} Y_3(w, y)^{\frac{3}{2}+a_3} \cdot y_1^{\frac{3}{2}+b_1} y_2^{2+b_2} y_3^{\frac{3}{2}+b_3} \cdot |\delta_w(y)| \\
& \cdot \frac{dy_1\, dy_2\, dy_3}{y_1^4\, y_2^5\, y_3^4} \cdot \int\limits_{U_w(\mathbb{Z}) \backslash U_w(\mathbb{R})} \int\limits_{\overline{U}_w(\mathbb{R})} \xi_1^{-\frac{1}{2} - a_1 + \frac{a_2}{2}} \cdot \xi_2^{-\frac{1}{2} + \frac{a_1}{2} - a_2 + \frac{a_3}{2}} \cdot \xi_3^{-\frac{1}{2} + \frac{a_2}{2} - a_3} \, |d^* u|.
\end{aligned} \tag{4.0.10}$$

It is known (see also [Jac67]) that the integral in ξ will converge, provided that each of the exponents of ξ_j for $j = 1, 2, 3$ is less than $-\frac{1}{2}$. Explicitly, we require that

$$2a_3, 2a_1 > a_2, \qquad \text{and} \qquad 2a_2 > a_1 + a_3. \tag{4.0.11}$$

Note that for any $r \ge 2$, the first of the conditions in (4.0.11) is ensured for *any* choice of a_1, a_2, a_3 such that

$$0 < |2r - a_j| < 1 \qquad \text{for each } j = 1, 2, 3. \tag{4.0.12}$$

In this range, the second condition (that $2a_2 > a_1 + a_3$) imposes an additional constraint, but we will see that it can be easily satisfied as well.

We next want to determine what additional restrictions on a_1, a_2, a_3 must be satisfied to guarantee the convergence of the sum over c. We require the following Kloosterman bounds.

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Proposition 4.0.13. *Let $w \neq w_1$. The sum*

$$\mathcal{K}(M,L;w,a) := \sum_{v\in V_4}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\sum_{c_3=1}^{\infty}\frac{|S_w(\psi_M,\psi_L^v,c)|}{c_1^{1+2a_1-a_2}c_2^{1-a_1+2a_2-a_3}c_3^{1-a_2+2a_3}},$$

which appears on the right-hand side of (4.0.10), satisfies the following bounds (with all implied constants independent of M, L, c, and a).

For $2 \le j \le 8$ and any $\varepsilon > 0$, we have

$$|\mathcal{K}(M,L;w_j,a)| \ll \sum_{c_1=1}^{\infty}\frac{1}{c_1^{2a_1-a_2}}\sum_{c_2=1}^{\infty}\frac{1}{c_2^{-a_1+2a_2-a_3}}\sum_{c_3=1}^{\infty}\frac{1}{c_3^{-a_2+2a_3}}.$$

Proof. As noted in Appendix B, we have

$$|S_{w_j}(\psi_M,\psi_L^v,c)| \ll c_1c_2c_3 \quad (2\le j\le 8). \tag{4.0.14}$$

Applying this to the definition of $\mathcal{K}(M,L,c;w,a)$ gives the desired result. □

Appendix B gives slightly better bounds for the Kloosterman sums in terms of c_1, c_2, c_3 at the expense of some powers of M, L. The bounds in Appendix B will likely be relevant in other applications. However, we thank the referee for pointing out that the proof of Proposition 4.0.13 only requires the trivial bound for Kloosterman sums proved in [DR98, Theorem 0.3 (i)].

Clearly, the series in the above proposition will converge as long as each c_j, for $1 \le j \le 3$, occurs to an exponent less than -1. This will be guaranteed by the following additional set of restrictions:

$$-2a_1 + a_2 < -1, \quad a_1 - 2a_2 + a_3 < -1, \quad a_2 - 2a_3 < -1. \tag{4.0.15}$$

Again, this is easy to arrange given that we may take any values of a_j satisfying the bounds of (4.0.12). Note, moreover, that (4.0.11) is a consequence of (4.0.15).

The next step is to show that the integral over $y = (y_1, y_2, y_3)$ also converges. An elementary calculation shows that

$$\frac{y_1^{\frac{3}{2}+b_1}y_2^{2+b_2}y_3^{\frac{3}{2}+b_3}Y_1(w,y)^{\frac{3}{2}+a_1}Y_2(w,y)^{2+a_2}Y_3(w,y)^{\frac{3}{2}+a_3}\delta_w(y)}{y_1^4y_2^5y_3^4}dy_1dy_2dy_3$$
$$= y_1^{b_1-e_1(w;a)}y_2^{b_2-e_2(w;a)}y_3^{b_3-e_3(w;a)}\frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dy_3}{y_3},$$

where $e_j(w;a)$ is given by

w	$e_1(w;a)$	$e_2(w;a)$	$e_3(w;a)$	(r_1', r_2', r_3')
w_2	a_3	$-a_1+a_3$	$-a_2+a_3$	$(r,0,0)$
w_3	a_1-a_2	a_1-a_3	a_1	$(0,0,r)$
w_4	a_2-a_3	a_2	$-a_1+a_2$	$(0,r,0)$
w_5	a_3	$a_1-a_2+a_3$	a_1	(r,r,r)
w_6	a_2-a_3	a_2	a_1	$(0,r,r)$
w_7	a_3	a_2	$-a_1+a_2$	$(r,r,0)$
w_8	a_3	a_2	a_1	(r,r,r)

(4.0.16)

The result for w_8, for example, is established using (4.0.9) and the fact that $\delta_{w_8}(y) = y_1^3y_2^4y_3^3$.

This means for each $j = 1, 2, 3$ and $k = 0, 1$, we are searching for a_1, a_2, a_3 satisfying (4.0.12) and (4.0.15). Simultaneously, we search for $r_1', r_2', r_3' \in \mathbb{Z}_{\ge 0}$ and b_1, b_2, b_3 satisfying the necessary bounds given in Theorem 4.0.3 such that the integral

$$\int_{I_k} y_j^{b_j - e_j(w;a)} \frac{dy_j}{y_j}$$

converges. Since $I_0 = (0,1]$ and $I_1 = (1,\infty)$, we require that $b_j - e_j(w;a)$ be positive if $k = 0$ and negative if $k = 1$. With this in mind, we let $\varepsilon > 0$ and choose b_1, b_2, b_3 such that

$$b_j - e_j(w;a) = \begin{cases} +\varepsilon & \text{if } k = 0 \\ -\varepsilon & \text{if } k = 1. \end{cases}$$

Note that having made this choice for b_j, the value of r'_j is forced to be that given in the final column of the table above. More precisely, given (r'_1, r'_2, r'_3) as indicated in the final column of the table, there exists some choice a_1, a_2, a_3 and $\varepsilon > 0$ for which conditions (4.0.12) and (4.0.15) are satisfied, and for which the necessary bounds on $b_j = e_j(w;a) \pm \varepsilon$ are also satisfied.

Except in the case that $w = w_2$ or $w = w_3$, the conditions on b_j are that

$$0 < |2r'_j(w) - e_j(w;a) \pm \varepsilon| < 1.$$

It is not hard to see that in these cases there exists an appropriate choice of parameters a_1, a_2, a_3.

On the other hand, if $w = w_2$, for example, we see that $(r'_1, r'_2, r'_3) = (r, 0, 0)$, and so the restriction on the quantities $b_2 = a_3 - a_1 \pm \varepsilon$ and $b_3 = a_3 - a_2 \pm \varepsilon$ (which plays the role of a_3 in Theorem 4.0.3) are

$$0 < |b_2| = |a_1 - a_3 \pm \varepsilon| < 1, \tag{4.0.17}$$

$$-1 + \varepsilon \leq b_3 = a_3 - a_2 \pm \varepsilon \leq -\varepsilon. \tag{4.0.18}$$

But we're assuming that $a_1 - 2a_2 + a_3 < -1$ and $a_1 - a_3 < 1 - \varepsilon$ (by equations (4.0.15) and (4.0.17)), which together imply that $a_3 - a_2 < -\varepsilon$ in any event. A similar analysis applies to the above restriction on b_1 in the case that $w = w_3$ and $(r'_1, r'_2, r'_3) = (0, 0, r)$.

The values of r'_j for $j = 1, 2, 3$ determine the exponent of T in (4.0.10), hence the values of $B_j(T)$ in the statement of the proposition. To complete the proof, we note that $a_j < 2r+1$ and $b_j < 2r'_j(w)+1 \leq 2r+1$, and therefore, the powers of $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$ in (4.0.10) are as claimed. □

5. Mellin transform of the GL(4) Whittaker function

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (i\mathbb{R})^4$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $\mathrm{Re}(s_j) > \varepsilon$ for $j = 1, 2, 3$ and $\varepsilon > 0$. Define the Mellin transform (denoted $\widetilde{W}_\alpha(s)$) of the $GL(4,\mathbb{R})$ Whittaker function W_α defined in Section 2.6 by the absolutely convergent integral

$$\widetilde{W}_\alpha(s) := \int_0^\infty \int_0^\infty \int_0^\infty y_1^{s_1 - \frac{3}{2}} y_2^{s_2 - 2} y_3^{s_3 - \frac{3}{2}} W_\alpha(y) \frac{dy_1 dy_2 dy_3}{y_1 y_2 y_3}. \tag{5.0.1}$$

5.1. Formula for Mellin transform of the Whittaker function

Here we present an expression for $\widetilde{W}_\alpha(s)$ as an integral, over an additional variable $t \in \mathbb{C}$, of a ratio of Gamma functions involving s and α. For the purposes of this section, we will assume that α is in general position, meaning that $\alpha_j \neq \alpha_k$ for all $j \neq k$. As explained in Remark 6.1.9, for our purposes this assumption is not prohibitive.

In the context of GL(4), this expression was first given in [Sta95]; that result was later generalized to $GL(n)$, in Theorem 3.1 of [Sta01]. The formula for $n = 4$ takes the form $\widetilde{W}_\alpha(s) = 2^{-3}\pi^{-s_1-s_2-s_3}\widehat{W}_{\frac{\alpha}{2}}\left(\frac{s}{2}\right)$,

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where, assuming that $\mathrm{Re}(s_j) \geq \varepsilon > 0$ for each $j = 1, 2, 3$,

$$\widehat{W}_\alpha(s) = \Gamma(s_1+\alpha_1)\Gamma(s_1+\alpha_2)\Gamma(s_2-\alpha_1-\alpha_2)\Gamma(s_2+\alpha_1+\alpha_2)\Gamma(s_3-\alpha_1)\Gamma(s_3-\alpha_2)$$
$$\cdot \frac{1}{2\pi i}\int_{\mathrm{Re}(t)=-\varepsilon'} \frac{\Gamma(-t+\alpha_3)\Gamma(-t+\alpha_4)\Gamma(t+s_1)\Gamma(t+s_2+\alpha_1)\Gamma(t+s_2+\alpha_2)\Gamma(t+s_3+\alpha_1+\alpha_2)}{\Gamma(t+s_1+s_2+\alpha_1+\alpha_2)\Gamma(t+s_2+s_3)}\, dt, \tag{5.1.1}$$

assuming that $0 < \varepsilon' < \varepsilon$.

5.2. Poles and residues of $\widetilde{W}_\alpha$

Following [Sta01], we introduce the sets

$$\begin{aligned} P_1 &:= \{-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4\}, \\ P_2 &:= \{\pm(\alpha_1+\alpha_2), \pm(\alpha_1+\alpha_3), \pm(\alpha_2+\alpha_3)\}, \\ P_3 &:= \{\alpha_1, \alpha_2, \alpha_2, \alpha_4\}. \end{aligned} \tag{5.2.1}$$

Note that if $p \in P_k$, then $-p \in P_{4-k}$.

It is shown in [ST21, Propositions 12 and 14] that the sets P_1, P_2, and P_3 determine the poles of $\widetilde{W}_\alpha(s)$, in the following sense.

Proposition 5.2.2. *The Mellin transform $\widetilde{W}_\alpha(s)$ extends to a meromorphic function of the variable $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with the following properties.*

(a) *$\widetilde{W}_\alpha(s)$ has a pole at $s_j = p - 2\delta$ for each $p \in P_j$ and $\delta \in \mathbb{Z}_{\geq 0}$. Moreover, $\widetilde{W}_\alpha(s)$ has no other poles or polar divisors in $\mathbb{C}^3$.*

(b) *The residue of $\widetilde{W}_\alpha(s)$ at any of the above poles is a polynomial times a ratio of Gamma functions. More specifically, for each $\delta \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $Q_\delta(b, c, d; e, f, g)$, of degree at most 3δ, such that the following are true.*

$$\operatorname*{Res}_{s_1=-\alpha_1-2\delta} \widetilde{W}_\alpha(s) = \frac{1}{\Gamma(\frac{s_2+s_3+\alpha_1}{2}+\delta)}\left[\prod_{k=2}^{4}\Gamma(\tfrac{\alpha_k-\alpha_1}{2}-\delta)\Gamma(\tfrac{s_2+\alpha_1+\alpha_k}{2})\Gamma(\tfrac{s_3-\alpha_k}{2})\right]$$
$$\cdot Q_\delta\left(\tfrac{s_3-\alpha_2}{2}, \tfrac{s_3-\alpha_3}{2}, \tfrac{s_3-\alpha_4}{2}; \tfrac{1+\alpha_1-s_2+s_3}{2}, \tfrac{s_3-\alpha_1}{2}-\delta, \tfrac{s_2+s_3+\alpha_1}{2}\right), \tag{5.2.3}$$

$$\operatorname*{Res}_{s_2=-\alpha_1-\alpha_4-2\delta} \widetilde{W}_\alpha(s) = \left[\prod_{j\in\{1,4\}}\Gamma(\tfrac{s_1+\alpha_j}{2})\right]\left[\prod_{k=2}^{3}\Gamma(\tfrac{s_3-\alpha_j}{2})\prod_{j\in\{1,4\}}\Gamma(\tfrac{\alpha_k-\alpha_j}{2}-\delta)\right]$$
$$\cdot Q_\delta\left(\tfrac{s_1+\alpha_1}{2}, \tfrac{\alpha_3-\alpha_4}{2}-\delta, \tfrac{s_3-\alpha_2}{2}; \tfrac{1+\alpha_1-\alpha_2}{2}, \tfrac{s_1+\alpha_3}{2}-\delta, \tfrac{s_3-\alpha_4}{2}-\delta\right), \tag{5.2.4}$$

and

$$\operatorname*{Res}_{s_3=\alpha_1-2\delta} \widetilde{W}_\alpha(s) = \frac{1}{\Gamma(\frac{s_1+s_2-\alpha_1}{2}+\delta)}\left[\prod_{k=2}^{4}\Gamma(\tfrac{\alpha_1-\alpha_k}{2}-\delta)\Gamma(\tfrac{s_1+\alpha_k}{2})\Gamma(\tfrac{s_2-\alpha_1-\alpha_k}{2})\right]$$
$$\cdot Q_\delta\left(\tfrac{s_1+\alpha_2}{2}, \tfrac{s_1+\alpha_3}{2}, \tfrac{s_1+\alpha_4}{2}; \tfrac{1-\alpha_1-s_2+s_1}{2}, \tfrac{s_1+\alpha_1}{2}-\delta, \tfrac{s_1+s_2-\alpha_1}{2}\right). \tag{5.2.5}$$

The formulas for the remaining residues are found from each of the above by permuting α (i.e., applying Weyl group transformations).

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It may be seen from [ST21, Equation (43)] that Q_0 is the constant polynomial $Q_0 \equiv 1$. Thus the case $\delta = 0$ of the above proposition yields the following special cases, also deduced in [Sta01, Theorem 3.2]:

$$\operatorname*{Res}_{s_1=-\alpha_1} \widetilde{W}_\alpha(s) = \frac{\prod_{k=2}^{4} \Gamma(\frac{\alpha_k-\alpha_1}{2})\Gamma(\frac{s_2+\alpha_1+\alpha_k}{2})\Gamma(\frac{s_3-\alpha_k}{2})}{\Gamma(\frac{s_2+s_3+\alpha_1}{2})}, \tag{5.2.6}$$

$$\operatorname*{Res}_{s_2=-\alpha_1-\alpha_4} \widetilde{W}_\alpha(s) = \left(\prod_{j=1}^{4} \Gamma(\tfrac{s_1+\alpha_j}{2})\right)\left(\prod_{k=2}^{3} \Gamma(\tfrac{s_3-\alpha_j}{2}) \prod_{j=1}^{4} \Gamma(\tfrac{\alpha_k-\alpha_j}{2})\right), \tag{5.2.7}$$

and

$$\operatorname*{Res}_{s_3=\alpha_1} \widetilde{W}_\alpha(s) = \frac{\prod_{k=2}^{4} \Gamma(\frac{\alpha_1-\alpha_k}{2})\Gamma(\frac{s_1+\alpha_k}{2})\Gamma(\frac{s_2-\alpha_1-\alpha_k}{2})}{\Gamma(\frac{s_1+s_2-\alpha_1}{2})}. \tag{5.2.8}$$

In Section 6.5 below, we will also need to consider double residues of $\widetilde{W}_\alpha(s)$ — that is, residues in any one of the three s_js of residues in either of the others. To this end we have the following, which is proved in [ST21, Proposition 15].

Proposition 5.2.9.

(a) *For each $\delta_1, \delta_2 \in \mathbb{Z}_{\geq 0}$, there exists $f_{\delta_1,\delta_2}(s_3, \alpha)$, a polynomial of degree at most $2\delta_1 + \delta_2$, such that*

$$\begin{aligned}&\operatorname*{Res}_{s_2=-\alpha_1-\alpha_4-2\delta_2}\left(\operatorname*{Res}_{s_1=-\alpha_1-2\delta_1} \widetilde{W}_\alpha(s)\right) \\ &= \Gamma(\tfrac{\alpha_4-\alpha_1}{2} - \delta_1)\left(\prod_{k=2}^{3} \Gamma(\tfrac{\alpha_k-\alpha_1}{2} - \delta_1)\Gamma(\tfrac{\alpha_k-\alpha_4}{2} - \delta_2)\Gamma(\tfrac{s_3-\alpha_k}{2})\right) f_{\delta_1,\delta_2}(s_3, \alpha).\end{aligned} \tag{5.2.10}$$

(b) *For each $\delta_1, \delta_3 \in \mathbb{Z}_{\geq 0}$, there is a polynomial $g_{\delta_1,\delta_3}(s_2, \alpha)$, of degree at most $2\delta_1 + \delta_3$, such that*

$$\begin{aligned}&\operatorname*{Res}_{s_3=\alpha_2-2\delta_3}\left(\operatorname*{Res}_{s_1=-\alpha_1-2\delta_1} \widetilde{W}_\alpha(s)\right) \\ &= \Gamma(\tfrac{\alpha_2-\alpha_1}{2} - \delta_1)\left(\prod_{k=3}^{4} \Gamma(\tfrac{\alpha_k-\alpha_1}{2} - \delta_1)\Gamma(\tfrac{\alpha_2-\alpha_k}{2} - \delta_3)\Gamma(\tfrac{s_2+\alpha_1+\alpha_k}{2})\right) g_{\delta_1,\delta_3}(s_2, \alpha).\end{aligned} \tag{5.2.11}$$

5.3. *Shift equations*

We would like to have an expression similar to (5.1.1) for $\widetilde{W}_\alpha(s)$ in the region where $\mathrm{Re}(s) < 0$. Although we cannot use the right-hand side of (5.1.1) directly when $\mathrm{Re}(s) < 0$, we can use the fact that $\widetilde{W}_\alpha(s)$ satisfies the following shift equation that (as will be shown) is a direct corollary of Propositions 7 and 9 from [ST21].

Proposition 5.3.1. *Let $s = (s_1, s_2, s_3) \in \mathbb{C}^3$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Suppose that $r_1, r_2, r_3 \geq 0$ are integers. Then*

$$\left|\widetilde{W}_\alpha(s)\right| \ll \sum_{\ell=0}^{r_1}\sum_{k=0}^{r_2}\sum_{j=0}^{r_3}\left|\frac{Q_{j,k,\ell}^{r_1,r_2,r_3}}{\mathcal{B}_1^{r_1}\mathcal{B}_2^{r_2}\mathcal{B}_3^{r_3}} \widetilde{W}_\alpha\big(s_1 + 2(r_1 + j + k), s_2 + 2r_2, s_3 + 2(r_3 + \ell)\big)\right|,$$

where

$$\begin{aligned}\mathcal{B}_1 &:= \mathcal{B}_1(\alpha, s) := (s_1+\alpha_1)(s_1+\alpha_2)(s_1+\alpha_3)(s_1+\alpha_4),\\ \mathcal{B}_2 &:= \mathcal{B}_2(\alpha, s) := (s_2+\alpha_1+\alpha_2)(s_2+\alpha_1+\alpha_3)(s_2+\alpha_1+\alpha_4)\\ &\qquad\qquad \cdot (s_2+\alpha_2+\alpha_3)(s_2+\alpha_2+\alpha_4)(s_2+\alpha_3+\alpha_4),\\ \mathcal{B}_3 &:= \mathcal{B}_3(\alpha, s) := (s_3-\alpha_1)(s_3-\alpha_2)(s_3-\alpha_3)(s_3-\alpha_4),\end{aligned}$$

and $Q^{r_1,r_2,r_3}_{j,k,\ell}$ *is a polynomial in* α *and* s *with combined degree* $2(r_1+2r_2+r_3-j-k-\ell)$.

Proof. We begin with Proposition 7(a) of [ST21] that states that

$$\widetilde{W}_\alpha(s_1,s_2,s_3) = \frac{q_0(\alpha,s)}{\mathcal{B}_1(\alpha,s_1)}\widetilde{W}_\alpha(s_1+2,s_2,s_3) + \frac{q_1(\alpha,s)}{\mathcal{B}_1(\alpha,s_1)}\widetilde{W}_\alpha(s_1+2,s_2,s_3+2),$$

where $\deg(q_i) = 2-2i$. Iterating this formula r_1 times gives

$$\left|\widetilde{W}_\alpha(s_1,s_2,s_3)\right| \ll \sum_{\ell=0}^{r_1}\left|\frac{\mathcal{Q}^{(\ell)}_{1,r_1}(\alpha,s)}{\mathcal{B}_1(\alpha,s_1)^{r_1}}\widetilde{W}_\alpha(s_1+2r_1,s_2,s_3+2\ell)\right|. \tag{5.3.2}$$

In order to have an equality here, we would need to keep track of various shifts of the polynomial $\mathcal{B}_1(\alpha,s_1)$, but since we are only interested in bounds, the version presented here suffices. It is easy to show that

$$\deg(\mathcal{Q}^{(\ell)}_{1,r_1}) = (r_1-\ell)\deg(q_0) + \ell\deg(q_1) = 2(r_1-\ell).$$

Similarly, Proposition 9 of [ST21] states that

$$\widetilde{W}_\alpha(s_1,s_2,s_3) = \frac{p_0(\alpha,s)}{\mathcal{B}_2(\alpha,s_2)}\widetilde{W}_\alpha(s_1,s_2+2,s_3) + \frac{p_1(\alpha,s)}{\mathcal{B}_2(\alpha,s_2)}\widetilde{W}_\alpha(s_1+2,s_2+2,s_3),$$

where $\deg(p_k) = 4-2k$. Iterating this formula r_2 times gives

$$\left|\widetilde{W}_\alpha(s_1,s_2,s_3)\right| \ll \sum_{k=0}^{r_2}\left|\frac{\mathcal{Q}^{(k)}_{2,r_2}(\alpha,s)}{\mathcal{B}_2(\alpha,s_2)^{r_2}}\widetilde{W}_\alpha(s_1+2k,s_2+2r_2,s_3)\right|, \tag{5.3.3}$$

and

$$\deg(\mathcal{Q}^{(k)}_{2,r_2}) = (r_2-k)\deg(p_0) + k\deg(p_1) = 4(r_2-k)+2k = 4r_2-2k.$$

Via the change of variables $(s_1,s_3,\alpha)\mapsto(s_3,s_1,-\alpha)$ that preserves $\widetilde{W}_\alpha$ applied to (5.3.2), we have

$$\left|\widetilde{W}_\alpha(s_1,s_2,s_3)\right| \ll \sum_{j=0}^{r_3}\left|\frac{\mathcal{Q}^{(j)}_{1,r_3}(-\alpha,\overline{s})}{\mathcal{B}_3(\alpha,s_3)^{r_3}}\widetilde{W}_\alpha(s_1+2j,s_2,s_3+2r_3)\right|. \tag{5.3.4}$$

Applying equations (5.3.2), (5.3.3) and (5.3.4) in succession gives the desired result with

$$\deg(Q^{r_1,r_2,r_3}_{j,k,\ell}) = \deg(\mathcal{Q}^{(\ell)}_{1,r_1}) + \deg(\mathcal{Q}^{(k)}_{2,r_2}) + \deg(\mathcal{Q}^{(j)}_{1,r_3}) = 2(r_1+2r_2+r_3-\ell-k-j),$$

as claimed. ◻

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5.4. Expressing W_α as the inverse Mellin transform of $\widetilde{W}_\alpha$

Given the equation (5.0.1) for the Mellin transform of W_α, we find by Mellin inversion that

$$W_\alpha(y) = \frac{1}{(2\pi i)^3} \iiint_{\mathrm{Re}(s)=u} y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3} \widetilde{W}_\alpha(s)\, ds, \tag{5.4.1}$$

provided that $u = (u_1, u_2, u_3)$ satisfies $u_j > 0$ for $j = 1, 2, 3$.

As a matter of notation, for $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ define

$$W_\alpha(y; u) := \frac{1}{(2\pi i)^3} \iiint_{\mathrm{Re}(s_j)=u_j} y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3} \widetilde{W}_\alpha(s)\, ds,$$

and if I is a nonempty subset of $\{1, 2, 3\}$ of cardinality r and $p_I = (p_i)_{i\in I} \in \mathbb{C}^r$ define

$$\mathcal{R}_\alpha^{p_I}(y; u) := \frac{1}{(2\pi i)^{3-r}} \int_{\substack{\mathrm{Re}(s_j)=u_j\\ j\notin I}} \operatorname*{Res}_{s_I=p_I} \left(y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3} \widetilde{W}_\alpha(s) \right) \prod_{j\notin I} ds_j,$$

where $\mathrm{Res}_{s_I=p_I}$ is the operator that evaluates the iterated residue at each of the points $s_i = p_i$ with $i \in I$: that is,

$$\operatorname*{Res}_{s_I=p_I} := \operatorname*{Res}_{s_{i_1}=p_{i_1}} \circ \cdots \circ \operatorname*{Res}_{s_{i_q}=p_{i_q}}, \qquad (I = (i_1, \ldots, i_q)).$$

We call $\mathcal{R}_\alpha^{p_I}$ a *single residue term* if $q = 1$, a *double residue term* if $q = 2$, and a *triple residue term* if $q = 3$. For $a = (a_1, a_2, a_3)$ with $a_j > 0$, we call $W_\alpha(y; -a)$ the *shifted main term*.

Note that if $u_1, u_2, u_3 > 0$, then $W_\alpha(y) = W_\alpha(y; u)$. We now shift the lines of integration in the s-variable to the left passing poles at $\mathrm{Re}(s_j) = 0, -2, -4, \ldots$. By the Cauchy Residue Theorem, this allows us to write $W_\alpha(y)$ in terms of $\mathcal{R}_\alpha^*$. For example, if $a = (a_1, a_2, a_3)$ with $0 < a_j < 2$ for each $j = 1, 2, 3$, then

$$W_\alpha(y) = W_\alpha(y; -a) + \sum_{\substack{p\in P_j\\ 1\le j\le 3}} \mathcal{R}_\alpha^{(p)}(y; -a) + \sum_{\substack{p\in P_j\\ q\in P_k\\ 1\le j<k\le 3}} \mathcal{R}_\alpha^{(p,q)}(y; -a) + \sum_{\substack{p\in P_1\\ q\in P_2\\ r\in P_3}} \mathcal{R}_\alpha^{(p,q,r)}(y; -a).$$

Letting $p_1 := -\alpha_1 - 2\delta_1$, $p_2 := -\alpha_1 - \alpha_2 - 2\delta_2$ and $p_3 := -\alpha_1 - \alpha_2 - \alpha_3 - 2\delta_3$ for some $\delta_1, \delta_2, \delta_3 \in \mathbb{Z}_{\ge 0}$, as given in Section 5.2, there are polynomials $f_{\delta_j}^{p_j}(s, \alpha)$ such that

$$\mathcal{R}_\alpha^{(p_1)}(y; -a) = \iint_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=2,3}} y_1^{\frac{3}{2}+p_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3} \frac{\prod_{j=2}^{4} \Gamma\left(\frac{\alpha_j-\alpha_1}{2}\right)\Gamma\left(\frac{s_2+\alpha_1+\alpha_j}{2}\right)\Gamma\left(\frac{s_3-\alpha_j}{2}\right)}{\Gamma\left(\frac{s_2+s_3+\alpha_1}{2}\right)} f_{\delta_1}^{p_1}(s, \alpha)\, ds_2 ds_3, \tag{5.4.2}$$

$$\begin{aligned}\mathcal{R}_\alpha^{(p_2)}(y; -a) = \iint_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=1,3}} & y_1^{\frac{3}{2}-s_1} y_2^{2+p_2} y_3^{\frac{3}{2}-s_3} \left(\prod_{j=1}^{2} \prod_{k=3}^{4} \Gamma\left(\frac{\alpha_k-\alpha_j}{2}\right) \right) f_{\delta_2}^{p_2}(s, \alpha) \\ & \cdot \Gamma\left(\frac{s_1+\alpha_1}{2}\right) \Gamma\left(\frac{s_1+\alpha_2}{2}\right) \Gamma\left(\frac{s_3-\alpha_3}{2}\right) \Gamma\left(\frac{s_3-\alpha_4}{2}\right) ds_1 ds_3. \end{aligned} \tag{5.4.3}$$

All other single residue terms can be obtained from these by observing that, first, the action of the Weyl group, which acts by permuting the set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, acts transitively on each of the sets P_1, P_2, P_3 and permutes the residue terms accordingly. Explicitly, if $\sigma = (12)$ represents the permutation interchanging

1 and 2, then $\mathcal{R}_\alpha^{(\sigma(p_1)-\delta_1)}(y;-a) = \mathcal{R}_{\sigma(\alpha)}^{(p_1-\delta_1)}(y;-a)$ has the same expression as $\mathcal{R}_\alpha^{(p_1-\delta_1)}(y;-a)$ except that the instances of α_1 and α_2 are interchanged.

Moreover, the involution $(\alpha, s_1, s_2, s_3) \mapsto (-\alpha, s_3, s_2, s_1)$ interchanges the set of single residues at, say, $s_1 = -\alpha_1 - 2\delta$ with those at $s_3 = \alpha_1 - 2\delta$ respecting the formulas of Proposition 5.2.9. So, in particular, we have $\mathcal{R}_\alpha^{(\alpha_1-2\delta)}(y_1, y_2, y_3; -a) = \mathcal{R}_{-\alpha}^{(-\alpha_1-2\delta)}(y_3, y_2, y_1; -a)$.

It can be similarly shown that for appropriate polynomials $f_{*,*}^{(*,*)}$ and $f_\delta^{(p_1,p_2,p_3)}$, every double residue term is equivalent to either

$$\mathcal{R}_\alpha^{(p_1,p_2)}(y;-a) = \int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}+p_1} y_2^{2+p_2} y_3^{\frac{3}{2}-s_3} f_{\delta_1,\delta_2}^{(p_1,p_2)}(s_3,\alpha) \cdot \left(\prod_{j=1}^{2} \prod_{k=j+1}^{4} \Gamma\left(\tfrac{\alpha_k-\alpha_j}{2}\right) \right) \Gamma\left(\tfrac{s_3-\alpha_4}{2}\right) \Gamma\left(\tfrac{s_3-\alpha_3}{2}\right) ds_3, \tag{5.4.4}$$

or

$$\mathcal{R}_\alpha^{(p_1,p_3)}(y;-a) = \int\limits_{\mathrm{Re}(s_2)=-a_3} y_1^{\frac{3}{2}+p_1} y_2^{2-s_2} y_3^{\frac{3}{2}+p_3} f_{\delta_1,\delta_3}^{(p_1,p_3)}(s_2,\alpha)\, \Gamma\left(\tfrac{s_2+\alpha_1+\alpha_2}{2}\right) \Gamma\left(\tfrac{s_2+\alpha_1+\alpha_3}{2}\right) \cdot \Gamma\left(\tfrac{\alpha_4-\alpha_3}{2}\right) \Gamma\left(\tfrac{\alpha_4-\alpha_2}{2}\right) \Gamma\left(\tfrac{\alpha_4-\alpha_1}{2}\right) \Gamma\left(\tfrac{\alpha_3-\alpha_1}{2}\right) \Gamma\left(\tfrac{\alpha_2-\alpha_1}{2}\right) ds_2, \tag{5.4.5}$$

and every triple residue term is equivalent to

$$\mathcal{R}_\alpha^{(p_1,p_2,p_3)}(y) = y_1^{3/2+p_1} y_2^{2+p_2} y_3^{3/2+p_3}\, f_\delta^{(p_1,p_2,p_3)}(\alpha) \prod_{1\le j<k\le 4} \Gamma\left(\tfrac{\alpha_k-\alpha_j}{2}\right). \tag{5.4.6}$$

It turns out that the bounds obtained from applying our methods only to the shifted integral $W_\alpha(y;-a)$ when $0 < a_j < 2$ are not sufficient for our needs. Instead, we will need to obtain bounds for various shifts of the form $2r_j - 1 < a_j < 2r_j$ for $r_1, r_2, r_3 \in \mathbb{Z}_{\ge 0}$. For a given choice of $(r_1, r_2, r_3) \in \mathbb{Z}_{\ge 0}^3$, we extrapolate the procedure outlined above obtaining poles corresponding to each case of

$$s_j = p_j - 2\delta_j, \qquad \text{for each} \qquad \delta_j = 0, \dots, r_j - 1.$$

Doing so, we arrive at the following result.

Proposition 5.4.7. *Let* $(r_1, r_2, r_3) \in \mathbb{Z}_{\ge 0}^3$ *be fixed. Let* $a = (a_1, a_2, a_3)$ *satisfy* $2r_j - 2 < a_j < 2r_j$ *be fixed* ($a_j \ne a_k$ *if* $j \ne k$). *Let* $P_1 = \{-\alpha_j \mid j = 1, \dots, 4\}$, $P_2 = \{-\alpha_j - \alpha_k \mid 1 \le j \ne k \le 4\}$, $P_3 = \{\alpha_j \mid j = 1, \dots, 4\}$, *as in* (5.2.1). *Then, after shifting contours of integration, the* GL(4)*-Whittaker function is given by* $W_\alpha(y) =$

$$W_\alpha(y;-a) + \sum_{\substack{p_j \in P_j - 2\delta_j \\ j=1,2,3 \\ \delta_j \in \{0,1,\dots,r_j-1\}}} \mathcal{R}_\alpha^{(p_j)}(y;-a) + \sum_{\substack{1\le j\ne k\le 4 \\ p_j \in P_j - 2\delta_j \\ p_k \in P_k - 2\delta_k \\ \delta_j \in \{0,1,\dots,r_j-1\} \\ \delta_k \in \{0,1,\dots,r_k-1\}}} \mathcal{R}_\alpha^{(p_j,p_k)}(y;-a) + \sum_{\substack{p_1 \in P_1 - 2\delta_1 \\ p_2 \in P_2 - 2\delta_2 \\ p_3 \in P_3 - 2\delta_3 \\ \delta_\ell \in \{0,1,\dots,r_\ell-1\} \\ \ell=1,2,3}} \mathcal{R}_\alpha^{(p_1,p_2,p_3)}(y).$$

6. Bounds for the test function $p_{T,R}$

Recall (see (3.1.3)) that $p_{T,R}$ is given by

$$p_{T,R}(y) = p_{T,R}(y_1,y_2,y_3) = \frac{1}{\pi^3} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} p_{T,R}^{\#}(\alpha)\, W_\alpha(y) \frac{d\alpha_1\, d\alpha_2\, d\alpha_3}{\prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}. \tag{6.0.1}$$

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Note that in [GK12], instead of $W_\alpha(y)$, one actually has its complex conjugate $\overline{W}_\alpha(y)$. However, one arrives at the formula here by noting that α is purely imaginary and, therefore, $\overline{W}_\alpha(y) = W_{-\alpha}(y)$. Hence, the change of variables $\alpha \mapsto -\alpha$ that leaves $p^\sharp_{T,R}(\alpha)$ and the measure $d\alpha_1\, d\alpha_2\, d\alpha_3 \Big/ \prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)$ invariant leads to the given formula.

6.1. Decomposition of $p_{T,R}$ in terms of poles and residues of $\widetilde{W}_\alpha$

Define

$$\Gamma_R(\alpha) := \prod_{1\le \ell\ne m\le 4} \frac{\Gamma\left(\frac{2+R+\alpha_\ell-\alpha_m}{4}\right)}{\Gamma\left(\frac{\alpha_\ell-\alpha_m}{2}\right)}. \tag{6.1.1}$$

We now replace $W_\alpha(y)$, on the right side of (6.0.1), by the expression given in Proposition 5.4.7. It follows from the definition of the test function $p^\#_{T,R}$ given in (3.1.1), that in doing so, we obtain a shifted $p_{T,R}$ term

$$p_{T,R}(y;-a) := \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, W_\alpha(s;-a)\, \mathfrak{F}_R(\alpha)\, \Gamma_R(\alpha)\, d\alpha, \tag{6.1.2}$$

single residue terms of the type

$$p^{j,\delta}_{T,R}(y) := \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, \mathfrak{F}_R(\alpha)\, \mathcal{R}^{(p_j-2\delta)}_\alpha(y;-a)\, \Gamma_R(\alpha)\, d\alpha,$$

double residue terms of the type

$$p^{jk,\delta}_{T,R}(y) := \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, \mathcal{R}^{(p_j-2\delta_j,\,p_k-2\delta_k)}_\alpha(y;-a)\, \mathfrak{F}_R(\alpha)\, \Gamma_R(\alpha)\, d\alpha,$$

and triple residue terms of the type

$$p^{123,\delta}_{T,R}(y) := \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, \mathcal{R}^{(p_1-2\delta_1,\,p_2-2\delta_2,\,p_3-2\delta_3)}_\alpha(y;-a)\, \mathfrak{F}_R(\alpha)\, \Gamma_R(\alpha)\, d\alpha,$$

where $p_1 = -\alpha_1$, $p_2 = -\alpha_1-\alpha_2$, $p_3 = -\alpha_1-\alpha_2-\alpha_3$ and $\delta_j \in \{0,1,\ldots,r-1\}$. Also, note that we use the notation $d\alpha := d\alpha_1\, d\alpha_2\, d\alpha_3$. In particular, by equations (5.4.2)–(5.4.6), we see that

$$p^{1,0}_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=2,3}} y_1^{\frac{3}{2}+p_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}\, \mathfrak{F}_R(\alpha)\, \Gamma_R(\alpha)$$
$$\cdot \frac{\prod\limits_{j=2}^{4} \Gamma\left(\frac{\alpha_j-\alpha_1}{2}\right)\Gamma\left(\frac{s_2+\alpha_1+\alpha_j}{2}\right)\Gamma\left(\frac{s_3-\alpha_j}{2}\right)}{\Gamma\left(\frac{s_2+s_3+\alpha_1}{2}\right)}\, ds_2 ds_3\, d\alpha, \tag{6.1.3}$$

$$p^{2,0}_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=1,3}} y_1^{\frac{3}{2}-s_1} y_2^{2+p_2} y_3^{\frac{3}{2}-s_3} \left(\prod_{j=1}^{2}\prod_{k=3}^{4} \Gamma\left(\frac{\alpha_k-\alpha_j}{2}\right)\right) \mathfrak{F}_R(\alpha)\, \Gamma_R(\alpha)$$
$$\cdot\, \Gamma\left(\tfrac{s_1+\alpha_1}{2}\right) \Gamma\left(\tfrac{s_1+\alpha_2}{2}\right) \Gamma\left(\tfrac{s_3-\alpha_3}{2}\right) \Gamma\left(\tfrac{s_3-\alpha_4}{2}\right)\, ds_1 ds_3\, d\alpha, \tag{6.1.4}$$

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$$p^{12,0}_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}+p_1} y_2^{2+p_2} y_3^{\frac{3}{2}-s_3}\, \mathcal{F}_R(\alpha)\, \Gamma_R(\alpha)$$
$$\cdot \left(\prod_{j=1}^{2}\prod_{k=j+1}^{4} \Gamma\big(\tfrac{\alpha_k-\alpha_j}{2}\big)\right) \Gamma\left(\tfrac{s_3-\alpha_4}{2}\right) \Gamma\left(\tfrac{s_3-\alpha_3}{2}\right) ds_3\, d\alpha, \tag{6.1.5}$$

$$p^{13,0}_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \int\limits_{\mathrm{Re}(s_2)=-a_3} y_1^{\frac{3}{2}+p_1} y_2^{2-s_2} y_3^{\frac{3}{2}+p_3}\, \Gamma\left(\tfrac{s_2+\alpha_1+\alpha_2}{2}\right) \Gamma\left(\tfrac{s_2+\alpha_1+\alpha_3}{2}\right)$$
$$\cdot\, \Gamma\left(\tfrac{\alpha_4-\alpha_3}{2}\right) \Gamma\left(\tfrac{\alpha_4-\alpha_2}{2}\right) \Gamma\left(\tfrac{\alpha_4-\alpha_1}{2}\right) \Gamma\left(\tfrac{\alpha_3-\alpha_1}{2}\right) \Gamma\left(\tfrac{\alpha_2-\alpha_1}{2}\right) ds_2\, \mathcal{F}_R(\alpha)\, \Gamma_R(\alpha)\, d\alpha, \tag{6.1.6}$$

$$p^{123,0}_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, y_1^{3/2+p_1} y_2^{2+p_2} y_3^{3/2+p_3} \left(\prod_{1\le j<k\le 4} \Gamma\big(\tfrac{\alpha_k-\alpha_j}{2}\big)\right) \mathcal{F}_R(\alpha)\, \Gamma_R(\alpha)\, d\alpha. \tag{6.1.7}$$

Since we are integrating over α on the right-hand side of (3.1.3) and the integrand (and measure) is invariant under the action of the Weyl group, there are explicitly computable constants c_1, c_{12}, c_{123} such that

$$p_{T,R}(y) = p_{T,R}(y;-a) \;+\; c_1 \sum_{\substack{1\le j\le 3\\ \delta\in\{0,1,\ldots,r-1\}}} p^{j,\delta}_{T,R}(y;-a) \;+\; c_{12} \sum_{\substack{1\le j<k\le 3\\ \delta\in\{0,1,\ldots,r-1\}^2}} p^{jk,\delta}_{T,R}(y;-a)$$
$$+\, c_{123} \sum_{\delta\in\{0,1,\ldots,r-1\}^3} p^{123,\delta}_{T,R}(y;-a). \tag{6.1.8}$$

Remark 6.1.9. There are no poles in the expressions of the residues given in (6.1.1) to (6.1.6) because the products of $\Gamma\big(\frac{\alpha_k-\alpha_j}{2}\big)$ are cancelled by the Gamma functions in the denominator of $\Gamma_R(\alpha)$.

6.2. Philosophy of the proof of Theorem 4.0.3

Recall that Theorem 4.0.3 gives sharp bounds for the $p_{T,R}$ function. Given (6.1.8), this can be achieved by bounding each of the terms on the right-hand side: $p_{T,R}(y;-a)$, $p^{1,\delta}_{T,R}(y;-a)$, $p^{2,\delta}_{T,R}(y;-a)$, $p^{12,\delta}_{T,R}(y;-a)$, $p^{23,\delta}_{T,R}(y;-a)$ and $p^{123,\delta}_{T,R}(y;-a)$ and each possible choice of δ. We obtain bounds for each of these terms in Sections 6.3–6.6, respectively. Theorem 4.0.3 then follows as an immediate consequence.

Before proceeding to those sections, we describe the idea in the special case that $a = (a_1, a_2, a_3)$ with $a_i < 0$ for each $i = 1, 2, 3$. In this case there are no residue terms at all: that is, we need only bound $p_{T,R}(y) = p_{T,R}(y;-a)$. Then the basic idea is to insert the formula (5.1.1) into (5.4.1) to get an expression for $W_\alpha(y)$ as an integral of the ratio of many Gamma functions. We then insert this into (3.1.3) and estimate each Gamma function using Stirling's approximation, which for fixed σ and $|t| \to \infty$ says that

$$\Gamma(\sigma + it) \sim \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}\, e^{-\frac{\pi}{2}|t|}. \tag{6.2.1}$$

We call $|t|^{\sigma-\frac{1}{2}}$ the *polynomial factor* of $\Gamma(\sigma + it)$, and $e^{-\frac{\pi}{2}|t|}$ is called the *exponential factor*. Through this process, we can replace the ratio of all of the Gamma factors by a rational function $\mathcal{P}(s, a)$ obtained as the product of the polynomial factors of the individual Gamma functions times $e^{-\mathcal{E}(s,\alpha)}$ that is the product of the exponential factors.

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To be completely explicit, after making a simple change of variables in (5.1.1), it is easy to see that (up to a constant) for $\varepsilon' > 0$ sufficiently small

$$\widetilde{W}_\alpha(s) = \int\limits_{\mathrm{Re}(t)=-\varepsilon'} \frac{\Gamma_0(t,\alpha)\Gamma_1(t,s_1,\alpha)\Gamma_2(t,s_2,\alpha)\Gamma_3(t,s_3,\alpha)}{\Gamma_{\mathrm{den}}(t,s,\alpha)}\,dt \tag{6.2.2}$$

where

$$\begin{aligned}
\Gamma_0(t,\alpha) &:= \Gamma\left(\tfrac{-t+\alpha_3}{2}\right)\Gamma\left(\tfrac{-t+\alpha_4}{2}\right),\\
\Gamma_1(t,s_1,\alpha) &:= \Gamma\left(\tfrac{s_1+\alpha_1}{2}\right)\Gamma\left(\tfrac{s_1+\alpha_2}{2}\right)\Gamma\left(\tfrac{s_1+t}{2}\right),\\
\Gamma_2(t,s_2,\alpha) &:= \Gamma\left(\tfrac{s_2+\alpha_1+\alpha_2}{2}\right)\Gamma\left(\tfrac{s_2+\alpha_1+t}{2}\right)\Gamma\left(\tfrac{s_2+\alpha_2+t}{2}\right)\Gamma\left(\tfrac{s_2+\alpha_3+\alpha_4}{2}\right),\\
\Gamma_3(t,s_3,\alpha) &:= \Gamma\left(\tfrac{s_3+\alpha_1+\alpha_2+t}{2}\right)\Gamma\left(\tfrac{s_3+\alpha_1+\alpha_3+\alpha_4}{2}\right)\Gamma\left(\tfrac{s_3+\alpha_2+\alpha_3+\alpha_4}{2}\right),\\
\Gamma_{\mathrm{den}}(t,s,\alpha) &:= \Gamma\left(\tfrac{s_1+s_2+\alpha_1+\alpha_2+t}{2}\right)\Gamma\left(\tfrac{s_2+s_3+t}{2}\right).
\end{aligned}$$

Thus by combining (5.4.1) and (6.2.2) into (3.1.3) as described above, we find that

$$\begin{aligned}
p_{T,R}(y) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}} \iiint\limits_{\mathrm{Re}(s)=\varepsilon} \int\limits_{\mathrm{Re}(t)=-\varepsilon'} y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}\,\mathcal{F}_R(\alpha)\,\Gamma_R(\alpha)\\
\cdot \frac{\Gamma_0(t,\alpha)\Gamma_1(t,s_1,\alpha)\Gamma_2(t,s_2,\alpha)\Gamma_3(t,s_3,\alpha)}{\Gamma_{\mathrm{den}}(t,s_1,s_2,s_3,\alpha)}\,dt\,ds\,d\alpha.
\end{aligned} \tag{6.2.3}$$

Applying Stirling's bound to each of the Gamma functions in (6.2.3) we see that (up to a constant factor depending at most on R, ε)

$$p_{T,R}(y) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \iiint\limits_{\mathcal{R}_T(0)} \cdot \iiint\limits_{\mathrm{Re}(s)=-a} \cdot \int\limits_{\mathrm{Re}(t)=\varepsilon'} \mathcal{P}(s,\alpha)\cdot\mathcal{F}_R(\alpha)\,\exp\left(-\frac{\pi}{4}\mathcal{E}(s,\alpha)\right)dt\,ds\,d\alpha,$$

where, for $\alpha_j = \kappa_j + i\tau_j$ $(j = 1,2,3,4)$,

$$\mathcal{R}_T(\kappa) := \left\{(i\tau_1+\kappa_1, i\tau_2+\kappa_2, i\tau_3) \mid -\tau_1-\tau_2-\tau_3 \le \tau_3 \le \tau_2 \le \tau_1 \le T^{1+\varepsilon}\right\}.$$

The Weyl group is isomorphic to S_4 and acts by permutations on the set $\{\alpha_j\}_{j=1}^4$ leaving the integrand for $p_{T,R}(y)$ invariant. Hence it suffices to restrict the integration over α to the set $\mathcal{R}_T(0)$.

We will prove below (see Lemma 6.2.5) that the integration in s and t can also be restricted to a finite volume set $\mathcal{R}$ that we call the *exponential zero set*. As s and t vary within this set, most of the polynomial terms can be uniformly bounded by a power of something of the form $(1+\tau_k-\tau_j)$, where $j<k$. We prove a very strong bound on the remaining terms (see Lemma A.3) that shows that it too is bounded by a product of similar factors. This implies that

$$\iiiint\limits_{(s,t)\in\mathcal{R}} \mathcal{P}(s,\alpha)\,dt\,ds \ll \prod_{1\le j<k\le 4} (1+\tau_j-\tau_k)^{b_{j,k}}$$

where each $b_{j,k} > 0$. Then the integration in α over the set $\mathcal{R}_T(0)$ can be estimated trivially. The main difference between this outline and the actual proof is that instead of using (6.2.2) directly as above, we replace it by the expression on the right-hand side of the result of Proposition 5.3.1. Also, instead of dealing with $p_{T,R}$ itself as given in (6.0.1), we individually bound each term on the right-hand side of (6.1.8).

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In order to describe the exponential zero set, note that since $\tau_1 \geq \tau_2 \geq \tau_3 \geq \tau_4$, if we let $\mathrm{Im}(s_j) =: \xi_j$ and $\mathrm{Im}(t_1) =: \rho$, the exponential term takes the simplified form

$$\begin{aligned}\mathcal{E}(s,\alpha) = &-6\tau_1 - 4\tau_2 - 2\tau_3 + |\rho - \tau_4| + |\rho - \tau_3| + |\xi_1 + \tau_1| + |\xi_1 + \tau_2| + |\xi_1 + \rho|\\ &+ |\xi_2 + \tau_1 + \tau_2)| + |\xi_2 + \tau_1 + \rho)| + |\xi_2 + \tau_2 + \rho)| + |\xi_2 + \tau_3 + \tau_4)|\\ &+ |\xi_3 + \tau_1 + \tau_2 + \rho| + |\xi_3 + \tau_1 + \tau_3 + \tau_4| + |\xi_3 + \tau_2 + \tau_3 + \tau_4|\\ &- |\xi_1 + \xi_2 + \tau_1 + \tau_2 + \rho| - |\rho + \xi_2 + \xi_3|.\end{aligned}$$

An important observation is the fact that $p_{T,R}(y)$ has been defined in such a way that there is at worst polynomial growth in the integrand. This means the exponential factor is never negative: that is, $\mathcal{E}(s,\alpha) \geq 0$ for all s. Since there will be exponential decay for any choice of s such that $\mathcal{E}(s,\alpha) > 0$, for the purposes of bounding $p_{T,R}(y)$, we need only determine when $\mathcal{E} = 0$. Note that this set depends only on the imaginary parts of s and α.

The expression for $\mathcal{E}$ above involves 14 absolute value terms. We can remove each of the 14 absolute values by replacing $|x|$ with $\pm x$ depending on whether x is positive or negative. This leads to an expression of the form

$$\begin{aligned}0 = &-6\tau_1 - 4\tau_2 - 2\tau_3 + \varepsilon_{t,1}(\rho - \tau_4) + \varepsilon_{t,2}(\rho - \tau_3) + \varepsilon_{1,0}(\xi_1 + \tau_1) + \varepsilon_{1,1}(\xi_1 + \tau_2)\\ &+ \varepsilon_{1,2}(\xi_1 + \rho) + \varepsilon_{2,0}(\xi_2 + \tau_1 + \tau_2) + \varepsilon_{2,1}(\xi_2 + \tau_1 + \rho) + \varepsilon_{2,2}(\xi_2 + \tau_2 + \rho)\\ &+ \varepsilon_{2,3}(\xi_2 + \tau_3 + \tau_4) + \varepsilon_{3,0}(\xi_3 + \tau_1 + \tau_2 + \rho) + \varepsilon_{3,1}(\xi_3 + \tau_1 + \tau_3 + \tau_4)\\ &+ \varepsilon_{3,2}(\xi_3 + \tau_2 + \tau_3 + \tau_4) - \varepsilon_{2-\frac{1}{2}}(\xi_1 + \xi_2 + \tau_1 + \tau_2 + \rho) - \varepsilon_{2+\frac{1}{2}}(\rho + \xi_2 + \xi_3),\end{aligned} \tag{6.2.4}$$

where each of the 14 εs is equal to ± 1. For a particular choice of εs either the sum on the right-hand side of (6.2.4) vanishes identically or not. If it does vanish, each ε_* determines an inequality, and the set of $\tau_1, \tau_2, \tau_3, \rho, \xi_1, \xi_2, \xi_3$ that satisfy all of these inequalities simultaneously is contained in the exponential zero set. The following lemma shows that there are three such choices of signs and each choice explicitly determines an exponential zero set.

Lemma 6.2.5. *Every solution* $\varepsilon = (\varepsilon_{t,1}, \varepsilon_{t,2}, \ldots, \varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) \in (\pm 1)^{14}$ *to* (6.2.4) *is of the form*

$$\varepsilon_{t,1} = +1, \quad \varepsilon_{t,2} = +1, \tag{6.2.6}$$

$$\varepsilon_{1,0} = +1, \quad \varepsilon_{1,1} = \varepsilon_{2-\frac{1}{2}}, \quad \varepsilon_{1,2} = -1, \tag{6.2.7}$$

$$\varepsilon_{2,0} = +1, \quad \varepsilon_{2,1} = \varepsilon_{2-\frac{1}{2}}, \quad \varepsilon_{2,2} = \varepsilon_{2+\frac{1}{2}}, \quad \varepsilon_{2,3} = +1, \tag{6.2.8}$$

$$\varepsilon_{3,0} = +1, \quad \varepsilon_{3,1} = \varepsilon_{2+\frac{1}{2}}, \quad \varepsilon_{3,2} = -1, \tag{6.2.9}$$

and $\varepsilon_{2-\frac{1}{2}} \geq \varepsilon_{2+\frac{1}{2}}$.

In particular, there are three possible exponential zero sets, which we denote as $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. *The first corresponds to the case of* $(\varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) = (+1, +1)$*:*

$$\mathcal{R}_1: \quad \begin{aligned} \tau_4 &\leq \rho \leq \tau_3\\ -\tau_2 &\leq \xi_1 \leq -\rho,\\ -\tau_2 - \rho &\leq \xi_2 \leq \tau_1 + \tau_2\\ \tau_2 &\leq \xi_3 \leq \tau_1,\end{aligned}$$

the second corresponds to $(\varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) = (+1, -1)$*:*

$$\mathcal{R}_2: \quad \begin{aligned} \tau_4 &\leq \rho \leq \tau_3\\ -\tau_2 &\leq \xi_1 \leq -\rho,\\ -\tau_1 - \rho &\leq \xi_2 \leq -\tau_2 - \rho\\ -(\tau_1 + \tau_2) - \rho &\leq \xi_3 \leq \tau_2,\end{aligned}$$

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and the third corresponds to $(\varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) = (-1, -1)$*:*

$$\begin{aligned} \mathcal{R}_3: \qquad\qquad \tau_4 &\le \rho \le \tau_3 \\ -\tau_1 &\le \xi_1 \le -\tau_2, \\ -(\tau_1+\tau_2) &\le \xi_2 \le -\tau_1-\rho \\ -(\tau_1+\tau_2)-\rho &\le \xi_3 \le \tau_2. \end{aligned}$$

Proof. Suppose that $\varepsilon = (\varepsilon_{t,1}, \varepsilon_{t,2}, \ldots, \varepsilon_{2+\frac{1}{2}}) \in (\pm 1)^{14}$ is a solution to (6.2.4). If we replace every instance of τ_4 in (6.2.4) with $-\tau_1 - \tau_2 - \tau_3$, notice that the coefficient of τ_3 is $(\varepsilon_{t,1} - \varepsilon_{t,2} - 2)$. This immediately implies (6.2.6), or equivalently, $\tau_4 \le \rho \le \tau_3$.

Recall that we are assuming that $\tau_1 \ge \tau_2 \ge \tau_3 \ge \tau_4$. This, together with the fact that $\tau_4 \le \rho \le \tau_3$, implies that

$$\xi_1 + \tau_1 \ge \xi_1 + \tau_2 \ge \xi_1 + \rho.$$

Since $\varepsilon_{1,0} = -1$ implies that $(\xi_1 + \tau_1) \le 0$, it follows that this would also imply that $\varepsilon_{1,1} = \varepsilon_{1,2} = -1$. But it can't be the case that all three $\varepsilon_{1,k}$ are -1 because if so, the coefficient of ξ_1 in (6.2.4) will not be zero. The same argument implies that $\varepsilon_{1,3} = -1$, and that the same relations hold for $\varepsilon_{3,j}$. Similarly, $+1 = \varepsilon_{2,0} \ge \varepsilon_{2,1} \ge \varepsilon_{2,2} \ge \varepsilon_{2,3} = -1$.

Using this information, we now rewrite (6.2.4) as

$$\begin{aligned} 0 &= -5\tau_1 - 3\tau_2 + (\tau_1 - \rho) + \varepsilon_{1,1}(\xi_1 + \tau_2) \\ &\quad + 2(\tau_1+\tau_2) + \varepsilon_{2,1}(\xi_2 + \tau_1 + \rho) + \varepsilon_{2,2}(\xi_2 + \tau_2 + \rho) \\ &\quad + (2\tau_1 + \tau_2 + \rho) + \varepsilon_{3,1}(\xi_3 + \tau_1 + \tau_3 + \tau_4) \\ &\quad - \varepsilon_{2-\frac{1}{2}}(\xi_1 + \xi_2 + \tau_1 + \tau_2 + \rho) + \varepsilon_{2+\frac{1}{2}}(\rho + \xi_2 + \xi_3) \\ &= \varepsilon_{1,1}(\xi_1 + \tau_2) + \varepsilon_{2,1}(\xi_2 + \tau_1 + \rho) + \varepsilon_{2,2}(\xi_2 + \tau_2 + \rho) + \varepsilon_{3,1}(\xi_3 - \tau_2) \\ &\quad - \varepsilon_{2-\frac{1}{2}}(\xi_1 + \xi_2 + \tau_1 + \tau_2 + \rho) - \varepsilon_{2+\frac{1}{2}}(\xi_2 + \xi_3 + \rho). \end{aligned} \tag{6.2.10}$$

Since the coefficient of ξ_1 is $(\varepsilon_{1,1} - \varepsilon_{2-\frac{1}{2}})$, we see that (6.2.7) is satisfied. By similarly looking at the coefficient of ξ_3, we see that (6.2.9) holds. Using this, (6.2.10) simplifies further to

$$0 = (\varepsilon_{2,2} - \varepsilon_{2-\frac{1}{2}})(\xi_2 + \tau_1 + \rho) + (\varepsilon_{2,3} - \varepsilon_{2+\frac{1}{2}})(\xi_2 + \tau_2 + \rho),$$

which is obviously true if and only if $\varepsilon_{2,2} = \varepsilon_{2-\frac{1}{2}}$ and $\varepsilon_{2,3} = \varepsilon_{2+\frac{1}{2}}$. This proves (6.2.8). Since it must be the case that $\varepsilon_{2,1} \ge \varepsilon_{2,2}$, it follows that $\varepsilon_{2-\frac{1}{2}} \ge \varepsilon_{2+\frac{1}{2}}$, as claimed.

Suppose that ε corresponds to one of the three admissible solutions to (6.2.10). Considering only the inequalities that are determined by the $\varepsilon_{j,k}$, the stated inequalities of the three solutions are immediate. So, in order to complete the proof, we must show that the inequalities imposed by $\varepsilon_{2-\frac{1}{2}}$ and $\varepsilon_{2+\frac{1}{2}}$ are superfluous: that is, they do not impose any further restriction on ξ_1, ξ_2 or ξ_3.

We check this first in the case that $(\varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) = (+1, +1)$, for which

$$\begin{aligned} -\tau_2 &\le \xi_1 \le -\rho, \\ -\tau_2 - \rho &\le \xi_2 \le \tau_1 + \tau_2, \\ -\tau_1 - \tau_2 - \rho &\le \xi_3 \le \tau_2. \end{aligned}$$

Combining the first and second sets of inequalities, we see that

$$0 \le \tau_1 - \tau_2 = (-\tau_2) + (-\tau_2 - \rho) + \tau_1 + \tau_2 + \rho \le \xi_1 + \xi_2 + \tau_1 + \tau_2 + \rho.$$

That is to say that $\varepsilon_{2-\frac{1}{2}}$ must be $+1$. In other words, the condition cut out by $\varepsilon_{2-\frac{1}{2}}$ is already a consequence of the fact that $\varepsilon_{1,1} = \varepsilon_{2,1} = \varepsilon_{2,2} = +1$. In like manner, we see that the inequality required by $\varepsilon_{2+\frac{1}{2}} = +1$ is already true by combining the second and third inequalities above.

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In each of the other two cases $(\varepsilon_{2-\frac{1}{2}}, \varepsilon_{2+\frac{1}{2}}) = (+1,-1)$ or $(-1,-1)$, one similarly shows that the inequalities imposed by $\varepsilon_{2\pm\frac{1}{2}}$ are already satisfied given those imposed by $\varepsilon_{j,k}$. □

Lemma 6.2.5 allows us to restrict the integration in (6.2.3) to the three possible bounded subsets in the s-variables, and then the integral can be bounded. However, the resulting bound is not strong enough for our application, thus requiring that we consider $a = (a_1, a_2, a_3)$ for which there exists at least one $i \in \{1, 2, 3\}$ with $a_i > 0$. In this case, (6.2.2) is no longer valid. Instead, we need to use Proposition 5.3.1. Although this introduces some technical difficulties, the Gamma-functions that occur are the same as in (6.2.2), and so Lemma 6.2.5 equally applies. This is carried out in detail in Section 6.3.

6.3. *Bounds for the shifted $p_{T,R}$ term*

Recall that

$$p^{\#}_{T,R}(\alpha) := e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}} \mathcal{F}_R(\alpha) \prod_{1\le j\ne k\le n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right). \tag{6.3.1}$$

By the inverse Lebedev-Whittaker transform (see [GK12]), we see that $p_{T,R}$ is given by

$$p_{T,R}(y) = p_{T,R}(y_1, y_2, y_3) = \frac{1}{\pi^3} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} p^{\#}_{T,R}(\alpha)\, W_\alpha(y) \frac{d\alpha_1\, d\alpha_2\, d\alpha_3}{\prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}. \tag{6.3.2}$$

To obtain a sharp bound for $p_{T,R}(y)$, we replace the Whittaker function $W_\alpha(y)$ on the right-hand side of (6.3.2) by its inverse Mellin transform

$$W_\alpha(y) = \frac{1}{(2\pi i)^3} \iiint\limits_{\mathrm{Re}(s_j)=\varepsilon} y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}\, \widetilde{W}_\alpha(s)\, ds.$$

Following (6.1.8), we can then shift the lines of integration $\mathrm{Re}(s) = \varepsilon$ to the left to $\mathrm{Re}(s) = -a = (-a_1, -a_2, -a_3)$ (with $a_1, a_2, a_3 > 0$) and express $p_{T,R}(y)$ as a sum of residues plus a shifted $p_{T,R}$ integral given by $p_{T,R}(y, -1) :=$

$$\frac{1}{(2\pi^2 i)^3} \iiint\limits_{\mathrm{Re}(s)=-a} y_1^{\frac{3}{2}-s_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} p^{\#}_{T,R}(\alpha)\, \widetilde{W}_\alpha(s) \frac{d\alpha_1\, d\alpha_2\, d\alpha_3}{\prod\limits_{1\le j\ne k\le 4} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\, ds. \tag{6.3.3}$$

Proposition 6.3.4. *Let $\varepsilon > 0$ and R sufficiently large be fixed. Let r_1, r_2, r_3 be integers. Given any choice of parameters a_1, a_2, a_3 for which $\varepsilon \le |2r_j - a_j| \le 1 - \varepsilon$ for each $j = 1, 2, 3$, we have the bound*

$$\left|p_{T,R}(y, -a)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \cdot T^{\varepsilon+4R+9+\sum\limits_{j=1}^{3}(\delta_{0,r_j}-r_j)},$$

where $\delta_{0,r_i} = 1$ if $r_i = 0$ and zero otherwise. The implicit constant depends on r, ε and R.

Proof. In order to obtain good bounds for $p_{T,R}(y, -a)$ when

$$a_i \in [2r_i - 1 + \varepsilon, 2r_i - \varepsilon] \cup [2r_i + \varepsilon, 2r_i + 1 - \varepsilon], \tag{6.3.5}$$

it is necessary to have sharp bounds for the growth of $\widetilde{W}_\alpha(s)$ on the lines $\mathrm{Re}(s_i) = -a_i$. Such bounds are known when $\mathrm{Re}(s_i) > 0$, and we can backtrack to this situation by use of the shift equation for $\widetilde{W}_\alpha(s)$ given in Proposition 5.3.1, which for any $t_1, t_2, t_3 \ge 0$ and any $s_1, s_2, s_3 \in \mathbb{C}$ states that

$$\left|\widetilde{W}_\alpha(s)\right| \ll \sum_{\ell=0}^{t_1} \sum_{k=0}^{t_2} \sum_{j=0}^{t_3} \left| \frac{Q^{t_1,t_2,t_3}_{j,k,\ell}}{\mathcal{B}_1^{t_1} \mathcal{B}_2^{t_2} \mathcal{B}_3^{t_3}} \widetilde{W}_\alpha\big(s_1 + 2(t_1 + j + k), s_2 + 2t_2, s_3 + 2(t_3 + \ell)\big) \right|, \tag{6.3.6}$$

where $\deg(Q^{t_1,t_2,t_3}_{j,k,\ell}) = 2(t_1 + 2t_2 + t_3 - j - k - \ell)$.

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Recall that

$$\mathcal{B}_1 := \mathcal{B}_1(\alpha, s) := (s_1+\alpha_1)(s_1+\alpha_2)(s_1+\alpha_3)(s_1+\alpha_4), \tag{6.3.7}$$

$$\mathcal{B}_2 := \mathcal{B}_2(\alpha, s) := (s_2+\alpha_1+\alpha_2)(s_2+\alpha_1+\alpha_3)(s_2+\alpha_1+\alpha_4) \cdot (s_2+\alpha_2+\alpha_3)(s_2+\alpha_2+\alpha_4)(s_2+\alpha_3+\alpha_4), \tag{6.3.8}$$

$$\mathcal{B}_3 := \mathcal{B}_3(\alpha, s) := (s_3-\alpha_1)(s_3-\alpha_2)(s_3-\alpha_3)(s_3-\alpha_4). \tag{6.3.9}$$

For each $i = 1, 2, 3$, let

$$t_i := \begin{cases} r_i & \text{if } a_i < 2r_i, \\ r_i + 1 & \text{if } a_i > 2r_i. \end{cases}$$

We see that for $a = (a_1, a_2, a_3)$,

$$p_{T,R}(y, -a) = \sum_{j=0}^{t_3}\sum_{k=0}^{t_2}\sum_{\ell=0}^{t_1} p_{T,R}^{(j,k,\ell)}(y, -a),$$

where $\left|p_{T,R}^{(j,k,\ell)}(y,-a)\right|$

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}} \cdot \left|\mathcal{F}_R(\alpha)\right| \cdot \left|\Gamma_R(\alpha)\right| \iiint\limits_{\mathrm{Re}(s)=-a} \left|\frac{Q_{j,k,\ell}^{t_1,t_2,t_3}}{\mathcal{B}_1^{t_1}\mathcal{B}_2^{t_2}\mathcal{B}_3^{t_3}}\right|$$
$$\cdot \int\limits_{\mathrm{Re}(t)=-\varepsilon'} \left|\frac{\Gamma_0(t,\alpha)\Gamma_1(t,s_1+2(t_1+j+k),\alpha)\Gamma_2(t,s_2+2t_2,\alpha)\Gamma_3(t,s_3+2(t_3+\ell),\alpha)}{\Gamma_{\mathrm{den}}\big(t,(s_1+2(t_1+j+k),s_2+2t_2,s_3+2(t_3+\ell)),\alpha\big)}\right| dt\, ds\, d\alpha.$$

It follows that

$$p_{T,R}^{(j,k,\ell)}(y,-a) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \cdot T^{\varepsilon+2t_1+2t_2+t_3-j-k-\ell)} \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T^2}} \iiint\limits_{\mathrm{Re}(s)=-a}$$
$$\cdot \left|\mathcal{B}_1^{-t_1}\mathcal{B}_2^{-t_2}\mathcal{B}_3^{-t_3}\right| \int\limits_{\mathrm{Re}(t)=-\varepsilon'} \left|\frac{\Gamma\left(\frac{-t+\alpha_3}{2}\right)\Gamma\left(\frac{-t+\alpha_4}{2}\right)\Gamma\left(\frac{s_1+2t_1+2j+2k+\alpha_1}{2}\right)\Gamma\left(\frac{s_1+2(t_1+j+k)+\alpha_2}{2}\right)\Gamma\left(\frac{s_1+2(t_1+j+k)+t}{2}\right)}{\Gamma\left(\frac{s_1+s_2+2(t_1+t_2+j+k)+\alpha_1+\alpha_2+t}{2}\right)\Gamma\left(\frac{s_2+s_3+2(t_2+t_3+\ell)+t}{2}\right)}\right|$$
$$\cdot \left|\Gamma\left(\frac{s_2+2t_2+\alpha_1+\alpha_2}{2}\right)\Gamma\left(\frac{s_2+2t_2+\alpha_1+t}{2}\right)\Gamma\left(\frac{s_2+2t_2+\alpha_2+t}{2}\right)\Gamma\left(\frac{s_2+2t_2+\alpha_3+\alpha_4}{2}\right)\right|$$
$$\cdot \left|\Gamma\left(\frac{s_3+2(t_3+\ell)+\alpha_1+\alpha_2+t}{2}\right)\Gamma\left(\frac{s_3+2(t_3+\ell)+\alpha_1+\alpha_3+\alpha_4}{2}\right)\Gamma\left(\frac{s_3+2(t_3+\ell)+\alpha_2+\alpha_3+\alpha_4}{2}\right)\right|$$
$$\cdot \left|\prod_{\sigma\in S_4}\left(1+\alpha_{\sigma(1)}+\alpha_{\sigma(2)}-\alpha_{\sigma(3)}-\alpha_{\sigma(4)}\right)\right|^{\frac{R}{24}} \prod_{1\le j<k\le 4} \frac{\left|\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)\right|^2}{\left|\Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)\right|^2}\, dt\, ds\, d\alpha, \tag{6.3.10}$$

where $-\varepsilon' = \mathrm{Re}(t)$ is such that the real part of the arguments of all of the Gamma functions appearing here are positive. Hence $0 < \varepsilon' < \varepsilon$.

Note that besides the presence of the additional polynomials $\mathcal{B}_i^{r_i}$, the Gamma factors occurring in (6.3.10) are the same as that of (6.2.3). In any event, the exponential zero set is precisely the same as that determined in Lemma 6.2.5 and can be any one of the three exponential zero sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ as given in Lemma 6.2.5.

Note that we can get from $\mathcal{R}_2$ to $\mathcal{R}_1$ by making the change of variables

$$(\xi_2, \xi_3) \mapsto (-\xi_3 - \rho, \xi_2 - \tau_1).$$

Similarly, to go from $\mathcal{R}_2$ to $\mathcal{R}_3$, one makes the change of variables

$$(\xi_1, \xi_2) \mapsto (\xi_2 - \tau_1, -\xi_1 + \rho).$$

Either of these transformations result in no change in (5.4.1) because it leaves both the measure ds and the region over which we are integrating $\mathrm{Re}(s) = -a$ invariant. This implies any bound obtained for a given $\mathcal{R}_i$ holds for each of the other choices as well.

Recall the exponential zero set $\mathcal{R}_2$ given by

$$\begin{aligned} \tau_4 \;\le\; \rho \;\le\; \tau_3, \quad -\tau_2 \;\le\; \xi_1 \;\le\; -\rho, \\ -\tau_1 - \rho \;\le\; \xi_2 \;\le\; -\tau_2 - \rho, \quad -(\tau_1 + \tau_2) - \rho \;\le\; \xi_3 \;\le\; \tau_2. \end{aligned}$$

Recall, also, that $\mathrm{Im}(\alpha_j) = \tau_j$ (for $j = 1, 2, 3, 4$ with $\tau_4 = -\tau_1 - \tau_2 - \tau_3$), $\mathrm{Im}(t) = \rho$, and $\mathrm{Im}(s_i) = \xi_i$, (for $i = 1, 2, 3$). Accordingly, we write $s = -a + i\xi,\ \alpha = i\tau$ with $\xi = (\xi_1, \xi_2, \xi_3)$ and $\tau = (\tau_1, \tau_2, \tau_3)$.

We now replace each Gamma factor in the integral (6.3.10) using the Stirling bound (6.2.1). A bound for the integral is given by integrating over $\mathcal{R}_2$ (the exponential zero set) and just using the polynomial bound coming from the Stirling bound (6.2.1). It follows (up to a constant dependant on a_k, R and $\varepsilon > 0$) that $p_{T,R}^{(j,k,\ell)}(y, -a)$ is bounded by

$$\begin{aligned}
& y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \cdot T^{\varepsilon+2(t_1+2t_2+t_3-j-k-\ell)} \\
& \cdot \iiint\limits_{\substack{\tau_1 \ge \tau_2 \ge \tau_3 \ge \tau_4 \\ |\tau_1|,|\tau_2|,|\tau_3|,|\tau_4| \ll T^{1+\varepsilon}}} \int\limits_{\rho=\tau_4}^{\tau_3} \int\limits_{\xi_1=-\tau_2}^{-\rho} \int\limits_{\xi_2=-\tau_1-\rho}^{-\tau_2-\rho} \int\limits_{\xi_3=-\tau_1-\tau_2-\rho}^{\tau_2} \left|\mathcal{B}_1^{-t_1} \mathcal{B}_2^{-t_2} \mathcal{B}_3^{-t_3}\right| \\
& \cdot \frac{(1+|\tau_3-\rho|)^{-\frac{1}{2}} \left(1+|\tau_4-\rho|\right)^{-\frac{1}{2}}}{(1+|\xi_1+\xi_2+\tau_1+\tau_2+\rho|)^{-\frac{1}{2}+j+k+t_1-\frac{a_1}{2}+t_2-\frac{a_2}{2}} (1+|\xi_2+\xi_3+\rho|)^{-\frac{1}{2}+\ell+t_3-\frac{a_3}{2}+t_2-\frac{a_2}{2}}} \\
& \cdot (1+|\xi_1+\tau_1|)^{j+k+t_1-\frac{a_1+1}{2}} (1+|\xi_1+\tau_2|)^{j+k+t_1-\frac{a_1+1}{2}} \left(1+|\xi_1+\rho|\right)^{j+k+t_1-\frac{a_1+1}{2}} \\
& \cdot (1+|\xi_2+\tau_1+\tau_2|)^{t_2-\frac{a_2+1}{2}} (1+|\xi_2+\tau_1+\rho|)^{t_2-\frac{a_2+1}{2}} \left(1+|\xi_2+\tau_2+\rho|\right)^{t_2-\frac{a_2+1}{2}} \\
& \cdot \left(1+|\xi_2+\tau_3+\tau_4|\right)^{t_2-\frac{a_2+1}{2}} \left(1+|\xi_3+\tau_1+\tau_2+\rho|\right)^{\ell+t_3-\frac{a_3+1}{2}} \left(1+|\xi_3+\tau_1+\tau_3+\tau_4|\right)^{\ell+t_3-\frac{a_3+1}{2}} \\
& \cdot \left(1+|\xi_3+\tau_2+\tau_3+\tau_4|\right)^{\ell+t_3-\frac{a_3+1}{2}} K_R(\tau)\, d\xi_3\, d\xi_2\, d\xi_1\, d\rho\, d\tau_3\, d\tau_2\, d\tau_1,
\end{aligned}$$

where

$$\begin{aligned} K_R(\tau) := \Big(1 + |\tau_1 + \tau_2 - \tau_3 - \tau_4|\Big)^{\frac{R}{3}} \Big(1 + |\tau_1 + \tau_3 - \tau_2 - \tau_4|\Big)^{\frac{R}{3}} \\ \cdot \Big(1 + |\tau_1 + \tau_4 - \tau_2 - \tau_3|\Big)^{\frac{R}{3}} \cdot \prod_{1 \le j < k \le 4} \big(1 + \tau_j - \tau_k\big)^{1+\frac{R}{2}}. \end{aligned} \tag{6.3.11}$$

Remark 6.3.12. The exponents of $(1 + |\tau_j - \rho|)$ etc. should be modified by $\pm\varepsilon$ as $\mathrm{Re}(t) = -\varepsilon$, but we have absorbed these into the term T^{ε}. To simplify the exposition, we drop these εs in what follows since they do not affect the conclusion.

Lemma 6.3.13. *Suppose that $a_i = 2r_i + \gamma_i a_i'$, where*

$$\gamma_i = \begin{cases} -1 & \text{if } a_i < 2r_i, \\ 1 & \text{if } a_i > 2r_i, \end{cases}$$

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and $0 < a_i' < 1$ *for each* $i = 1, 2, 3$. *Throughout the domain of integration, in the integrand for* $p_{T,R}^{(j,k,\ell)}(y, -a)$ *above, the expression*

$$\frac{\left(1+|\xi_2+\tau_1+\tau_2|\right)^{t_2-\frac{a_2+1}{2}}\left(1+|\xi_2+\tau_1+\rho|\right)^{t_2-\frac{a_2+1}{2}}\left(1+|\xi_2+\tau_2+\rho|\right)^{t_2-\frac{a_2+1}{2}}\left(1+|\xi_2+\tau_3+\tau_4|\right)^{t_2-\frac{a_2+1}{2}}}{\left(1+|\xi_1+\xi_2+\tau_1+\tau_2+\rho|\right)^{t_2-\frac{a_2+1}{2}}\left(1+|\xi_2+\xi_3+\rho|\right)^{t_2-\frac{a_2+1}{2}}}$$

is bounded by a constant multiple of $T^{\varepsilon+\frac{1+\gamma_2}{2}}$. *Similarly,*

$$\frac{\left(1+|\xi_1+\tau_1|\right)^{j+k+t_1-\frac{a_1+1}{2}}\left(1+|\xi_1+\tau_2|\right)^{j+k+t_1-\frac{a_1+1}{2}}\left(1+|\xi_1+\rho|\right)^{j+k+t_1-\frac{a_1+1}{2}}}{\left(1+|\xi_1+\xi_2+\tau_1+\tau_2+\rho|\right)^{j+k+t_1-\frac{a_1}{2}}} \ll T^{\varepsilon+2(j+k)+\frac{1+\gamma_1}{2}},$$

and

$$\frac{\left(1+|\xi_3+\tau_1+\tau_2+\rho|\right)^{\ell+t_3-\frac{a_3+1}{2}}\left(1+|\xi_3+\tau_1+\tau_3+\tau_4|\right)^{\ell+t_3-\frac{a_3+1}{2}}\left(1+|\xi_3+\tau_2+\tau_3+\tau_4|\right)^{\ell+t_3-\frac{a_3+1}{2}}}{\left(1+|\xi_2+\xi_3+\rho|\right)^{\ell+t_3-\frac{a_3}{2}}} \ll T^{\varepsilon+2\ell+\frac{1+\gamma_3}{2}}.$$

Proof. Note that $t_i = r_i + \frac{1+\gamma_i}{2}$. This allows us to simplify many of the exponents. For example,

$$\left(1+|\xi_i+f(\tau,\rho)|\right)^{t_i-\frac{a_i+1}{2}} = \left(1+|\xi_i+f(\tau,\rho)|\right)^{\gamma_i\frac{1-a_i'}{2}} \ll \left(1+|\xi_i+f(\tau,\rho)|\right)^{\frac{1+\gamma_i}{4}} \tag{6.3.14}$$

for each of the cases $i = 1, 2, 3$ and $f(\tau, \rho)$ appearing above.

In a similar manner, one sees that

$$\begin{aligned}\left(1+|\xi_1+\xi_2+\tau_1+\tau_2|\right)^{\frac{1}{2}-j-k-t_1+\frac{a_2}{2}} &\ll \left(1+|\xi_1+\xi_2+\tau_1+\tau_2|\right)^{\frac{1}{2}-j-k-\frac{1+\gamma_1}{4}} \\ &\ll \frac{T^{\varepsilon+\frac{1}{2}}}{\left(1+|\xi_1+\xi_2+\tau_1+\tau_2|\right)^{j+k+\frac{1+\gamma_1}{4}}},\end{aligned} \tag{6.3.15}$$

$$\left(1+|\xi_2+\xi_3+\rho|\right)^{\frac{1}{2}-\ell-t_3+\frac{a_3}{2}} \ll \left(1+|\xi_2+\xi_3+\rho|\right)^{\frac{1}{2}-\ell-\frac{1+\gamma_3}{4}} \ll \frac{T^{\varepsilon+\frac{1}{2}}}{\left(1+|\xi_2+\xi_3+\rho|\right)^{\ell+\frac{1+\gamma_3}{4}}}. \tag{6.3.16}$$

Successively making the change of variables

$$\xi_1 \mapsto \xi_1 - \tau_2, \quad \xi_2 \mapsto \xi_2 - \tau_1 - \rho, \quad \xi_3 \mapsto \xi_3 - \tau_1 - \tau_2 - \rho, \quad \rho \mapsto \rho + \tau_4,$$

and the substitutions

$$T_1 = \tau_1 - \tau_2, \quad T_2 = \tau_2 - \tau_3, \quad T_3 = \tau_3 - \tau_4 = \tau_1 + \tau_2 + 2\tau_3,$$

the bounds of integration in $p_{T,R}^{(j,k,\ell)}(y, -a)$ become $0 \le T_1, T_2, T_3 \le T^{1+\varepsilon}$, $0 \le \rho \le T_3$, $0 \le \xi_1 \le T_2 + T_3 - \rho$, $0 \le \xi_2 \le T_1$ and $0 \le \xi_3 \le T_2 + \rho$, and the terms involving ξ_1 are

$$\frac{\left(1+T_1+\xi_1\right)^{j+k+\frac{1+\gamma_1}{4}}\left(1+\xi_1\right)^{j+k+\frac{1+\gamma_1}{4}}\left(1+T_2+T_3-\rho-\xi_1\right)^{j+k+\frac{1+\gamma_1}{4}}}{\left(1+\xi_1+\xi_2\right)^{j+k+\frac{1+\gamma_1}{4}}},$$

which, since $0 \le j, k$ and $0 \le T_i \le T^{1+\varepsilon}$, is bounded by $T^{\varepsilon+\frac{1}{2}+2(j+k)+\frac{1+\gamma_1}{2}}$, as claimed. The bound for ξ_3 follows in precisely the same fashion, and that for ξ_2 is similar. □

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Putting this together, we obtain the bound

$$\begin{aligned}\left|p_{T,R}^{(j,k,\ell)}(y,-a)\right| \ll\ & y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+2(t_1+2t_2+t_3)+\sum_{i=1}^{3}\frac{1+\gamma_i}{2}} \\ &\cdot \iiint\limits_{\substack{\tau_1\ge\tau_2\ge\tau_3\ge\tau_4\\ |\tau_1|,|\tau_2|,|\tau_3|,|\tau_4|\ll T^{1+\varepsilon}}} \int_{\rho=\tau_4}^{\tau_3}\int_{\xi_1=-\tau_2}^{-\rho}\int_{\xi_2=-\tau_1-\rho}^{-\tau_2-\rho}\int_{\xi_3=-\tau_1-\tau_2-\rho}^{\tau_2} (1+|\tau_3-\rho|)^{-\frac{1}{2}}\,(1+|\tau_4-\rho|)^{-\frac{1}{2}} \\ &\cdot\Big[(1+|\xi_1+\tau_1|)(1+|\xi_1+\tau_2|)(1+|\xi_1+\tau_3|)(1+|\xi_1+\tau_4|)\Big]^{-t_1} \\ &\cdot\Big[(1+|\xi_2+\tau_1+\tau_2|)(1+|\xi_2+\tau_1+\tau_3|)(1+|\xi_2+\tau_1+\tau_4|)(1+|\xi_2+\tau_2+\tau_3|)\Big]^{-t_2} \\ &\cdot\Big[(1+|\xi_2+\tau_2+\tau_4|)(1+|\xi_2+\tau_3+\tau_4|)\Big]^{-t_2} \\ &\cdot\Big[(1+|\xi_3-\tau_1|)(1+|\xi_3-\tau_2|)(1+|\xi_3-\tau_3|)(1+|\xi_3-\tau_4|)\Big]^{-t_3} \\ &\cdot K_R(\tau)\, d\xi_3\, d\xi_2\, d\xi_1\, d\rho\, d\tau_3\, d\tau_2\, d\tau_1.\end{aligned}$$

Next, we successively make the change of variables

$$\xi_1\mapsto\xi_1-\tau_2,\qquad \xi_2\mapsto\xi_2-\tau_1-\rho,\qquad \xi_3\mapsto\xi_3-\tau_1-\tau_2-\rho,\qquad \rho\mapsto\rho+\tau_4,$$

and the substitutions

$$T_1=\tau_1-\tau_2,\quad T_2=\tau_2-\tau_3,\quad T_3=\tau_3-\tau_4=\tau_1+\tau_2+2\tau_3.$$

Then by abuse of notation we may replace $K_R(\tau)$ by $K_R(T_1,T_2,T_3)$, where

$$\begin{aligned}K_R(T_1,T_2,T_3) :=\ & \Big(1+T_1+2T_2+T_3\Big)^{\frac{R}{3}}\Big(1+T_1+T_3\Big)^{\frac{R}{3}}\Big(1+|T_1-T_3|\Big)^{\frac{R}{3}} \\ &\cdot\Big(1+T_1\Big)^{1+\frac{R}{2}}\Big(1+T_2\Big)^{1+\frac{R}{2}}\Big(1+T_3\Big)^{1+\frac{R}{2}}\Big(1+T_1+T_2\Big)^{1+\frac{R}{2}} \\ &\cdot\Big(1+T_2+T_3\Big)^{1+\frac{R}{2}}\Big(1+T_1+T_2+T_3\Big)^{1+\frac{R}{2}} \\ \ll\ & T^{\varepsilon+3+\frac{13R}{6}}\cdot\Big(1+|T_1-T_3|\Big)^{\frac{R}{3}}\Big(1+T_1\Big)^{1+\frac{R}{2}}\Big(1+T_2\Big)^{1+\frac{R}{2}}\Big(1+T_3\Big)^{1+\frac{R}{2}}.\end{aligned}\tag{6.3.17}$$

It follows that

$$\begin{aligned}\left|p_{T,R}^{(j,k,\ell)}(y,-a)\right| \ll\ & y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+\frac{13R}{6}+3}+2(t_1+2t_2+t_3)+\sum_{i=1}^{3}\frac{1+\gamma_i}{2} \\ &\cdot\iiint\limits_{0\le T_1,T_2,T_3\le T^{1+\varepsilon}}\int_{\rho=0}^{T_3}\int_{\xi_1=0}^{T_2+T_3-\rho}\int_{\xi_2=0}^{T_1}\int_{\xi_3=0}^{T_2+\rho}(1+T_3-\rho)^{-\frac{1}{2}}\,(1+\rho)^{-\frac{1}{2}} \\ &\cdot\Big[(1+\xi_1+T_1)(1+\xi_1)(1+|\xi_1-T_2|)(1+T_2+T_3-\xi_1)\Big]^{-t_1} \\ &\cdot\Big[(1+\xi_2+T_2+T_3-\rho)(1+\xi_2+T_3-\rho)(1+|\xi_2-\rho|)\Big]^{-t_2} \\ &\cdot\Big[(1+|\xi_2-T_1+T_3-\rho|)(1+T_1+\rho-\xi_2)(1+T_1-\xi_2+T_2+\rho)\Big]^{-t_2} \\ &\cdot\Big[(1+T_1+T_2+\rho-\xi_3)(1+T_2+\rho-\xi_3)(1+|\xi_3-\rho|)(1+\xi_3+T_3-\rho)\Big]^{-t_3} \\ &(1+|T_1-T_3|)^{\frac{R}{3}}(1+T_1)^{1+\frac{R}{2}}(1+T_2)^{1+\frac{R}{2}}(1+T_3)^{1+\frac{R}{2}}\, d\xi_3\, d\xi_2\, d\xi_1\, d\rho\, dT_3\, dT_2\, dT_1.\end{aligned}$$

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Now for $0 \le \xi_1 \le T_2 + T_3 - \rho$, $0 \le \xi_2 \le T_1$, $0 \le \xi_3 \le T_2 + \rho$ and $0 \le \rho \le T_3$,

$$\begin{aligned} 1+\xi_1+T_1 &\ge 1+T_1 \\ 1+T_2+T_3-\rho+\xi_2 &\ge 1+T_2+T_3-\rho \ \ge\ 1+T_2 \\ 1+T_1+T_2+\rho-\xi_2 &\ge 1+T_2+\rho \ \ge\ 1+T_2 \\ 1+T_1+T_2+\rho-\xi_3 &\ge 1+T_1. \end{aligned}$$

Hence, we have the bounds

$$\begin{aligned} (1+\xi_1+T_1)^{-t_1} &\le (1+T_1)^{-t_1} \\ (1+T_2+T_3-\rho+\xi_2)^{-t_2} &\le (1+T_2)^{-t_2} \\ (1+T_1+T_2+\rho-\xi_2)^{-t_2} &\le (1+T_2)^{-t_2} \\ (1+T_1+T_2+\rho-\xi_3)^{-t_3} &\le (1+T_1)^{-t_3}. \end{aligned}$$

Inserting these bounds, we see that

$$\begin{aligned} \left|p_{T,R}^{(j,k,\ell)}(y,-a)\right| \ll\ & y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+\frac{13R}{6}+3+2(t_1+2t_2+t_3)+\sum_{i=1}^{3}\frac{1+\gamma_i}{2}} \\ & \cdot \iiint\limits_{0\le T_1,T_2,T_3\le T^{1+\varepsilon}} (1+T_1)^{-t_1-t_3}(1+T_2)^{-2t_2} \int\limits_{\rho=0}^{T_3} (1+T_3-\rho)^{-\frac{1}{2}}(1+\rho)^{-\frac{1}{2}} \\ & \cdot \int\limits_{\xi_1=0}^{T_2+T_3-\rho} (1+\xi_1)^{-t_1}(1+|\xi_1-T_2|)^{-t_1}(1+T_2+T_3-\xi_1)^{-t_1}\, d\xi_1 \\ & \cdot \int\limits_{\xi_2=0}^{T_1} (1+\xi_2+T_3-\rho)^{-t_2}(1+|\xi_2-\rho|)^{-t_2}(1+|\xi_2-T_1+T_3-\rho|)^{-t_2} \\ & \qquad \cdot (1+T_1+\rho-\xi_2)^{-t_2}\, d\xi_2 \\ & \cdot \int\limits_{\xi_3=0}^{T_2+\rho} (1+T_2+\rho-\xi_3)^{-t_3}(1+|\xi_3-\rho|)^{-t_3}(1+\xi_3+T_3-\rho)^{-t_3}\, d\xi_3\, d\rho \\ & \cdot (1+|T_1-T_3|)^{\frac{R}{3}}(1+T_1)^{1+\frac{R}{2}}(1+T_2)^{1+\frac{R}{2}}(1+T_3)^{1+\frac{R}{2}}\, dT_3\, dT_2\, dT_1. \end{aligned} \tag{6.3.18}$$

Let us denote the integral in ξ_i $(1 \le i \le 3)$, in (6.3.18), by $\mathcal{I}_i(r_i)$. We have

Lemma 6.3.19. *For each $i = 1, 2, 3$, $\mathcal{I}_i(0) \ll T^{\varepsilon+1}$. Otherwise, if $r \ge 1$, we have the following bounds:*

$$\begin{aligned} \mathcal{I}_1(r), \mathcal{I}_3(r) &\ll (1+T_2)^{-r}(1+T_3)^{-r}, \\ \mathcal{I}_2(r) &\ll \left((1+T_3)^{-2r} + (1+T_1)^{-2r}\right)(1+|T_1-T_3|)^{-r}. \end{aligned}$$

Proof. The case of $r = 0$ is clear given that $0 \le T_i \le T^{1+\varepsilon}$. Thus, we may assume now that $r \ge 1$. By expanding the region of integration, since the integrand is positive, we have that

$$\begin{aligned} \mathcal{I}_1 &\ll \int_{\xi_1=0}^{T_2+T_3} (1+\xi_1)^{-r}(1+|\xi_1-T_2|)^{-r}(1+T_2+T_3-\xi_1)^{-r}\, d\xi_1 \\ &= \int_{\xi_1=B_1}^{B_3} (1+|\xi_1-B_1|)^{-r}(1+|\xi_1-B_2|)^{-r}(1+|\xi_1-B_3|)^{-r}\, d\xi_1, \end{aligned}$$

where

$$B_1 = 0, \qquad B_2 = T_2, \qquad B_3 = T_2 + T_3.$$

Applying Lemma A.3, it follows that

$$\mathcal{I}_1 \ll \left(1+T_1\right)^{-r}\left(1+T_2\right)^{-r}\left(1+T_3\right)^{-r}.$$

Replacing ρ by $T_3 - \rho$ in $\mathcal{I}_1$, one finds that $\mathcal{I}_3 = \mathcal{I}_1$, and hence the same bound holds for $i = 3$.

To bound $\mathcal{I}_2$, we also expand the region of integration, whence

$$\begin{aligned}\mathcal{I}_2(r_2) \ll \int\limits_{\xi_2=\rho-T_3}^{T_1+\rho} & \left(1+\xi_2+T_3-\rho\right)^{-r}\left(1+|\xi_2-\rho|\right)^{-r} \\ & \cdot \left(1+|\xi_2-T_1+T_3-\rho|\right)^{-r}\left(1+T_1+\rho-\xi_2\right)^{-r}\, d\xi_2 \\ = \int\limits_{\xi_2=0}^{T_1} & \prod_{\ell=1}^{4}\left(1+|\xi_2-B_\ell|\right)^{-r}\, d\xi_2,\end{aligned}$$

where

$$B_1 = \rho - T_3, \qquad B_2 = \rho, \qquad B_3 = \rho + T_1 - T_3, \qquad B_4 = \rho + T_1.$$

Note that

$$B_2 \le B_3 \quad \Longleftrightarrow \quad T_3 \le T_1.$$

In either event, using Lemma A.3 gives the claimed bound. □

Inserting these bounds into (6.3.18), the integral in the ρ-variable is

$$\int\limits_{\rho=0}^{T_3} (1+\rho)^{-\frac{1}{2}}(1+T_3-\rho)^{-\frac{1}{2}}\, d\rho \ll 1,$$

where this bound is obtained via a direct application of Lemma A.1.

Plugging these bounds into (6.3.18), it follows that

$$\begin{aligned}\left|p_{T,R}^{(j,k,\ell)}(y,-a)\right| \ll{}& y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \cdot T^{\varepsilon+\frac{13R}{6}+3+2(r_1+2r_2+r_3)+\delta_{0,t_1}+\delta_{0,t_2}+\delta_{0,t_3}+\sum_{i=1}^{3}\frac{1+\gamma_i}{2}} \\ & \cdot \iiint\limits_{0\le T_1,T_2,T_3\le T^{1+\varepsilon}} \left(1+T_1\right)^{1+\frac{R}{2}-t_1-t_3}\left(\left(1+T_3\right)^{-2t_2}+\left(1+T_1\right)^{-2t_2}\right) \\ & \cdot \left(1+T_3\right)^{1+\frac{R}{2}-t_1-t_3}\left(1+|T_1-T_3|\right)^{\frac{R}{3}-t_2}\left(1+T_2\right)^{1+\frac{R}{2}-t_1-2t_2-t_3}\, dT_3\, dT_2\, dT_1. \qquad (6.3.20)\end{aligned}$$

Remark 6.3.21. As long as

$$1+\frac{R}{2} \ge t_1 + 2t_2 + t_3 \qquad \text{and} \qquad \frac{R}{3} \ge t_2,$$

each of the terms in the integrand, in (6.3.20), is of the form $\left(1+T_j\right)^{\alpha_j}$ for $j = 1, 2, 3$ and some $\alpha_j \ge 0$. Since the integral is over the domain $0 \le T_1, T_2, T_3 \ll T^{1+\varepsilon}$, it follows that each of these terms is bounded by T^{α}.

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In light of the above remark, we obtain the bound

$$\left|p_{T,R}^{(j,k,\ell)}(y,-a)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} \cdot T^{\varepsilon+4R+9+\delta_{0,t_1}+\delta_{0,t_2}+\delta_{0,t_3}-(r_1+r_2+r_3)}. \tag{6.3.22}$$

Since $0 \le r_i \le t_i$, it follows that $\delta_{0,t_i} \le \delta_{0,r_i}$, and thus (6.3.22) implies the desired result. □

6.4. Bounds for the single residue terms

We need to deal with two types of single residue terms. These are defined in equations (6.4.1) and (6.4.17). We show in Propositions 6.4.2 and 6.4.18, respectively, that the bounds from each of these are small.

The first single residue term that we need to consider is

$$p_{T,R}^{1,\delta}(y;(-a_2,-a_3)) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=2,3}} y_1^{\frac{3}{2}-p_1} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}\, Q_\delta(s,\alpha)$$
$$\cdot\, \mathfrak{F}_R(\alpha)\,\Gamma_R(\alpha)\cdot \frac{\prod\limits_{j=2}^{4}\Gamma(\frac{\alpha_j-\alpha_1}{2}-\delta)\Gamma(\frac{s_2+\alpha_1+\alpha_j}{2})\Gamma(\frac{s_3-\alpha_j}{2})}{\Gamma(\frac{s_2+s_3+\alpha_1}{2}+\delta)}\, ds_2 ds_3\, d\alpha, \tag{6.4.1}$$

where Q_δ is a polynomial (see Section 5.2) of degree $\le 3\delta$, $\mathfrak{F}_R(\alpha)$ is as in (3.1.2), $\Gamma_R(\alpha)$ is as in (6.1.1), and $p_1 = -\alpha_1 - 2\delta$ for $\delta = 0, 1, \ldots, r_1 - 1$.

Proposition 6.4.2. *Let $r_1 \ge 1$, $r_2, r_3 \ge 0$ be integers, and $0 < \varepsilon < 1$. Suppose a_1, a_2, a_3 satisfy the hypotheses of Theorem 4.0.3. If $0 \le \delta \le r_1 - 1$, then*

$$\left|p_{T,r}^{1,\delta}(y;(-a_2,-a_3))\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+9+\delta_{0,r_2}+\delta_{0,r_3}-(r_1+r_2+r_3)}. \tag{6.4.3}$$

Proof. In order to bound $p_{T,R}^{1,\delta}(y;(-a_2,-a_3))$, we will need to shift the lines of integration in the α_1 variable. In doing so, we will pick up residues. In other words, we may write

$$p_{T,R}^{1,\delta}(y;(-a_2,-a_3)) = p_{T,R}^{1,\delta}(y;(-a_2,-a_3),\kappa) + \sum \text{Residues},$$

where

$$p_{T,R}^{1,\delta}(y;(-a_2,-a_3),\kappa) := \iiint\limits_{\mathrm{Re}(\alpha)=\kappa} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=2,3}} y_1^{\frac{3}{2}+\alpha_1+2\delta} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}$$
$$\cdot \frac{\Gamma\left(\frac{\alpha_2-\alpha_1}{2}-\delta\right)}{\Gamma\left(\frac{\alpha_2-\alpha_1}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_1-\alpha_2}{4}\right)\Gamma\left(\frac{2+R+\alpha_2-\alpha_1}{4}\right)}{\Gamma\left(\frac{\alpha_1-\alpha_2}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_2-\alpha_3}{4}\right)\Gamma\left(\frac{2+R+\alpha_3-\alpha_2}{4}\right)}{\Gamma\left(\frac{\alpha_2-\alpha_3}{2}\right)\Gamma\left(\frac{\alpha_3-\alpha_2}{2}\right)}$$
$$\cdot \frac{\Gamma\left(\frac{\alpha_3-\alpha_1}{2}-\delta\right)}{\Gamma\left(\frac{\alpha_3-\alpha_1}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_1-\alpha_3}{4}\right)\Gamma\left(\frac{2+R+\alpha_3-\alpha_1}{4}\right)}{\Gamma\left(\frac{\alpha_1-\alpha_3}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_2-\alpha_4}{4}\right)\Gamma\left(\frac{2+R+\alpha_4-\alpha_2}{4}\right)}{\Gamma\left(\frac{\alpha_2-\alpha_4}{2}\right)\Gamma\left(\frac{\alpha_4-\alpha_2}{2}\right)}$$
$$\cdot \frac{\Gamma\left(\frac{\alpha_4-\alpha_1}{2}-\delta\right)}{\Gamma\left(\frac{\alpha_4-\alpha_1}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_1-\alpha_4}{4}\right)\Gamma\left(\frac{2+R+\alpha_4-\alpha_1}{4}\right)}{\Gamma\left(\frac{\alpha_1-\alpha_4}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+\alpha_4-\alpha_3}{4}\right)\Gamma\left(\frac{2+R+\alpha_3-\alpha_4}{4}\right)}{\Gamma\left(\frac{\alpha_4-\alpha_3}{2}\right)\Gamma\left(\frac{\alpha_3-\alpha_4}{2}\right)}$$

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$$\cdot \frac{\Gamma\left(\frac{s_2+\alpha_1+\alpha_2}{2}\right)\Gamma\left(\frac{s_2+\alpha_1+\alpha_3}{2}\right)\Gamma\left(\frac{s_2+\alpha_1+\alpha_4}{2}\right)\Gamma\left(\frac{s_3-\alpha_2}{2}\right)\Gamma\left(\frac{s_3-\alpha_3}{2}\right)\Gamma\left(\frac{s_3-\alpha_4}{2}\right)}{\Gamma(\frac{s_2+s_3+\alpha_1}{2}+\delta)}$$
$$\cdot \mathcal{F}_R(\alpha)\, Q_\delta(s,\alpha)\, ds_2\, ds_3\, d\alpha. \tag{6.4.4}$$

and the residues that appear depend on the particular choice of $\kappa = (\kappa_1, \kappa_2, \kappa_3)$. For example, if $\kappa_j = 0$ for $j = 1, 2, 3$, then equations (6.4.1) and (6.4.4) are the same, meaning there are no residues.

Our goal will be for the given value of $2r_j - 1 + \varepsilon \le a_j \le 2r_j - \varepsilon$ to shift the lines of integration of the α variables from $\mathrm{Re}(\alpha) = 0$ to $\mathrm{Re}(\alpha) = \kappa = (\kappa_1, \kappa_2, \kappa_3)$ with

$$\mathrm{Re}(\alpha_1 + 2\delta) = \kappa_1 + 2\delta = a_1, \qquad \kappa_2 = 0 = \kappa_3.$$

To help clarify the structure of the proof, we now give a brief outline of what is to follow. As described above, in order for the exponent of y_1 to be correct, we need to shift the line of integration in the variable α_1 from $\mathrm{Re}(\alpha_1) = 0$ to $\mathrm{Re}(\alpha_1) = \kappa_1 = a_1 - 2\delta$. In Lemma 6.4.5, we identify the poles that are passed in making this shift, of which there are three types.

After establishing this lemma, we bound the shifted integral (6.4.4). The residue at any one of the three types of poles is essentially the same as that at any of the other two up to a simple transformation that doesn't effect the rest of the argument. Hence, it suffices to pick any one of the types of poles from Lemma 6.4.5. Having made a choice, we then show that it is, similar to before, necessary to shift in the variable α_2 in order for the exponent of y_1 to be correct. Unlike before, however, we are able to show in Lemma 6.4.13 that in shifting α_2, no further poles are encountered. Hence, it suffices to bound the shifted terms, which we then do. Since both (6.4.4) and the shifted residue term satisfy (6.4.3), taken together, this proves the proposition.

Lemma 6.4.5. *In shifting the line of integration in* α_1 *from* $\mathrm{Re}(\alpha_1) = 0$ *to* $\mathrm{Re}(\alpha_1) = a_1 - 2\delta$*, poles are crossed only at the points* $\alpha_1 = q$ *for*

$$q \in \left\{ \begin{array}{c} -s_2 - \alpha_2 - 2(r_2 - \delta_2), \\ -s_2 - \alpha_3 - 2(r_2 - \delta_2), \\ -s_3 - \alpha_2 - \alpha_3 - 2(r_3 - \delta_3) \end{array} \middle| \begin{array}{c} 0 \le \delta_j \le r_j \\ \delta_j \le r_1 - \delta \end{array} \right\}. \tag{6.4.6}$$

Proof. We first consider the ratio

$$\frac{\Gamma(\frac{\alpha_j-\alpha_1}{2} - \delta)}{\Gamma(\frac{\alpha_j-\alpha_1}{2})\Gamma(\frac{\alpha_1-\alpha_j}{2})},$$

for $j = 2, 3$, or 4. The numerator has a simple pole when $\frac{\alpha_j-\alpha_1}{2} - \delta$ is a nonpositive integer, but then $\frac{\alpha_j-\alpha_1}{2}$ is also an integer, so that there is a pole in the denominator as well. In other words, this ratio is holomorphic in α_1.

For each of the terms

$$\Gamma\left(\frac{2 + R + \alpha_j - \alpha_k}{4}\right), \qquad (1 \le j \ne k \le 4)$$

if R is sufficiently large, again no poles are crossed.

Examining the term $\Gamma(\frac{s_2+\alpha_1+\alpha_j}{2})$ with $j = 2$ or $j = 3$, assuming that $\mathrm{Re}(\alpha_j) = 0$ and $\mathrm{Re}(s_2) = -a_2$, we see that poles occur at $\mathrm{Re}(\alpha_1) = \kappa_1$, where $\kappa_1 = a_2 - 2(r_2 - \delta_2)$ for nonnegative integers $r_2 - \delta_2$. (Note that we have chosen this notation because it enumerates the δ_2-th pole that is crossed as one starts at $\mathrm{Re}(\alpha_1) = 0$ and moves to $\mathrm{Re}(\alpha_1) = a_1 - 2\delta$.) Hence

$$0 \le \mathrm{Re}(\alpha_1) = a_2 - 2(r_2 - \delta_2) < 2r_2 + 1 - 2(r_2 - \delta_2) = 1 + 2\delta_2,$$

which implies that $\delta_2 > -1/2$, hence $\delta_2 \ge 0$ (since δ_2 is an integer). And of course $\delta_2 \le r_2$, since we are assuming that $r_2 - \delta_2$ is nonnegative.

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However, since the shift in α_1 is going from $\kappa_1 = 0$ to $\kappa_1 = a_1 - 2\delta$, the largest value of δ_2 that can in fact yield a pole is the one for which

$$a_2 - 2(r_2 - \delta_2) \le a_1 - 2\delta < a_2 - 2(r_2 - (\delta_2 + 1)).$$

It's readily checked that these inequalities imply $\delta_2 \le r_1 - \delta$.

The details for evaluating the poles of $\Gamma(\frac{s_3-\alpha_4}{2})$ are similar. We leave the details to the reader. □

Step 1: Bounding the shifted integral $p^{1,\delta}_{T,R}$

Before dealing with the residues, we first bound $p^{1,\delta}_{T,R}(y;(-a_2,-a_3),(\kappa_1,0,0))$.

Notice that we may interchange τ_1, τ_2 and τ_3 without affecting the integrand in (6.4.4). Therefore, we may assume

$$-\tau_1 - \tau_2 - \tau_3 = \tau_4 \le \tau_3 \le \tau_2.$$

It follows that the exponential factor is

$$\begin{aligned}\mathcal{E} = -2\tau_1 - 4\tau_2 - 2\tau_3 - |\xi_2+\xi_3+\tau_1| + |\xi_2+\tau_1+\tau_2| + |\xi_2+\tau_1+\tau_3| \\ + |\xi_2+\tau_1+\tau_4| + |\xi_3-\tau_2| + |\xi_3-\tau_3| + |\xi_3-\tau_4|,\end{aligned}$$

and using the method of Lemma 6.2.5, it is easy to show that there are two possible exponential zero sets:

$$\mathcal{R}_+ := \left\{ (-a_2+i\xi_2, -a_3+i\xi_3) \in \mathbb{C}^2 \,\middle|\, \begin{array}{c} -\tau_1-\tau_3 \le \xi_2 \le \tau_2+\tau_3 \\ \tau_3 \le \xi_3 \le \tau_2 \end{array} \right\},$$
$$\mathcal{R}_- := \left\{ (-a_2+i\xi_2, -a_3+i\xi_3) \in \mathbb{C}^2 \,\middle|\, \begin{array}{c} -\tau_1-\tau_2 \le \xi_2 \le -\tau_1-\tau_3 \\ \tau_4 \le \xi_3 \le \tau_3 \end{array} \right\}.$$

The change of variables $(\xi_2,\xi_3,\tau_2,\tau_4) \mapsto (\xi_3,\xi_2,\tau_4,\tau_2)$ relates $\mathcal{R}_+$ and $\mathcal{R}_-$, so it suffices to consider just the case of $\mathcal{R}_+$.

We replace the Gamma factors with their corresponding polynomial terms to obtain

$$\begin{aligned}\left| p^{1,\delta}_{T,R}(y;(-a_2,-a_3),\kappa) \right| \ll y_1^{\frac32+a_1} y_2^{2+a_2} y_3^{\frac32+a_3} \cdot T^{\varepsilon+R+3\delta} \iiint\limits_{\substack{|\tau_1|,|\tau_2|,|\tau_3| \le T^{1+\varepsilon} \\ 0\le\tau_1+\tau_2+2\tau_3\le\tau_1+2\tau_2+\tau_3}} \big(1+|\tau_1-\tau_2|\big)^{\frac{R+1}{2}-\frac{a_1}{2}} \\ \cdot\big(1+|\tau_1-\tau_3|\big)^{\frac{R+1}{2}-\frac{a_1}{2}} \big(1+|\tau_1-\tau_4|\big)^{\frac{R+1}{2}-a_1+\delta} \big(1+\tau_2-\tau_3\big)^{1+\frac{R}{2}} \big(1+\tau_2-\tau_4\big)^{1+\frac{R}{2}} \\ \cdot\big(1+\tau_3-\tau_4\big)^{1+\frac{R}{2}} \iint\limits_{\substack{-\tau_1-\tau_3\le\xi_2\le\tau_2+\tau_3 \\ \tau_3\le\xi_3\le\tau_2}} \frac{\big(1+|\xi_2+\tau_1+\tau_2|\big)^{-\frac{1+a_2}{2}+\frac{a_1}{2}-\delta} \big(1+|\xi_2+\tau_1+\tau_3|\big)^{-\frac{1+a_2}{2}+\frac{a_1}{2}-\delta}}{\big(1+|\xi_2+\xi_3+\tau_1|\big)^{-\frac{1+a_2+a_3-a_1}{2}}} \\ \cdot\big(1+|\xi_2-\tau_2-\tau_3|\big)^{-\frac{1+a_2}{2}} \big(1+|\xi_3-\tau_2|\big)^{-\frac{1+a_3}{2}} \big(1+|\xi_3-\tau_3|\big)^{-\frac{1+a_3}{2}} \big(1+|\xi_3-\tau_4|\big)^{-\frac{1+a_3}{2}+\frac{a_1}{2}-\delta} \\ \cdot\ d\xi_2 d\xi_3\ d\tau. \qquad (6.4.7)\end{aligned}$$

We make the change of variables

$$\xi_2 \mapsto \xi_2 - \tau_1 - \tau_3,\ \ \xi_3 \mapsto \xi_3 + \tau_3,$$

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and $T_j = \tau_j - \tau_{j+1}$ $(1 \le j \le 3)$, to get

$$\left|p^{1,\delta}_{T,R}(y;(-a_2,-a_3),\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+3\delta} \iiint\limits_{0\le|T_1|,T_2,T_3\le T^{1+\varepsilon}} \left(1+|T_1|\right)^{\frac{1+R-a_1}{2}}$$
$$\cdot\left(1+T_2\right)^{1+\frac{R}{2}}\left(1+T_3\right)^{1+\frac{R}{2}}\left(1+|T_1+T_2|\right)^{\frac{1+R-a_1}{2}}\left(1+|T_1+T_2+T_3|\right)^{\frac{1+R}{2}-a_1+\delta}\left(1+T_2+T_3\right)^{1+\frac{R}{2}}$$
$$\cdot\int\limits_{\xi_3=0}^{T_2}\int\limits_{\xi_2=0}^{T_3} \frac{\left(1+\xi_2+T_2\right)^{-\frac{1+a_2}{2}+\frac{a_1}{2}-\delta}\left(1+\xi_2\right)^{-\frac{1+a_2}{2}+\frac{a_1}{2}-\delta}\left(1+T_3-\xi_2\right)^{-\frac{1+a_2}{2}}}{\left(1+\xi_2+\xi_3\right)^{-\frac{1+a_2+a_3-a_1}{2}}}\left(1+T_2-\xi_3\right)^{-\frac{1+a_3}{2}}$$
$$\cdot\left(1+\xi_3\right)^{-\frac{1+a_3}{2}}\left(1+\xi_3+T_3\right)^{-\frac{1+a_3}{2}+\frac{a_1}{2}-\delta}\, d\xi_2\, d\xi_3\, dT_1\, dT_2\, dT_3. \qquad (6.4.8)$$

We first examine (6.4.8) in the case $r_2 = r_3 = 0$. In this case, we have

$$1 < 1+\varepsilon \le 2(\delta+1)-1+\varepsilon \le 2r_1 - 1 + \varepsilon \le a_1 \le 2r_1 + 1 - \varepsilon,$$

and $-1 < -1+\varepsilon \le a_j \le 1-\varepsilon < 1 \quad (2 \le j \le 3)$. If we now denote the integral in ξ_2 and ξ_3, in (6.4.8), by $\mathcal{I}(\tau, a, \delta)$ (where $\tau = (\tau_1, \tau_2, \tau_3)$ and $a = (a_1, a_2, a_3)$), then the above estimates on a_1, a_2, a_3 imply that

$$\mathcal{I}(\tau,a,\delta) \ll T\int_{\xi_2=0}^{T_3}\int_{\xi_3=0}^{T_2}\left(1+\xi_2+T_2\right)^{\frac{a_1}{2}-\delta}\left(1+\xi_2\right)^{\frac{a_1}{2}-\delta}\left(1+\xi_2+\xi_3\right)^{\frac{1-a_1}{2}}\left(1+\xi_3+T_3\right)^{\frac{a_1}{2}-\delta}\, d\xi_3\, d\xi_2.$$

We note that $a_1/2 - \delta > 0$ and $1 - a_1 < 0$. Two applications of Lemma A.3 (to the integrals in ξ_3 and ξ_2, respectively) then yield

$$\mathcal{I}(\tau,a,\delta) \ll T\left(1+T_2\right)^{\varepsilon+1}\left(1+T_2+T_3\right)^{a_1-2\delta}\left(1+T_3\right)^{\varepsilon+\frac{3}{2}-\delta}.$$

Putting this back into (6.4.8) yields

$$\left|p^{1,\delta}_{T,R}(y;(-a_2,-a_3),\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+1+R+3\delta} \iiint\limits_{0\le|T_1|,T_2,T_3\le T^{1+\varepsilon}} \left(1+|T_1|\right)^{\frac{1+R-a_1}{2}}$$
$$\cdot\left(1+T_3\right)^{\varepsilon+\frac{5}{2}+\frac{R}{2}-\delta}\left(1+|T_1+T_2|\right)^{\frac{1+R-a_1}{2}}\left(1+|T_1+T_2+T_3|\right)^{\frac{1+R}{2}-a_1+\delta}$$
$$\cdot\left(1+T_2\right)^{\varepsilon+2+\frac{R}{2}}\left(1+T_2+T_3\right)^{1+\frac{R}{2}+a_1-2\delta}\, dT_1\, dT_2\, dT_3. \qquad (6.4.9)$$

Assuming that R is large enough to ensure that the exponents in the above integrand are positive, we find that

$$\begin{aligned}\left|p^{1,\delta}_{T,R}(y;(-a_2,-a_3),\kappa)\right| &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+1+R+3\delta}\cdot T^3\cdot T^{\varepsilon+3R+7-a_1-2\delta}\\ &= y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+11+\delta-a_1}\\ &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+11-r_1},\end{aligned}$$

where in the last step we have used that $\delta \le r_1 - 1$ and $2r_1 - 1 + \varepsilon \le a_1$ imply that $\delta - a_1 \le -r_1$. Again, we're presently in the case $r_2 = r_3 = 0$, whence

$$11 - r_1 = 9 + \delta_{0,r_2} + \delta_{0,r_3} - r_1 - r_2 - r_3.$$

So in this case, $p^{1,\delta}_{T,R}(y;(-a_2,-a_3),\kappa)$ satisfies the bound on $p^{1,\delta}_{T,r}(y;(-a_2,-a_3))$ given by (6.4.3).

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We now demonstrate that this is true for $r_2 + r_3 \geq r_1$ as well. Because

$$1 + a_i > 2r_1 \geq 0 \quad (1 \leq i \leq 3) \quad \text{and} \quad \frac{a_1}{2} - \delta > \frac{2r_1 - 1}{2} - (r_1 - 1) = 1/2 > 0,$$

we have the bounds

$$\left(1 + \xi_j\right)^{-\frac{1+a_j}{2}} \leq \left(1 + \xi_j\right)^{-r_j} \quad (2 \leq j \leq 3),$$
$$\left(1 + \xi_j + T_j\right)^{-\frac{1+a_j}{2} + \frac{a_1}{2} - \delta} \leq \left(1 + T_j\right)^{-r_j} \left(1 + T_2 + T_3\right)^{\frac{a_1}{2} - \delta} \quad (2 \leq j \leq 3).$$

Further,

$$\frac{\left(1 + \xi_2\right)^{\frac{a_1}{2} - \delta}}{\left(1 + \xi_2 + \xi_3\right)^{\frac{a_1}{2} - \frac{1+a_2+a_3}{2}}} = \left(\frac{1 + \xi_2}{1 + \xi_2 + \xi_3}\right)^{\frac{a_1}{2} - \delta} \left(1 + \xi_2 + \xi_3\right)^{\frac{1+a_2+a_3}{2} - \delta} \ll T^{\varepsilon + \frac{1+a_2+a_3}{2} - \delta},$$

the last step because $r_2 + r_3 \geq r_1$ implies that $\frac{1+a_2+a_3}{2} - \delta > 0$. So (6.4.8) yields

$$\left|p^{1,\delta}_{T,R}(y, (-a_2, -a_3), \kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon + R + 2\delta + \frac{1+a_2+a_3}{2}} \iiint\limits_{0 \leq |T_1|, T_2, T_3 \leq T^{1+\varepsilon}} \left(1 + |T_1|\right)^{\frac{1+R-a_1}{2}}$$
$$\cdot \left(1 + T_2\right)^{1 + \frac{R}{2} - r_2} \left(1 + T_3\right)^{1 + \frac{R}{2} - r_3} \left(1 + |T_1 + T_2|\right)^{\frac{1+R-a_1}{2}} \left(1 + |T_1 + T_2 + T_3|\right)^{\frac{1+R}{2} - a_1 + \delta}$$
$$\cdot \left(1 + T_2 + T_3\right)^{1 + \frac{R}{2} + a_1 - 2\delta} \int_{\xi_3 = 0}^{T_2} \left(1 + \xi_3\right)^{-r_3} \left(1 + T_2 - \xi_3\right)^{-r_3} d\xi_3$$
$$\cdot \int_{\xi_2 = 0}^{T_3} \left(1 + \xi_2\right)^{-r_2} \left(1 + T_3 - \xi_2\right)^{-r_2} d\xi_2 \, dT_1 \, dT_2 \, dT_3$$

or, by Lemma A.1 applied to the above integrals in ξ_2 and ξ_3,

$$\left|p^{1,\delta}_{T,R}(y, (-a_2, -a_3), \kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon + R + 2\delta + \frac{1+a_2+a_3}{2}}$$
$$\cdot \iiint\limits_{0 \leq |T_1|, T_2, T_3 \leq T^{1+\varepsilon}} \left(1 + |T_1|\right)^{\frac{1+R-a_1}{2}} \left(1 + T_2\right)^{1 + \delta_{r_2,0} + \frac{R}{2} - r_2 - r_3} \left(1 + T_3\right)^{1 + \delta_{r_3,0} + \frac{R}{2} - r_2 - r_3}$$
$$\cdot \left(1 + |T_1 + T_2|\right)^{\frac{1+R-a_1}{2}} \left(1 + |T_1 + T_2 + T_3|\right)^{\frac{1+R}{2} - a_1 + \delta}$$
$$\cdot \left(1 + T_2 + T_3\right)^{1 + \frac{R}{2} + a_1 - 2\delta} dT_1 \, dT_2 \, dT_3. \tag{6.4.10}$$

Assuming again that R is sufficiently large that the exponents in the above integrals in T_1, T_2, and T_3 are positive, we find that $\left|p^{1,\delta}_{T,R}(y, (-a_2, -a_3), \kappa)\right|$

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon + 4R + \delta + \frac{1+a_2+a_3}{2} + 3 + \varepsilon + 3R + \frac{9}{2} - a_1 + \delta_{0,r_2} + \delta_{0,r_3} - 2r_2 - 2r_3}.$$

Since $a_j \leq 2r_j + 1$ for $2 \leq j \leq 3$ and $\delta - a_1 \leq r_1 - 1 - (2r_1 - 1) = -r_1$, we then conclude that

$$\left|p^{1,\delta}_{T,R}(y, (-a_2, -a_3), \kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon + 4R + 9 - r_1 + \delta_{0,r_2} + \delta_{0,r_3} - r_2 - r_3}.$$

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So, in the case $r_2 + r_3 \geq r_1$, we again have a bound on $p^{1,\delta}_{T,R}(y, (-a_2, -a_3), \kappa)$ that is consistent with the statement of Proposition 6.4.2.

Step 2: Bounding the residue term at $\alpha_1 = -s_2 - \alpha_2 - 2(r_2 - \delta_2)$

The next step of the proof is to show that all of the residues at the poles $\alpha_1 = -s_2 - \alpha_2 - 2(r_2 - \delta_2)$ contribute smaller bounds. The residue we must bound is:

$$\iiint\limits_{\substack{\mathrm{Re}(\alpha_2)=0\\ \mathrm{Re}(\alpha_3)=0\\ \mathrm{Re}(s_2)=-a_2}} e^{\frac{(s_2+\alpha_2+2(r_2-\delta_2))^2+\alpha_2^2+\alpha_3^2+(s_2-\alpha_3+2(r_2-\delta_2))^2}{2T^2}} \int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}-s_2-\alpha_2-2(r_2-\delta_2-\delta)} y_2^{2-s_2} y_3^{\frac{3}{2}-s_3}$$
$$\cdot \frac{\Gamma\left(\frac{2+R+\alpha_2-\alpha_3}{4}\right)\Gamma\left(\frac{2+R+\alpha_3-\alpha_2}{4}\right)}{\Gamma\left(\frac{\alpha_2-\alpha_3}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R+s_2-2\alpha_3+2(r_2-\delta_2)}{4}\right)\Gamma\left(\frac{2+R-s_2+2\alpha_3-2(r_2-\delta_2)}{4}\right)}{\Gamma\left(\frac{-s_2+2\alpha_3-2(r_2-\delta_2)}{2}\right)\Gamma\left(\frac{s_2-2\alpha_3+2(r_2-\delta_2)}{2}\right)} \cdot \frac{\Gamma\left(\frac{s_2+2\alpha_2}{2}+r_2-\delta_2-\delta\right)}{\Gamma\left(\frac{s_2+2\alpha_2}{2}+r_2-\delta_2\right)}$$
$$\cdot \frac{\Gamma\left(\frac{2+R-s_2+\alpha_2+\alpha_3-2(r_2-\delta_2)}{4}\right)\cdot\Gamma\left(\frac{2+R+s_2-\alpha_2-\alpha_3+2(r_2-\delta_2)}{4}\right)}{\Gamma\left(\frac{-s_2+\alpha_2+\alpha_3-2(r_2-\delta_2)}{2}\right)} \frac{\Gamma\left(\frac{2+R-s_2-\alpha_2-\alpha_3-2(r_2-\delta_2)}{4}\right)\Gamma\left(\frac{2+R+s_2+\alpha_2+\alpha_3+2(r_2-\delta_2))}{4}\right)}{\Gamma\left(\frac{-s_2-\alpha_2-\alpha_3-2(r_2-\delta_2)}{2}\right)}$$
$$\cdot \frac{\Gamma\left(\frac{2s_2+\alpha_2-\alpha_3+4(r_2-\delta_2)}{2}-\delta\right)}{\Gamma\left(\frac{2s_2+\alpha_2-\alpha_3+4(r_2-\delta_2)}{2}\right)} \cdot \frac{\Gamma\left(\frac{2+R-s_2-2\alpha_2-2(r_2-\delta_2)}{4}\right)\Gamma\left(\frac{2+R+s_2+2\alpha_2+2(r_2-\delta_2)}{4}\right)}{\Gamma\left(\frac{-s_2-2\alpha_2-2(r_2-\delta_2)}{2}\right)} \cdot \frac{\Gamma\left(\frac{s_2+\alpha_2+\alpha_3}{2}+r_2-\delta_2-\delta\right)}{\Gamma\left(\frac{s_2+\alpha_2+\alpha_3}{2}+r_2-\delta_2\right)}$$
$$\cdot \frac{\Gamma\left(\frac{2+R-2s_2-\alpha_2+\alpha_3-4(r_2-\delta_2)}{4}\right)\Gamma\left(\frac{2+R+2s_2+\alpha_2-\alpha_3+4(r_2-\delta_2)}{4}\right)}{\Gamma\left(\frac{-2s_2-\alpha_2+\alpha_3-4(r_2-\delta_2)}{2}\right)} \cdot \frac{\Gamma\left(\frac{\alpha_3-\alpha_2}{2}-r_2+\delta_2\right)}{\Gamma\left(\frac{\alpha_3-\alpha_2}{2}\right)} \frac{\Gamma\left(\frac{s_2-\alpha_2-\alpha_3}{2}\right)}{\Gamma\left(\frac{s_2-\alpha_2-\alpha_3}{2}+r_2-\delta_2\right)}$$
$$\cdot \frac{\Gamma\left(\frac{s_3-\alpha_2}{2}\right)\Gamma\left(\frac{s_3-\alpha_3}{2}\right)\Gamma\left(\frac{s_3-s_2+\alpha_3}{2}-r_2+\delta_2\right)}{\Gamma\left(\frac{s_3-\alpha_2}{2}-r_2+\delta_2+\delta\right)} \left(\mathcal{F}_R(\alpha)\, Q_\delta(s,\alpha)\right)\Big|_{\alpha_1=-s_2-\alpha_2-2(r_2-\delta_2)} ds_2\, ds_3\, d\alpha. \tag{6.4.11}$$

We need to shift the line of integration in α_2 so that the real part of the exponent of y_1 is a_1. That is, we require that

$$2r_1 - a_1' = a_1 = a_2 - \mathrm{Re}(\alpha_2) - 2(r_2 - \delta_2 - \delta) = a_2' - \mathrm{Re}(\alpha_2) + 2(\delta_2 + \delta),$$

where $0 < a_1', a_2' < 1$. In other words, given the bounds from Lemma 6.4.5, we shift the line in α_2 to

$$\mathrm{Re}(\alpha_2) = \kappa_2 := a_2' - a_1' - 2(r_1 - \delta - \delta_2). \tag{6.4.12}$$

Lemma 6.4.13. *In shifting the line of integration in* (6.4.11) *in the variable α_2 from* $\mathrm{Re}(\alpha_2) = 0$ *to* $\mathrm{Re}(\alpha_2) = \kappa_2$ *as in* (6.4.12)*, no poles are crossed.*

Proof. As in the proof of Lemma 6.4.5, we assume that R is sufficiently large. Then for each of the Gamma factors involving R, no poles are crossed. Further, for reasons also described in that proof, each of the factors of the form

$$\frac{\Gamma\left(-\frac{\alpha_2}{2} + z - \delta\right)}{\Gamma\left(-\frac{\alpha_2}{2} + z\right)\Gamma\left(\frac{\alpha_2}{2} - z\right)}$$

is holomorphic in α_2. Thus, in moving the line of integration in α_2, only the term $\Gamma\left(\frac{s_3-\alpha_2}{2}\right)$ might contribute poles.

Regarding this term, we consider two cases. The first is when $r_2 \geq r_1$. In this case, since $\delta_2 \leq r_1 - \delta$ by Lemma 6.4.5, we have $\delta_2 \leq r_2 - \delta$, so the pole in the denominator factor $\Gamma(\frac{s_3-\alpha_2}{2} - r_2 + \delta + \delta_2)$ cancels the pole in $\Gamma\left(\frac{s_3-\alpha_2}{2}\right)$.

Finally, then, we need to consider the case $r_2 < r_1$. Of the triples (r_1, r_2, r_3) under consideration, the only one satisfying this criterion is the triple $(r, 0, 0)$. Recall that, in the case of this triple, we are assuming that $\mathrm{Re}(a_3) < 0$. Also, note that our integral in s_3, in (6.4.11), is over the line $\mathrm{Re}(s_3) = -a_3$. So on our original line of integration in α_2, namely $\mathrm{Re}(\alpha_2) = 0$, we have that $\mathrm{Re}(s_3 - \alpha_2) > 0$. But note

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that we are moving this line to the left, since (6.4.12) and the fact that $\delta_2 = 0$ (since $r_2 = 0$) together imply that the new line of integration is

$$\mathrm{Re}(\alpha_2) = a_2' - a_1' - 2(r_1 - \delta) \le a_2' - a_1' - 2 < 0.$$

Therefore, in moving this line we are only increasing the real part of $s_3 - \alpha_2$ and consequently are not passing any poles of $\Gamma(\frac{s_3-\alpha_2}{2})$. □

In order to bound (6.4.11), we first remark that the exponential factor is

$$\mathcal{E} = -|\xi_2 - 2\tau_3| + |\xi_3 - \tau_3| + |\xi_3 - \xi_2 + \tau_3|,$$

and it follows that the exponential zero set is

$$\xi_2 - \tau_3 \le \xi_3 \le \tau_3 \qquad \text{or} \qquad \tau_3 \le \xi_3 \le \xi_2 - \tau_3.$$

Using the first of these,[2] and assuming as before that $\tau_2 \ge \tau_3$, it is not hard to see that the shifted version of (6.4.11), to $\mathrm{Re}(\alpha_2) = \kappa_2 := a_2 - a_1 - 2(r_2 - \delta - \delta_2)$, is bounded by

$$\begin{aligned} y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+3\delta} &\iint\limits_{-\infty<\tau_3\le\tau_2<\infty} \big(1+\tau_2-\tau_3\big)^{\frac{1+R+a_1-a_2}{2}-\delta} \int\limits_{\xi_2=-\infty}^{2\tau_3} e^{-\frac{(\xi_2+\tau_2)^2+\tau_2^2+\tau_3^2+(\xi_2-\tau_3)^2}{2T^2}} \\ &\cdot \big(1+2\tau_3-\xi_2\big)^{1+\frac{R}{2}} \big(1+|\xi_2+2\tau_2|\big)^{\frac{1+R-2a_1+a_2}{2}-r_2+\delta_2+\delta} \big(1+\tau_2+\tau_3-\xi_2\big)^{\frac{1+R+a_1-2a_2}{2}+r_2-\delta-\delta_2} \\ &\cdot \big(1+|2\xi_2+\tau_2-\tau_3|\big)^{\frac{1+R-a_1-a_2}{2}+r_2-\delta_2} \big(1+|\xi_2+\tau_2+\tau_3|\big)^{\frac{1+R-a_1}{2}} \\ &\cdot \int\limits_{\xi_3=\xi_2-\tau_3}^{\tau_3} \big(1+\tau_2-\xi_3\big)^{r_2-\delta_2-\delta} \big(1+\tau_3-\xi_3\big)^{-\frac{1+a_3}{2}} \big(1+\xi_3-(\xi_2-\tau_3)\big)^{\frac{a_2-a_3-1}{2}-r_2+\delta_2} d\xi_3\, d\xi_2\, d\tau_2\, d\tau_3. \end{aligned} \tag{6.4.14}$$

Since

$$\big(1+\tau_2-\xi_3\big)^{-\delta} \le \big(1+\tau_2-\tau_3\big)^{-\delta} \text{ and } \big(1+\xi_3-(\xi_2-\tau_3)\big)^{\delta_2+1/2} \le \big(1+2\tau_3-\xi_2\big)^{\delta_2+1/2},$$

we find that (6.4.14) is

$$\begin{aligned} \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+3\delta} &\iint\limits_{-\infty<\tau_3\le\tau_2<\infty} \big(1+\tau_2-\tau_3\big)^{\frac{1+R+a_1-a_2}{2}-2\delta} \int\limits_{\xi_2=-\infty}^{2\tau_3} e^{-\frac{(\xi_2+\tau_2)^2+\tau_2^2+\tau_3^2+(\xi_2-\tau_3)^2}{2T^2}} \\ &\cdot \big(1+2\tau_3-\xi_2\big)^{\frac{3+R}{2}+\delta_2} \big(1+|\xi_2+2\tau_2|\big)^{\frac{1+R-2a_1+a_2}{2}-r_2+\delta_2+\delta} \big(1+\tau_2+\tau_3-\xi_2\big)^{\frac{1+R+a_1-2a_2}{2}+r_2-\delta-\delta_2} \\ &\cdot \big(1+|2\xi_2+\tau_2-\tau_3|\big)^{\frac{1+R-a_1-a_2}{2}+r_2-\delta_2} \big(1+|\xi_2+\tau_2+\tau_3|\big)^{\frac{1+R-a_1}{2}} \\ &\cdot \int\limits_{\xi_3=\xi_2-\tau_3}^{\tau_3} \big(1+\tau_2-\xi_3\big)^{r_2-\delta_2} \big(1+\tau_3-\xi_3\big)^{-\frac{1+a_3}{2}} \big(1+\xi_3-(\xi_2-\tau_3)\big)^{\frac{a_2-a_3-1}{2}-r_2} d\xi_3\, d\xi_2\, d\tau_2\, d\tau_3. \end{aligned} \tag{6.4.15}$$

Since $r_2 - \delta_2 \ge 0$, $\frac{1+a_3}{2} > 0$, and $\frac{a_2-a_3-2}{2} - r_2 < 0$ we find, using Lemma A.0.3, that the integral in ξ_3, in (6.4.15), is

$$\ll \big(1+2\tau_3-\xi_2\big)^{\delta_{0,r_3}-\frac{1+a_3}{2}} \cdot \big(1+\tau_2+\tau_3-\xi_2\big)^{r_2-\delta_2}.$$

[2]For the other exponential zero set, the answer is virtually identical.

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But then (6.4.15) is

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+3\delta} \iint\limits_{-\infty<\tau_3\le\tau_2<\infty} \left(1+\tau_2-\tau_3\right)^{\frac{1+R+a_1-a_2}{2}-2\delta} \int\limits_{\xi_2=-\infty}^{2\tau_3} e^{-\frac{(\xi_2+\tau_2)^2+\tau_2^2+\tau_3^2+(\xi_2-\tau_3)^2}{2T^2}}$$
$$\cdot\left(1+2\tau_3-\xi_2\right)^{\frac{2+R-a_3}{2}+\delta_2+\delta_{r_3,0}} \left(1+|\xi_2+2\tau_2|\right)^{\frac{1+R-2a_1+a_2}{2}-r_2+\delta_2+\delta} \left(1+\tau_2+\tau_3-\xi_2\right)^{\frac{1+R+a_1-2a_2}{2}+2r_2-\delta-2\delta_2}$$
$$\cdot\left(1+|2\xi_2+\tau_2-\tau_3|\right)^{\frac{1+R-a_1-a_2}{2}+r_2-\delta_2} \left(1+|\xi_2+\tau_2+\tau_3|\right)^{\frac{1+R-a_1}{2}}\, d\xi_2\, d\tau_2\, d\tau_3. \tag{6.4.16}$$

Substituting $\tau_j \to \tau_j T$ $(1 \le j \le 2)$ and $\xi_3 \to \xi_3 T$, we find that (6.4.16) is

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+3\delta+3+\frac{7}{2}+3R-\frac{2a_1+3a_2+a_3}{2}+2r_2-2\delta-\delta_2+\delta_{r_3,0}},$$

which, because $-a_j < 1-2r_j$ for $1 \le j \le 3$, is

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+\delta+\frac{19}{2}-2r_1-r_2-r_3-\delta_2+\delta_{r_3,0}}$$
$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+\frac{17}{2}-r_1-r_2-r_3+\delta_{r_3,0}},$$

where in the final step we have used the facts that $r_1-\delta \ge 1$ and $\delta_2 \ge 0$. So the residue term (6.4.11) also satisfies the bound given in Proposition 6.4.2. The proof of that proposition is therefore complete. □

The other type of single residue term that we have to consider is

$$p_{T,R}^{2,\delta}(y;(-a_1,-a_3)) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=1,3}} y_1^{\frac{3}{2}-s_1} y_2^{2-p_2} y_3^{\frac{3}{2}-s_3}\, \mathcal{F}_R(\alpha)\, \Gamma_R(\alpha)\, Q_\delta(s,\alpha)$$
$$\cdot\left(\prod_{j=1}^{2}\prod_{k=3}^{4}\Gamma\left(\frac{\alpha_k-\alpha_j}{2}-\delta\right)\right)\Gamma\left(\tfrac{s_1+\alpha_1}{2}\right)\Gamma\left(\tfrac{s_1+\alpha_2}{2}\right)\Gamma\left(\tfrac{s_3-\alpha_3}{2}\right)\Gamma\left(\tfrac{s_3-\alpha_4}{2}\right) ds_1 ds_3\, d\alpha \tag{6.4.17}$$

where Q_δ is a polynomial (see Section 5.2) of degree $\le 3\delta$ and $p_2 = -\alpha_1-\alpha_2-2\delta$.

Proposition 6.4.18. *Let $r_2 \ge 1$, $r_1, r_3 \ge 0$ be integers, and $0 < \varepsilon < 1$. Suppose a_1, a_2, a_3 satisfy the hypotheses of Theorem 4.0.3. If $0 \le \delta \le r_2-1$, then*

$$\left|p_{T,R}^{2,\delta}(y;(-a_1,-a_3))\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+9+\delta_{0,r_1}+\delta_{0,r_3}-(r_1+r_2+r_3)}. \tag{6.4.19}$$

Proof. We first rearrange the terms on the right-hand side of (6.4.17) as follows.

$$p_{T,R}^{2,\delta}(y;(-a_1,-a_3)) = \iiint\limits_{\mathrm{Re}(\alpha_j)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=1,3}} y_1^{\frac{3}{2}-s_1} y_2^{2+\alpha_1+\alpha_2+2\delta} y_3^{\frac{3}{2}-s_3}\, \mathcal{F}_R(\alpha)$$
$$\cdot\, Q_\delta(s,\alpha)\left(\prod_{j=1}^{2}\prod_{k=3}^{4}\frac{\Gamma(\frac{\alpha_k-\alpha_j}{2}-\delta)}{\Gamma(\frac{\alpha_k-\alpha_j}{2})}\,\frac{\Gamma(\frac{2+R+\alpha_j-\alpha_k}{4})\Gamma(\frac{2+R+\alpha_k-\alpha_j}{4})}{\Gamma(\frac{\alpha_j-\alpha_k}{2})}\right)$$
$$\cdot\,\frac{\Gamma(\frac{2+R+\alpha_1-\alpha_2}{4})\Gamma(\frac{2+R+\alpha_2-\alpha_1}{4})}{\Gamma(\frac{\alpha_1-\alpha_2}{2})\Gamma(\frac{\alpha_2-\alpha_1}{2})}\cdot\frac{\Gamma(\frac{2+R+\alpha_3-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_3}{4})}{\Gamma(\frac{\alpha_3-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_3}{2})}$$
$$\cdot\,\Gamma\left(\tfrac{s_1+\alpha_1}{2}\right)\Gamma\left(\tfrac{s_1+\alpha_2}{2}\right)\Gamma\left(\tfrac{s_3-\alpha_3}{2}\right)\Gamma\left(\tfrac{s_3-\alpha_4}{2}\right) ds_1\, ds_3\, d\alpha.$$

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The proof follows the very same outline as that of Proposition 6.4.2. First, in order for the exponent of y_2 to match that in the statement of the proposition, we will shift the integration in the α variables from real part zero to $\mathrm{Re}(\alpha_j) = \kappa_j$, such that

$$a_2 = \mathrm{Re}(\alpha_1 + \alpha_2 + 2\delta) = \kappa_1 + \kappa_2 + 2\delta$$

lies in the interval $(2r_2 - 1, 2r_2 + 1)$. We do this by defining

$$\kappa_1 := a_2 - 2\delta \qquad \text{and} \qquad \kappa_2 := 0 =: \kappa_3.$$

Let $\tau_j = \mathrm{Im}(\alpha_j)$. Note that since the above integral is invariant under each of the change of coordinates

$$(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_1), \quad \text{and} \quad (\alpha_3, \alpha_4) \mapsto (\alpha_4, \alpha_3),$$

we may assume that $\tau_1 \geq \tau_2$ and $\tau_3 \geq \tau_4$.

As before, we now use Stirling's formula to write the Gamma factors in the above integrand as a product of linear and exponential terms. The exponential factor is $e^{\frac{\pi}{4}\mathcal{E}(\xi,\tau)}$, where

$$\mathcal{E} = 2\tau_1 + 2\tau_3 - |\xi_1 + \tau_2| - |\xi_1 + \tau_1| - |\xi_3 + \tau_1 + \tau_2 + \tau_3| - |\xi_3 - \tau_3|,$$

and from this we readily deduce that the exponential zero set is

$$\mathcal{R} = \big\{(\xi_1, \xi_3) \mid -\tau_1 \leq \xi_1 \leq -\tau_2,\ \tau_4 \leq \xi_3 \leq \tau_3\big\}.$$

The polynomial factor is

$$\mathcal{P}(\xi,\tau) = \big(1+|\tau_1-\tau_2|\big)^{1+\frac{R}{2}} \big(1+|\tau_3-\tau_4|\big)^{1+\frac{R}{2}} \left(\prod_{j=1}^{2}\prod_{k=3}^{4}\big(1+|\tau_k-\tau_j|\big)^{\frac{1+R}{2}-(\frac{\kappa_j-\kappa_k}{2})-\delta}\right)$$
$$\cdot\big(1+|\xi_1+\tau_1|\big)^{-\frac{1+a_1-\kappa_1}{2}} \cdot\big(1+|\xi_1+\tau_2|\big)^{-\frac{1+a_1-\kappa_2}{2}} \cdot\big(1+|\xi_3-\tau_3|\big)^{-\frac{1+a_3+\kappa_3}{2}} \cdot\big(1+|\xi_3-\tau_4|\big)^{-\frac{1+a_3+\kappa_4}{2}}.$$

Plugging in the values of κ_j given above, and bounding the resulting terms as before, we find that

$$\Big|p^{2,\delta}_{T,R}(y;(-a_1,-a_3),\kappa)\Big| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+3\delta} \iint\limits_{\substack{-T^{1+\varepsilon}\leq\tau_4\leq\tau_3\leq T^{1+\varepsilon}\\ -T^{1+\varepsilon}\leq\tau_2\leq\tau_1\leq T^{1+\varepsilon}}} \big(1+|\tau_2-\tau_3|\big)^{\frac{1+R}{2}-\delta}$$
$$\cdot\,(1+\tau_1-\tau_2)^{1+\frac{R}{2}}(1+\tau_3-\tau_4)^{1+\frac{R}{2}}\big(1+|\tau_1-\tau_4|\big)^{\frac{3+R}{2}-2r_2+\delta}\big(1+|\tau_1-\tau_3|\big)^{\frac{2+R}{2}-r_2}$$
$$\cdot\,\big(1+|\tau_2-\tau_4|\big)^{\frac{2+R}{2}-r_2}\int_{\xi_1=-\tau_1}^{-\tau_2}\big(1+|\xi_1+\tau_1|\big)^{\frac{1}{2}-r_1+r_2-\delta}\big(1+|\xi_1+\tau_2|\big)^{-r_1}\,d\xi_1$$
$$\cdot\int_{\xi_3=\tau_4}^{\tau_3}\big(1+|\xi_3-\tau_3|\big)^{-r_3}\big(1+|\xi_3-\tau_4|\big)^{\frac{1}{2}-r_3+r_2-\delta}\,d\xi_3\,d\tau_2\,d\tau_3. \tag{6.4.20}$$

Note that the integral in ξ_3 is bounded by $T^{\varepsilon+\frac{1}{2}+r_2-\delta}(1+\tau_3-\tau_4)^{\delta_{0,r_3}-r_3}$, and the integral in ξ_1 is bounded by $T^{\varepsilon+\frac{1}{2}+r_2-\delta}(1+\tau_1-\tau_2)^{\delta_{0,r_1}-r_1}$. Plugging in these bounds and simplifying, we find that this

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shifted term is bounded as follows:

$$\left|p^{2,\delta}_{T,R}(y;(-a_1,-a_3),\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+1+2r_2+\delta}$$

$$\cdot \iint\limits_{\substack{-T^{1+\varepsilon}\le\tau_4\le\tau_3\le T^{1+\varepsilon}\\ -T^{1+\varepsilon}\le\tau_2\le\tau_1\le T^{1+\varepsilon}}} \left(1+|\tau_2-\tau_3|\right)^{\frac{1+R}{2}-\delta}\left(1+\tau_1-\tau_2\right)^{1+\frac{R}{2}+\delta_{0,r_1}-r_1}\left(1+\tau_3-\tau_4\right)^{1+\frac{R}{2}+\delta_{0,r_3}-r_3}$$

$$\cdot\left(1+|\tau_1-\tau_4|\right)^{\frac{3+R}{2}-2r_2+\delta}\left(1+|\tau_1-\tau_3|\right)^{\frac{2+R}{2}-r_2}\left(1+|\tau_2-\tau_4|\right)^{\frac{2+R}{2}-r_2}\,d\tau_2\,d\tau_3$$

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+10+\delta_{0,r_1}+\delta_{0,r_3}-r_1-r_2-r_3-(r_2-\delta)}$$

$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+9+\delta_{0,r_1}+\delta_{0,r_3}-r_1-r_2-r_3},$$

since $r_2-\delta\ge 1$.

In analogy to Lemma 6.4.5, it is easy to show that in shifting $\mathrm{Re}(\alpha_1)$ from zero to κ_1, poles are crossed at $\alpha_1 = q$ for

$$q\in\left\{\begin{array}{c} -s_1-2(r_1-\delta_1)\\ -s_3-\alpha_2-\alpha_3-2(r_3-\delta_3)\end{array}\;\middle|\;\begin{array}{l}0\le\delta_j\le r_j\\ \delta_j\le r_2-\delta\end{array}\right\}.$$

In what follows, the method holds equally well for either of these two types of residues. For concreteness, we consider the residue at $\alpha_1 = -s_1-2(r_1-\delta_1)$. This gives

$$\iint\limits_{\substack{\mathrm{Re}(\alpha_j)=\kappa_j\\ j=2,3}} e^{\frac{(s_1+2(r_1-\delta_1))^2+\alpha_2^2+\alpha_3^2+(s_1+\alpha_2+\alpha_3+2(r_1-\delta_1))^2}{2T^2}} \iint\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j=1,3}} y_1^{\frac{3}{2}-s_1} y_2^{2-s_1+\alpha_2-2(r_1-\delta_1-\delta)} y_3^{\frac{3}{2}-s_3}$$

$$\cdot\left(\mathcal{F}_R(\alpha)\,Q_\delta(s,\alpha)\right)\Big|_{\alpha_1=-s_1-2(r_1-\delta_1)} \frac{\Gamma\left(\frac{\alpha_3-\alpha_2}{2}-\delta\right)}{\Gamma\left(\frac{\alpha_3-\alpha_2}{2}\right)} \frac{\Gamma\left(\frac{2+R+\alpha_2-\alpha_3}{4}\right)\Gamma\left(\frac{2+R+\alpha_3-\alpha_2}{4}\right)}{\Gamma\left(\frac{\alpha_2-\alpha_3}{2}\right)}$$

$$\cdot\frac{\Gamma\left(\frac{\alpha_3+s_1}{2}+r_1-\delta_1-\delta\right)}{\Gamma\left(\frac{\alpha_3+s_1}{2}+r_1-\delta_1\right)} \frac{\Gamma\left(\frac{2+R-s_1-\alpha_3-2(r_1-\delta_1)}{4}\right)\Gamma\left(\frac{2+R+\alpha_3+s_1+2(r_1-\delta_1)}{4}\right)}{\Gamma\left(\frac{-s_1-\alpha_3}{2}-r_1+\delta_1\right)} \frac{\Gamma\left(\frac{2s_1-\alpha_2-\alpha_3}{2}+2(r_1-\delta_1)-\delta\right)}{\Gamma\left(\frac{2s_1-\alpha_2-\alpha_3}{2}+2(r_1-\delta_1)\right)}$$

$$\cdot\frac{\Gamma\left(\frac{2+R-2s_1+\alpha_2+\alpha_3}{4}-r_1+\delta_1\right)\Gamma\left(\frac{2+R+2s_1-\alpha_2-\alpha_3}{4}+r_1-\delta_1\right)}{\Gamma\left(\frac{-2s_1+\alpha_2+\alpha_3}{2}-2(r_1-\delta_1)\right)} \frac{\Gamma\left(\frac{2+R-s_1-2(r_1-\delta_1)+2\alpha_2+\alpha_3}{4}\right)\Gamma\left(\frac{2+R+s_1+2(r_1-\delta_1)-2\alpha_2-\alpha_3}{4}\right)}{\Gamma\left(\frac{-s_1+2\alpha_2+\alpha_3}{2}-r_1+\delta_1\right)}$$

$$\cdot\frac{\Gamma\left(\frac{s_1-2\alpha_2-\alpha_3}{2}+r_1-\delta_1-\delta\right)}{\Gamma\left(\frac{s_1-2\alpha_2-\alpha_3}{2}+r_1-\delta_1\right)} \frac{\Gamma\left(\frac{2+R-s_1-2(r_1-\delta_1)-\alpha_2}{4}\right)\Gamma\left(\frac{2+R+s_1+2(r_1-\delta_1)+\alpha_2}{4}\right)\Gamma\left(\frac{2+R-s_1-2(r_1-\delta_1)+\alpha_2+2\alpha_3}{4}\right)}{\Gamma\left(\frac{-s_1-2(r_1-\delta_1)-\alpha_2}{2}\right)\Gamma\left(\frac{s_1+2(r_1-\delta_1)+\alpha_2}{2}\right)\Gamma\left(\frac{-s_1+\alpha_2+2\alpha_3}{2}-r_1+\delta_1\right)}$$

$$\cdot\frac{\Gamma\left(\frac{2+R+s_1+2(r_1-\delta_1)-\alpha_2-2\alpha_3}{4}\right)}{\Gamma\left(\frac{s_1-\alpha_2-2\alpha_3}{2}+r_1-\delta_1\right)}\cdot\Gamma\left(\frac{s_1+\alpha_2}{2}\right)\Gamma\left(\frac{s_3-\alpha_3}{2}\right)\Gamma\left(\frac{s_3-s_1+\alpha_2+\alpha_3}{2}-r_1+\delta_1\right)\,ds_1ds_3\,d\alpha.$$

Strictly speaking, this is what we want to bound in the case that $\kappa_2=\kappa_3=0$, but as before we need to shift the line of integration in the α_2 variable to $\mathrm{Re}(\alpha_2)=\kappa_2$ such that the real part of the exponent of y_2 is $2+a_2$. This means $a_2=\mathrm{Re}(-s_1+\alpha_2-2(r_1-\delta_1-\delta)) = a_1+\kappa_2-2(r_1-\delta_1-\delta)$, or in other words,

$$\kappa_2 = 2(r_2-\delta-\delta_1)+a_2'-a_1' \tag{6.4.21}$$

where $-1<a_j' = a_j-2r_j<1\,(j=1,2)$.

This implies that $\mathrm{Re}(\alpha_2)$ gets shifted to the right. Just as in the case of Lemma 6.4.13, one can show in this case that no poles are crossed in moving $\mathrm{Re}(\alpha_2)$. So it suffices to bound the above for these values of κ_2 and κ_3.

The exponential factor is $e^{-\frac{\pi}{4}\mathcal{E}}$, where

$$\mathcal{E} = |\xi_1-\tau_2-2\tau_3|-|\xi_3-\tau_3|-|\xi_3-\xi_1+\tau_2+\tau_3|,$$

which leads to two exponential zero sets: the first is

$$\mathcal{R}: \qquad \tau_3 \le \xi_3 \le \xi_1 - \tau_2 - \tau_3,$$

and the second is similar but the inequalities are reversed.

The polynomial factor (coming from the Gamma factors specifically) is

$$\begin{aligned}\mathcal{P} =& \big(1+|\xi_1+\tau_2|\big)^{\frac{1+R+2a_1'-a_2'}{2}-r_1+r_2-\delta-\delta_1}\big(1+|\tau_2-\tau_3|\big)^{\frac{1+R-a_1'+a_2'}{2}-r_2+\delta_1}\\ &\cdot\big(1+|\xi_1+\tau_3|\big)^{\frac{1+R+a_1'}{2}-\delta-\delta_1}\big(1+|-2\xi_1+\tau_2+\tau_3|\big)^{\frac{1+R+a_1'+a_2'}{2}-r_2-\delta_1}\\ &\cdot\big(1+|-\xi_1+2\tau_2+\tau_3|\big)^{\frac{1+R-a_1'+2a_2'}{2}-2r_2+\delta+\delta_1}\big(1+|-\xi_1+\tau_2+2\tau_3|\big)^{1+\frac{R}{2}}\\ &\cdot\big(1+|\xi_3-\tau_3|\big)^{\frac{-1+a_3'}{2}-r_3}\big(1+|-\xi_1+\xi_3+\tau_2+\tau_3|\big)^{\frac{-1-a_2'+a_3'}{2}+r_2-r_3-\delta}.\end{aligned}$$

Note that the presence of the exponential term means we can restrict the integral to the set of $|\tau_2|, |\tau_3|, |\xi_1| \le T^{1+\varepsilon}$. Then the integral that we seek to bound is

$$\begin{aligned}&\ll y_1^{\frac{3}{2}+a_1}y_2^{2+a_2}y_3^{\frac{3}{2}+a_3}\,T^{\varepsilon+R+3\delta+\frac{1}{2}}\iiint\limits_{\substack{|\tau_2|,|\tau_3|,|\xi_1|\le T^{1+\varepsilon}\\ \tau_2+2\tau_3\le\xi_1}}\big(1+|\xi_1+\tau_2|\big)^{\frac{1+R+2a_1'-a_2'}{2}-r_1+r_2-\delta-\delta_1}\\ &\quad\cdot\big(1+|\tau_2-\tau_3|\big)^{\frac{1+R-a_1'+a_2'}{2}-r_2+\delta_1}\big(1+|\xi_1+\tau_3|\big)^{\frac{1+R+a_1'}{2}-\delta-\delta_1}\big(1+|-2\xi_1+\tau_2+\tau_3|\big)^{\frac{1+R+a_1'+a_2'}{2}-r_2-\delta_1}\\ &\quad\cdot\big(1+|-\xi_1+2\tau_2+\tau_3|\big)^{\frac{1+R-a_1'+2a_2'}{2}-2r_2+\delta+\delta_1}\big(1+|-\xi_1+\tau_2+2\tau_3|\big)^{1+\frac{R}{2}}\\ &\quad\cdot\int\limits_{\xi_3=\tau_3}^{\xi_1-\tau_2-\tau_3}\big(1+|\xi_3-\tau_3|\big)^{-r_3}\big(1+|-\xi_1+\xi_3+\tau_2+\tau_3|\big)^{r_2-r_3-\delta}\,d\xi_3\,d\xi_1\,d\alpha_2\,d\alpha_3\\ &\ll y_1^{\frac{3}{2}+a_1}y_2^{2+a_2}y_3^{\frac{3}{2}+a_3}\,T^{\varepsilon+4R+3+\frac{8}{2}+a_1'+\frac{3}{2}a_2'+\delta_{0,r_3}-r_1-r_2-r_3-(r_2-\delta)-\delta_1}\\ &\ll y_1^{\frac{3}{2}+a_1}y_2^{2+a_2}y_3^{\frac{3}{2}+a_3}\,T^{\varepsilon+4R+\frac{17}{2}+\delta_{0,r_3}-r_1-r_2-r_3},\end{aligned}$$

since $r_2 - \delta \ge 1$, $\delta_1 \ge 0$. □

6.5. Bounds for the double residue terms

We need to consider two types of double residue terms: $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3)$ and $p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2)$. They are obtained by taking residues at $s_2 = -\alpha_1 - \alpha_4 - 2\delta_2$ and $s_3 = \alpha_2 - 2\delta_3$, respectively, of the single residue term $p_{T,R}^{1,(\delta_1)}(y)$ defined by (6.4.1).

Specifically, write

$$p_1 = -\alpha_1 - 2\delta_1, \quad p_2 = -\alpha_1 - \alpha_4 - 2\delta_2, \quad p_3 = \alpha_2 - 2\delta_3, \tag{6.5.1}$$

where $0 \le \delta_j \le r_j - 1$ for $1 \le j \le 3$. Then we find from Proposition 5.2.9 that

$$\begin{aligned}p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3) = &\iiint\limits_{\mathrm{Re}(\alpha)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}-p_1}y_2^{2-p_2}y_3^{\frac{3}{2}-s_3}\,\mathcal{F}_R(\alpha)\,\Gamma_R(\alpha)\\ &\cdot f_{\delta_1,\delta_2}(s_3,\alpha)\Gamma\big(\tfrac{\alpha_2-\alpha_1}{2}-\delta_1\big)\Gamma\big(\tfrac{\alpha_3-\alpha_1}{2}-\delta_1\big)\Gamma\big(\tfrac{\alpha_4-\alpha_1}{2}-\delta_1\big)\\ &\cdot\Gamma\big(\tfrac{\alpha_2-\alpha_4}{2}-\delta_2\big)\Gamma\big(\tfrac{\alpha_3-\alpha_4}{2}-\delta_2\big)\Gamma\big(\tfrac{s_3-\alpha_2}{2}\big)\Gamma\big(\tfrac{s_3-\alpha_3}{2}\big)\,ds_3\,d\alpha,\end{aligned} \tag{6.5.2}$$

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and

$$p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2) = \iiint\limits_{\mathrm{Re}(\alpha)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \int\limits_{\mathrm{Re}(s_2)=-a_2} y_1^{\frac{3}{2}-p_1} y_2^{2-s_2} y_3^{\frac{3}{2}-p_3}\, \mathcal{F}_R(\alpha)\,\Gamma_R(\alpha)$$
$$\cdot\, g_{\delta_1,\delta_3}(s_2,\alpha)\Gamma(\tfrac{\alpha_2-\alpha_1}{2}-\delta_1)\Gamma(\tfrac{\alpha_3-\alpha_1}{2}-\delta_1)\Gamma(\tfrac{\alpha_4-\alpha_1}{2}-\delta_1)$$
$$\cdot\, \Gamma(\tfrac{\alpha_2-\alpha_3}{2}-\delta_3)\Gamma(\tfrac{\alpha_2-\alpha_4}{2}-\delta_3)\Gamma(\tfrac{s_2+a_1+\alpha_3}{2})\Gamma(\tfrac{s_2+a_1+\alpha_4}{2})\, ds_2\, d\alpha, \tag{6.5.3}$$

where $\deg f_{\delta_1,\delta_2} \le 2\delta_1+\delta_2$ and $\deg g_{\delta_1,\delta_3} \le 2\delta_1+\delta_3$. (See Section 5.2.)

In what follows, we show that the bounds on (6.5.2) and (6.5.3) are ‘small.’

We begin with (6.5.2). We have:

Proposition 6.5.4. *Let $r_1, r_2 \ge 1$, $r_3 \ge 0$ be integers, and $0 < \varepsilon < 1$. Suppose a_1, a_2, a_3 satisfy the hypotheses of Theorem 4.0.3. If $0 \le \delta_j \le r_j - 1$ for $1 \le j \le 2$, then*

$$\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+\frac{13}{2}+\delta_{r_3,0}-r_1-r_2-r_3}. \tag{6.5.5}$$

Proof. The proof is similar, in spirit and in many of the details, to that of Proposition 6.4.2.

More specifically: to obtain the desired bound on $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3)$, we will need to shift the lines of integration in both the α_1 and α_2 variables, so that the resulting exponents of y_1 and y_2 have real parts as stated in the proposition. In doing so, we will pick up residues. That is, we will have

$$p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3) = p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa) + \sum \text{Residues}, \tag{6.5.6}$$

where

$$p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa) := \iiint\limits_{\mathrm{Re}(\alpha)=\kappa} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}+\alpha_1+2\delta_1} y_2^{2+\alpha_1+\alpha_4+2\delta_2} y_3^{\frac{3}{2}-s_3}$$
$$\frac{\Gamma(\frac{2+R+\alpha_1-\alpha_2}{4})\Gamma(\frac{2+R+\alpha_2-\alpha_1}{4})\Gamma(\frac{\alpha_2-\alpha_1}{2}-\delta_1)\Gamma(\frac{2+R+\alpha_2-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_2}{4})\Gamma(\frac{s_3-\alpha_2}{2})\Gamma(\frac{s_3-\alpha_3}{2})}{\Gamma(\frac{\alpha_1-\alpha_2}{2})\Gamma(\frac{\alpha_2-\alpha_1}{2})\Gamma(\frac{\alpha_2-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_2}{2})}$$
$$\cdot\, \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_1}{4})\Gamma(\frac{\alpha_3-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_1}{2})}\, \frac{\Gamma(\frac{2+R+\alpha_2-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_2}{4})\Gamma(\frac{\alpha_2-\alpha_4}{2}-\delta_2)}{\Gamma(\frac{\alpha_2-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_2}{2})}$$
$$\cdot\, \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_1}{4})\Gamma(\frac{\alpha_4-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_1}{2})}\, \frac{\Gamma(\frac{2+R+\alpha_3-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_3}{4})\Gamma(\frac{\alpha_3-\alpha_4}{2}-\delta_2)}{\Gamma(\frac{\alpha_3-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_3}{2})}$$
$$\cdot\, \mathcal{F}_R(\alpha)\, f_{\delta_1,\delta_2}(s_3,\alpha)\, ds_3\, d\alpha, \tag{6.5.7}$$

and the residues that appear depend on the particular choice of $\kappa = (\kappa_1,\kappa_2,\kappa_3)$. We've grouped the Gamma factors, above, in a manner that will be convenient for what follows.

Because we want the exponents of y_1 and y_2, in (6.5.7), to have real parts $3/2 + a_1$ and $2 + a_2$, respectively, we will choose $\kappa = (\kappa_1,\kappa_2,\kappa_3) \in \mathbb{R}^3$ such that

$$\mathrm{Re}(\alpha_1+2\delta_1) = \kappa_1 + 2\delta_1 = a_1, \quad \mathrm{Re}(\alpha_1+\alpha_4+2\delta_2) = -\kappa_2-\kappa_3+2\delta_2 = a_2.$$

Specifically, we will define

$$\kappa = (\kappa_1,\kappa_2,\kappa_3) = (a_1-2\delta_1, -a_2+2\delta_2, 0) \quad (\text{and } \kappa_4 = -\kappa_1-\kappa_2-\kappa_3). \tag{6.5.8}$$

For this value of κ, we will obtain an estimate of the desired magnitude for $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)$.

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It will remain to estimate the residues that appear in (6.5.6). To do so we first identify the poles; see also Lemma 6.5.15 below. We then show that, for these residue terms, the desired exponents on the y_js can be obtained by shifting lines of integration, without passing additional poles. Finally, using this information, we show that the residue terms are small.

Step 1: Bounding the shifted integral $p_{T,R}^{12,(\delta_1,\delta_2)}$

Before estimating the residue terms in (6.5.6), we obtain a bound of the desired magnitude on the shifted integral $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)$, with $\kappa=(\kappa_1,\kappa_2,\kappa_3)$ as in (6.5.8).

Note that, from any of the grouped combinations of Gamma functions in (6.5.7) *except for the second one*, the contribution to the exponential factor in Stirling's formula is zero. This is because absolute values of imaginary parts from the numerator of any such combination cancel those from the denominator. So, again, the only one of these terms that contributes to the exponential factor is the second one, which contributes a factor of

$$e^{-\frac{\pi}{2}(|\tau_2-\tau_3|/4+|\tau_3-\tau_2|/4+|\xi_3-\tau_2|/2+|\xi_3-\tau_3|/2-|\tau_2-\tau_3|/2-|\tau_3-\tau_2|/2)}=e^{-\frac{\pi}{4}(|\xi_3-\tau_2|+|\xi_3-\tau_3|-|\tau_2-\tau_3|)}.$$

But the integrand in (6.5.7) is invariant under $\alpha_2\leftrightarrow\alpha_3$, so we may assume that $\tau_2\geq\tau_3$, whence the exponential factor in question is simply

$$e^{-\frac{\pi}{4}(|\xi_3-\tau_2|+|\xi_3-\tau_3|-\tau_2+\tau_3)}.$$

It is then easily seen that there is just one exponential zero set, namely

$$\mathcal{R}:=\{(-a_3+i\xi_3)\in\mathbb{C}|\tau_3\leq\xi_3\leq\tau_2\}.$$

Replacing the Gamma factors with their corresponding polynomial terms, in (6.5.7), then gives

$$\begin{aligned}\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right| &\ll y_1^{\frac{3}{2}+a_1}y_2^{2+a_2}y_3^{\frac{3}{2}+a_3}\,T^{\varepsilon+R+2\delta_1+\delta_2}\\ &\cdot\iiint\limits_{\substack{|\tau_1|,|\tau_2|,|\tau_3|\leq T^{1+\varepsilon}\\ 0\leq\tau_2-\tau_3}}\big(1+|\tau_1-\tau_2|\big)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\big(1+|\tau_1-\tau_3|\big)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\big(1+|\tau_1-\tau_4|\big)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}\\ &\cdot\big(1+\tau_2-\tau_3\big)^{\frac{2+R}{2}}\big(1+|\tau_2-\tau_4|\big)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_2}\big(1+|\tau_3-\tau_4|\big)^{\frac{R+1+\kappa_3-\kappa_4}{2}-\delta_2}\\ &\cdot\int_{\xi_3=\tau_3}^{\tau_2}\big(1+\tau_2-\xi_3\big)^{\frac{-1-a_3-\kappa_2}{2}}\big(1+\xi_3-\tau_3\big)^{\frac{-1-a_3-\kappa_3}{2}}\,d\xi_3\,d\tau.\end{aligned}\tag{6.5.9}$$

The change of variables

$$\xi_3\mapsto\xi_3+\tau_3,\qquad T_j=\tau_j-\tau_{j+1}\quad(1\leq j\leq 3)$$

applied to (6.5.9) then gives

$$\begin{aligned}\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right| &\ll y_1^{\frac{3}{2}+a_1}y_2^{2+a_2}y_3^{\frac{3}{2}+a_3}\,T^{\varepsilon+R+2\delta_1+\delta_2}\iiint\limits_{0\leq|T_1|,T_2,|T_3|\leq T^{1+\varepsilon}}\\ &\cdot\big(1+|T_1|\big)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\big(1+|T_1+T_2|\big)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\big(1+|T_1+T_2+T_3|\big)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}\big(1+T_2\big)^{\frac{2+R}{2}}\\ &\cdot\big(1+|T_2+T_3|\big)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_2}\big(1+|T_3|\big)^{\frac{R+1+\kappa_3-\kappa_4}{2}-\delta_2}\\ &\cdot\int_{\xi_3=0}^{T_2}\big(1+T_2-\xi_3\big)^{\frac{-1-a_3-\kappa_2}{2}}\big(1+\xi_3\big)^{\frac{-1-a_3-\kappa_3}{2}}\,d\xi_3\;dT_1\;dT_2\;dT_3.\end{aligned}\tag{6.5.10}$$

Now Lemma A.1 and the fact that $\kappa_3 = 0$ (see also (6.5.8)) tell us that

$$\int_{\xi_3=0}^{T_2} (1+T_2-\xi_3)^{\frac{-1-a_3-\kappa_2}{2}} (1+\xi_3)^{\frac{-1-a_3-\kappa_3}{2}}\, d\xi_3$$
$$\ll (1+T_2)^{-\min\left\{\frac{1+a_3+\kappa_2}{2},\, \frac{1+a_3+\kappa_3}{2},\, a_3+\frac{\kappa_2+\kappa_3}{2}\right\}+\varepsilon}$$
$$= (1+T_2)^{-\frac{1+a_3+\kappa_2}{2}+\frac{1}{2}\max\{0,\,\kappa_2,\,1-a_3\}+\varepsilon}. \tag{6.5.11}$$

But

$$\kappa_2 = -a_2 + 2\delta_2 \le 1 - 2r_2 - \varepsilon + 2(r_2-1) = -1-\varepsilon < 0,$$

and

$$-2r_3 + \varepsilon \le 1 - a_3 \le 2 - 2r_3 - \varepsilon;$$

from this information, it follows that

$$\max\{0,\ \kappa_2,\ 1-a_3\} \le 2\delta_{r_3,0}.$$

So by (6.5.11),

$$\int_{\xi_3=0}^{T_2} (1+T_2-\xi_3)^{\frac{-1-a_3-\kappa_2}{2}} (1+\xi_3)^{\frac{-1-a_3-\kappa_3}{2}}\, d\xi_3 \ll (1+T_2)^{-\frac{1+a_3+\kappa_2}{2}+\delta_{r_3,0}+\varepsilon}. \tag{6.5.12}$$

Then (6.5.10) gives

$$\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_2} \iiint\limits_{0\le|T_1|,T_2,|T_3|\le T^{1+\varepsilon}} (1+|T_1|)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}$$
$$\cdot\left(1+T_2\right)^{\frac{R+1-a_3-\kappa_2}{2}+\delta_{r_3,0}+\varepsilon} (1+|T_1+T_2|)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1} (1+|T_1+T_2+T_3|)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}$$
$$\cdot\left(1+|T_2+T_3|\right)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_2} \left(1+|T_3|\right)^{\frac{R+1+\kappa_3-\kappa_4}{2}-\delta_2}\, dT_1\, dT_2\, dT_3. \tag{6.5.13}$$

It's straightforward to estimate the above integral, using the facts that, on the indicated domains of integration,

$$T_1, T_2, T_3 \ll T^{1+\varepsilon},$$

and the length of each domain of integration is also $\ll T^{1+\varepsilon}$. We thereby find that

$$\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+6+\delta_1+\delta_2+\delta_{r_3,0}-a_1-a_2-\frac{a_3}{2}}.$$

But we're assuming that $-a_3 \le 1 - 2r_3$, and

$$\delta_j - a_j \le \delta_j + 1 - 2r_j = (\delta_j + 1 - r_j) - r_j \le -r_j \quad \text{for} \quad j = 1, 2.$$

So our above estimate reads

$$\left|p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+\frac{13}{2}+\delta_{r_3,0}-r_1-r_2-r_3}, \tag{6.5.14}$$

which gives us a bound of the desired magnitude of $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)$ in (6.5.6).

Step 2: Bounding the residue terms

Our next step is to estimate the residues in (6.5.6). Recall: these are the residues at the poles that one crosses in moving the lines of integration in (6.5.2), to transform it into (6.5.7).

We first locate these poles.

Lemma 6.5.15. *Suppose the lines of integration, in* (6.5.2)*, are shifted from* $\mathrm{Re}(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$ *to* $\mathrm{Re}(\alpha_1, \alpha_2, \alpha_3) = (a_1 - 2\delta_1, -a_2 + 2\delta_2, 0)$. *Then:*

(a) *No poles are crossed in the* α_1 *variable.*
(b) *For a fixed* s_3*, any pole crossed in the* α_2 *variable belongs to the set*

$$\{s_3 + 2\delta_3 \,|\, \delta_3 \in \mathbb{Z}_{\geq 0},\ \max\{0, r_3 - (r_2 - \delta_2)\} \leq \delta_3 \leq r_3\}\,. \tag{6.5.16}$$

Proof. First we consider the factors

$$\Gamma\big(\tfrac{2+R+\alpha_j-\alpha_k}{4}\big) \qquad (1 \leq j \neq k \leq 4),$$

in the integrand of (6.5.7). If R is sufficiently large, then no poles of these factors will be crossed in moving our lines of integration in α_1 and α_2.

Nor do any of the terms of the form

$$\frac{\Gamma(\frac{\alpha_k-\alpha_j}{2} - \delta_n)}{\Gamma(\frac{\alpha_j-\alpha_k}{2})\Gamma(\frac{\alpha_k-\alpha_j}{2})}$$

give rise to any poles. Indeed, if the numerator of this term has a pole, then $\frac{\alpha_k-\alpha_j}{2} - \delta_n$ is a nonpositive integer, whence $\frac{\alpha_k-\alpha_j}{2} \in \mathbb{Z}$, so this numerator pole will be cancelled by a pole from either $\Gamma(\frac{\alpha_j-\alpha_k}{2})$ or $\Gamma(\frac{\alpha_k-\alpha_j}{2})$ in the denominator.

The only factors remaining to consider are the factors

$$\Gamma(\tfrac{s_3-\alpha_2}{2}) \quad \text{and} \quad \Gamma(\tfrac{s_3-\alpha_3}{2}),$$

and since we are not shifting the line of integration in α_3, we need only examine the former of these factors.

In particular, part (a) of our lemma is proved.

Regarding $\Gamma(\frac{s_3-\alpha_2}{2})$: for fixed s_3, this factor has poles, as a function of α_2, whenever

$$\alpha_2 = s_3 + 2\delta_3 \quad (\delta_3 \in \mathbb{Z}_{\geq 0}). \tag{6.5.17}$$

But for such an α_2 to lie between the initial line of integration $\mathrm{Re}\,\alpha_2 = 0$ and the terminal line $\mathrm{Re}\,\alpha_2 = -a_2 + 2\delta_2$, we must have

$$-a_2 + 2\delta_2 \leq \mathrm{Re}\,\alpha_2 = \mathrm{Re}\,(s_3 + 2\delta_3) = -a_3 + 2\delta_3 \leq 0. \tag{6.5.18}$$

But

$$-a_3 + 2\delta_3 \geq -a_2 + 2\delta_2 \implies \delta_3 \geq \frac{a_3 - a_2}{2} + \delta_2 \geq r_3 - r_2 + \delta_2 - 1 + \varepsilon,$$

the last inequality because $2r_j - 1 + \varepsilon \leq a_j \leq 2r_j + 1 - \varepsilon$. Since δ_3 is an integer, this implies $\delta_3 \geq r_3 - r_2 + \delta_2$. On the other hand,

$$-a_3 + 2\delta_3 \leq 0 \implies \delta_3 \leq \frac{a_3}{2} \leq r_3 + \frac{1}{2} - \varepsilon, \tag{6.5.19}$$

so that $\delta_3 \leq r_3$. So part (b) of our lemma is proved. □

To complete our proof of Proposition 6.5.4, then, we need only show that the residue at each of the above poles in the variable α_2 is sufficiently small.

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For ease of notation, let us denote such a pole by $\widehat{\alpha}_2$, for some fixed δ_3 as described in the above lemma. We also write $\widehat{\alpha}_4 := -\alpha_1 - \widehat{\alpha}_2 - \alpha_3$, and $\widehat{\alpha} := (\alpha_1, \widehat{\alpha}_2, \alpha_3, \widehat{\alpha}_4)$. Then by (6.5.2) and (6.5.7), the residue at $\widehat{\alpha}_2$ has the following form:

$$\begin{aligned}
&\operatorname*{Res}_{\alpha_2=\widehat{\alpha}_2}\left(p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right)\\
&= \frac{(-1)^{\delta_3}}{\delta_3!} \iint\limits_{\substack{\mathrm{Re}(\alpha_1)=a_1-2\delta_1\\ \mathrm{Re}(\alpha_3)=0}} e^{\frac{\alpha_1^2+\widehat{\alpha}_2^2+\alpha_3^2+\widehat{\alpha}_4^2}{2T^2}} \int\limits_{\mathrm{Re}(s_3)=-a_3} y_1^{\frac{3}{2}+\alpha_1+2\delta_1} y_2^{2-\widehat{\alpha}_2-\alpha_3+2\delta_2} y_3^{\frac{3}{2}-s_3}\\
&\cdot \frac{\Gamma(\frac{2+R+\alpha_1-\widehat{\alpha}_2}{4})\Gamma(\frac{2+R+\widehat{\alpha}_2-\alpha_1}{4})\Gamma(\frac{\widehat{\alpha}_2-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\widehat{\alpha}_2}{2})\Gamma(\frac{\widehat{\alpha}_2-\alpha_1}{2})} \frac{\Gamma(\frac{2+R+\widehat{\alpha}_2-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\widehat{\alpha}_2}{4})\Gamma(\frac{s_3-\alpha_3}{2})}{\Gamma(\frac{\widehat{\alpha}_2-\alpha_3}{2})\Gamma(\frac{\alpha_3-\widehat{\alpha}_2}{2})}\\
&\cdot \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_1}{4})\Gamma(\frac{\alpha_3-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_1}{2})} \frac{\Gamma(\frac{2+R+\widehat{\alpha}_2-\widehat{\alpha}_4}{4})\Gamma(\frac{2+R+\widehat{\alpha}_4-\widehat{\alpha}_2}{4})\Gamma(\frac{\widehat{\alpha}_2-\widehat{\alpha}_4}{2}-\delta_2)}{\Gamma(\frac{\widehat{\alpha}_2-\widehat{\alpha}_4}{2})\Gamma(\frac{\widehat{\alpha}_4-\widehat{\alpha}_2}{2})}\\
&\cdot \frac{\Gamma(\frac{2+R+\alpha_1-\widehat{\alpha}_4}{4})\Gamma(\frac{2+R+\widehat{\alpha}_4-\alpha_1}{4})\Gamma(\frac{\widehat{\alpha}_4-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\widehat{\alpha}_4}{2})\Gamma(\frac{\widehat{\alpha}_4-\alpha_1}{2})} \frac{\Gamma(\frac{2+R+\alpha_3-\widehat{\alpha}_4}{4})\Gamma(\frac{2+R+\widehat{\alpha}_4-\alpha_3}{4})\Gamma(\frac{\alpha_3-\widehat{\alpha}_4}{2}-\delta_2)}{\Gamma(\frac{\alpha_3-\widehat{\alpha}_4}{2})\Gamma(\frac{\widehat{\alpha}_4-\alpha_3}{2})}\\
&\qquad\cdot \mathcal{F}_R(\widehat{\alpha}) f_{\delta_1,\delta_2}(s_3,\widehat{\alpha})\, ds_3\, d\alpha_3\, d\alpha_1. \qquad (6.5.20)
\end{aligned}$$

We want our bound on (6.5.20) to contain the factor $y_1^{3/2+a_1} y_2^{2+a_2} y_3^{3/2+a_3}$, as usual. To effect this, we will move the line of integration in α_3, in (6.5.20), from $\mathrm{Re}\,\alpha_3 = 0$ to $\mathrm{Re}(2-\widehat{\alpha}_2-\alpha_3+2\delta_2) = 2+a_2$, which is to say, to the line

$$\mathrm{Re}(\alpha_3) = -a_2 - \widehat{\alpha}_2 + 2\delta_2 = -a_2 + a_3 + 2\delta_2 - 2\delta_3. \qquad (6.5.21)$$

The crucial observation here is that, in moving this line, we do not cross any poles. This is by arguments very similar to those employed in the proof of the above lemma. The only additional argument we need to make here regards the term $\Gamma(\frac{s_3-\alpha_3}{2})$, in (6.5.20). But if this factor contributes a pole, then $s_3-\alpha_3 \in 2\mathbb{Z}$; since $s_3 - \widehat{\alpha}_2 = -2\delta_3$ is also in $2\mathbb{Z}$, we conclude that $\alpha_3 - \widehat{\alpha}_2 \in 2\mathbb{Z}$, whence the pole from either the term $\Gamma(\frac{\widehat{\alpha}_2-\alpha_3}{2})$ or the term $\Gamma(\frac{\alpha_3-\widehat{\alpha}_2}{2})$ in the denominator of (6.5.20) cancels the pole from $\Gamma(\frac{s_3-\alpha_3}{2})$.

So in estimating (6.5.20), we may replace the line of integration $\mathrm{Re}(\alpha_3) = 0$ with the line given by (6.5.21). The estimation is then similar to that of (6.5.7).

Specifically: as was the case with (6.5.7), the only grouped combination of Gamma functions in (6.5.20) that contributes to the exponential factor in Stirling's formula is the second one. In the present case, since $\mathrm{Im}\,\widehat{\alpha}_2 = \mathrm{Im}\,s_3 = \xi_3$, these Gamma functions contribute a factor of

$$e^{-\frac{\pi}{2}(|\xi_3-\tau_3|/2-|\xi_3-\tau_3|/2)} = e^0.$$

In other words, our exponential zero set entails no restrictions on our integration in $\xi_3 = \mathrm{Im}(s_3)$.

We now write

$$\begin{aligned}
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &:= (\mathrm{Re}\,\alpha_1, \mathrm{Re}\,\widehat{\alpha}_2, \mathrm{Re}\,\alpha_3, \mathrm{Re}\,\widehat{\alpha}_4)\\
&= (a_1-2\delta_1, -a_3+2\delta_3, -a_2+a_3+2\delta_2-2\delta_3, -a_1+a_2+2\delta_1-2\delta_2). \qquad (6.5.22)
\end{aligned}$$

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Then (6.5.20) yields

$$\operatorname*{Res}_{\alpha_2=\widehat{\alpha}_2}\left(p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+2\delta_1+\delta_2}$$
$$\cdot \iiint\limits_{\tau_1,\tau_3,\xi_3\in\mathbb{R}} e^{\frac{\alpha_1^2+\widehat{\alpha}_2^2+\alpha_3^2+\widehat{\alpha}_4^2}{2T^2}} \left(1+|\tau_1-\xi_3|\right)^{\frac{R+1+\lambda_2-\lambda_1}{2}-\delta_1} \left(1+|\tau_1-\tau_3|\right)^{\frac{R+1+\lambda_3-\lambda_1}{2}-\delta_1}$$
$$\cdot \left(1+|2\tau_1+\tau_3+\xi_3|\right)^{\frac{R+1+\lambda_4-\lambda_1}{2}-\delta_1} \left(1+|\xi_3-\tau_3|\right)^{\frac{1+R-a_3-\lambda_3}{2}} \left(1+|2\xi_3+\tau_1+\tau_3|\right)^{\frac{R+1+\lambda_2-\lambda_4}{2}-\delta_2}$$
$$\cdot \left(1+|2\tau_3+\tau_1+\xi_3|\right)^{\frac{R+1+\lambda_3-\lambda_4}{2}-\delta_2} \, d\xi_3 \, d\tau_3 \, d\tau_1. \tag{6.5.23}$$

The factor

$$e^{\frac{\alpha_1^2+\widehat{\alpha}_2^2+\alpha_3^2+\widehat{\alpha}_4^2}{2T^2}}$$

in (6.5.23) is of exponential decay in τ_1 if $|\tau_1| \gg T^{1+\varepsilon}$, and similarly for the variables τ_3 and ξ_3. So for our estimate, we may restrict attention to the domain where $|\tau_1|, |\tau_3|, |\xi_3| \ll T^{1+\varepsilon}$. On such a domain, each of the other factors in our integrand is $\ll T^{c+\varepsilon}$, where c is the exponent on that factor. So (6.5.23) implies

$$\operatorname*{Res}_{\alpha_2=\widehat{\alpha}_2}\left(p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+R+2\delta_1+\delta_2}$$
$$\cdot \iiint\limits_{|\tau_1|,|\tau_3|,|\xi_3|\ll T^{1+\varepsilon}} T^{\varepsilon+4R+3+\frac{-a_3-3\lambda_1+2\lambda_2+\lambda_3-\lambda_4}{2}-3\delta_1-2\delta_2} \, d\xi_3 \, d\tau_3 \, d\tau_1$$
$$\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+6-a_1-a_2-a_3+\delta_1+\delta_2+\delta_3}. \tag{6.5.24}$$

By (6.5.19), we have $\delta_3 - a_3 \le \frac{1}{2} - r_3$; also, $\delta_j - a_j \le r_j - 1 + 1 - 2r_j = -r_j$ for $j = 1, 2$. Then (6.5.24) yields

$$\operatorname*{Res}_{\alpha_2=\widehat{\alpha}_2}\left(p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3,\kappa)\right) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+\frac{13}{2}-r_1-r_2-r_3}.$$

In other words, the sum of the residue terms in (6.5.6) also has a bound of the magnitude stipulated in Proposition 6.5.4. This completes the proof of that proposition. □

We now turn to our estimate of the term (6.5.3). The analysis here is similar to that of (6.5.2), but different enough that some detail is merited.

We have:

Proposition 6.5.25. *Let $r_1, r_3 \ge 1$, $r_2 \ge 0$ be integers, and $0 < \varepsilon < 1$. Suppose a_1, a_2, a_3 satisfy the hypotheses of Theorem 4.0.3. If $0 \le \delta_j \le r_j - 1$ for $j = 1, 3$, then*

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+\frac{13}{2}+\delta_{r_2,0}-r_1-r_2-r_3}. \tag{6.5.26}$$

Proof. To obtain the desired bound on $p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_3)$, we will need to shift the lines of integration in both the α_1 and α_2 variables, so that the exponents of y_1 and y_3 become $3/2 + a_1$ and $3/2 + a_3$, respectively. In doing so, we will pick up residues, whence

$$p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2) = p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa) + \sum \text{Residues}, \tag{6.5.27}$$

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where

$$p^{13,(\delta_1,\delta_3)}_{T,R}(y;-a_2,\kappa) := \iiint_{\mathrm{Re}(\alpha)=\kappa} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}} \int_{\mathrm{Re}(s_2)=-a_2} y_1^{\frac{3}{2}+\alpha_1+2\delta_1} y_2^{2-s_2} y_3^{\frac{3}{2}-\alpha_2+2\delta_3}$$
$$\cdot \mathcal{F}_R(\alpha)\, g_{\delta_1,\delta_3}(s_2,\alpha) \cdot \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_2}{4})\Gamma(\frac{2+R+\alpha_2-\alpha_1}{4})\Gamma(\frac{\alpha_2-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_2}{2})\Gamma(\frac{\alpha_2-\alpha_1}{2})}$$
$$\cdot \frac{\Gamma(\frac{2+R+\alpha_2-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_2}{4})\Gamma(\frac{\alpha_2-\alpha_3}{2}-\delta_3)}{\Gamma(\frac{\alpha_2-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_2}{2})} \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_1}{4})\Gamma(\frac{\alpha_3-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_1}{2})}$$
$$\cdot \frac{\Gamma(\frac{2+R+\widehat{\alpha}_2-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\widehat{\alpha}_2}{4})\Gamma(\frac{\widehat{\alpha}_2-\alpha_4}{2}-\delta_3)}{\Gamma(\frac{\widehat{\alpha}_2-\alpha_4}{2})\Gamma(\frac{\alpha_4-\widehat{\alpha}_2}{2})} \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_1}{4})\Gamma(\frac{\alpha_4-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_1}{2})}$$
$$\cdot \frac{\Gamma(\frac{2+R+\alpha_4-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_4}{4})\Gamma(\frac{s_2+\alpha_1+\alpha_3}{2})\Gamma(\frac{s_2+\alpha_1+\alpha_4}{2})}{\Gamma(\frac{\alpha_4-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_4}{2})}\, ds_2\, d\alpha, \qquad (6.5.28)$$

and the residues that appear depend on the particular choice of $\kappa = (\kappa_1,\kappa_2,\kappa_3)$. As before, we've grouped the Gamma factors in an auspicious manner.

To obtain the desired exponents on y_1 and y_3, in (6.5.28), we will choose $\kappa = (\kappa_1,\kappa_2,\kappa_3) \in \mathbb{R}^3$ such that

$$\mathrm{Re}(\alpha_1 + 2\delta_1) = \kappa_1 + 2\delta_1 = a_1, \qquad \mathrm{Re}(-\alpha_2 + 2\delta_3) = -\kappa_2 + 2\delta_3 = a_3,$$

by putting

$$\kappa = (\kappa_1,\kappa_2,\kappa_3) = (a_1 - 2\delta_1, -a_3 + 2\delta_3, 0) \quad (\text{and } \kappa_4 = -\kappa_1 - \kappa_2 - \kappa_3). \qquad (6.5.29)$$

For this value of κ, we will obtain an estimate of the desired magnitude for $p^{13,(\delta_1,\delta_3)}_{T,R}(y;-a_2,\kappa)$. Subsequently we will, as before, show that the residues in (6.5.27) are small.

Step 1: Bounding the shifted integral $p^{13,(\delta_1,\delta_2)}_{T,R}$

Here we estimate the term $p^{13,(\delta_1,\delta_3)}_{T,R}(y;-a_2,\kappa)$, with $\kappa = (\kappa_1,\kappa_2,\kappa_3)$ as in (6.5.29).

Only the last grouped combination of Gamma functions, in (6.5.28), contributes to the exponential factor in Stirling's formula, for the same reasons as we discussed in the proof of Proposition 6.5.4. In the present case, this last term contributes a factor of

$$e^{-\frac{\pi}{2}(|\tau_4-\tau_3|/4+|\tau_3-\tau_4|/4+|\xi_2+\tau_1+\tau_3|/2+|\xi_2+\tau_1+\tau_4|/2-|\tau_4-\tau_3|/2-|\tau_3-\tau_4|/2)} = e^{-\frac{\pi}{4}(|\xi_2+\tau_1+\tau_3|+|\xi_2+\tau_1\tau_4|-|\tau_3-\tau_4|)}.$$

As the integrand in (6.5.28) is invariant under $\alpha_3 \leftrightarrow \alpha_4$, we may assume that $\tau_3 \geq \tau_4$, so that the exponential factor in question equals

$$e^{-\frac{\pi}{4}(|\xi_2+\tau_1+\tau_3|+|\xi_2+\tau_1+\tau_4|-\tau_3+\tau_4)}.$$

Then the corresponding exponential zero set is seen to be

$$\mathcal{R} := \{(-a_2 + i\xi_2) \in \mathbb{C} | -\tau_1 - \tau_3 \leq \xi_2 \leq -\tau_1 - \tau_4\}.$$

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We replace the Gamma factors in (6.5.28) with their corresponding polynomial terms; we get

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_3} \iiint\limits_{\substack{|\tau_1|,|\tau_2|,|\tau_3|\le T^{1+\varepsilon}\\ 0\le\tau_1+\tau_2+2\tau_3}}$$
$$\cdot\left(1+|\tau_1-\tau_2|\right)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\left(1+|\tau_1-\tau_3|\right)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\left(1+|\tau_1-\tau_4|\right)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}$$
$$\cdot\left(1+|\tau_2-\tau_3|\right)^{\frac{R+1+\kappa_2-\kappa_3}{2}-\delta_3}\left(1+|\tau_2-\tau_4|\right)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_3}\left(1+\tau_3-\tau_4\right)^{\frac{2+R}{2}}$$
$$\cdot\int_{\xi_2=-\tau_1-\tau_3}^{-\tau_1-\tau_4}\left(1+\xi_2+\tau_1+\tau_3\right)^{\frac{-1-a_2+\kappa_1+\kappa_3}{2}}\left(1-(\xi_2+\tau_1+\tau_4)\right)^{\frac{-1-a_2+\kappa_1+\kappa_4}{2}}\, d\xi_2\, d\tau. \tag{6.5.30}$$

The change of variables

$$\xi_2 \mapsto \xi_2-\tau_1-\tau_3, \qquad T_j=\tau_j-\tau_{j+1} \quad (1\le j\le 3)$$

applied to (6.5.30) then gives

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_3} \iiint\limits_{0\le|T_1|,|T_2|,T_3\le T^{1+\varepsilon}}$$
$$\cdot\left(1+|T_1|\right)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\left(1+|T_1+T_2|\right)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\left(1+|T_1+T_2+T_3|\right)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}\left(1+|T_2|\right)^{\frac{R+1+\kappa_2-\kappa_3}{2}-\delta_3}$$
$$\cdot\left(1+|T_2+T_3|\right)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_3}\left(1+T_3\right)^{\frac{2+R}{2}}$$
$$\cdot\int_{\xi_2=0}^{T_3}\left(1+\xi_2\right)^{\frac{-1-a_2+\kappa_1+\kappa_3}{2}}\left(1+T_3-\xi_2\right)^{\frac{-1-a_2+\kappa_1+\kappa_4}{2}}\, d\xi_2\, dT_1\, dT_2\, dT_3. \tag{6.5.31}$$

Now by Lemma A.1, we find that

$$\int_{\xi_2=0}^{T_3}\left(1+\xi_2\right)^{\frac{-1-a_2+\kappa_1+\kappa_3}{2}}\left(1+T_3-\xi_2\right)^{\frac{-1-a_2+\kappa_1+\kappa_4}{2}}\, d\xi_2$$
$$\ll (1+T_3)^{-\min\left\{\frac{1+a_2-\kappa_1-\kappa_3}{2},\,\frac{1+a_2-\kappa_1-\kappa_4}{2},\,a_2-\frac{2\kappa_1+\kappa_3+\kappa_4}{2}\right\}+\varepsilon}$$
$$=(1+T_3)^{\frac{-1-a_2+2\kappa_1+\kappa_3+\kappa_4}{2}+\frac{1}{2}\max\{-\kappa_1-\kappa_4,\,-\kappa_1-\kappa_3,\,1-a_2\}+\varepsilon}. \tag{6.5.32}$$

Further, it follows from (6.5.29) that

$$-\kappa_1-\kappa_4,\ -\kappa_1-\kappa_3 \le 0,$$

and that $1-a_2<0$ unless $r_2=0$, in which case $1-a_2<2$. In either case, $1-a_2<2\delta_{r_2,0}$. So (6.5.31) and (6.5.32) give

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_3} \iiint\limits_{0\le|T_1|,|T_2|,T_3\le T^{1+\varepsilon}}$$
$$\cdot\left(1+|T_1|\right)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\left(1+|T_1+T_2|\right)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\left(1+|T_1+T_2+T_3|\right)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}\left(1+|T_2|\right)^{\frac{R+1+\kappa_2-\kappa_3}{2}-\delta_3}$$
$$\cdot\left(1+|T_2+T_3|\right)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_3}\left(1+T_3\right)^{\frac{R+1-a_2+2\kappa_1+\kappa_3+\kappa_4}{2}+\delta_{r_2,0}+\varepsilon}\, dT_1\, dT_2\, dT_3. \tag{6.5.33}$$

Then, because

$$T_1, T_2, T_3 \ll T^{1+\varepsilon}$$

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on the indicated domains of integration, and because the length of each domain of integration is also $\ll T^{1+\varepsilon}$, we see that

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+6+\delta_1+\delta_3+\delta_{r_2,0}-a_1-\frac{a_2}{2}-a_3}.$$

But our assumptions on the a_js and δ_js imply that $\delta_j - a_j \le -r_j$ for $j = 1, 3$ and $-\frac{a_2}{2} \le \frac{1}{2} - r_2$, so we conclude that

$$\left|p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3} T^{\varepsilon+4R+\frac{13}{2}+\delta_{r_2,0}-r_1-r_2-r_3}, \tag{6.5.34}$$

which gives us a bound of the desired magnitude in (6.5.27).

Step 2: Bounding the residue terms

Next, we estimate the residues in (6.5.27), which arise from moving the lines of integration in (6.5.3), to get (6.5.28).

The locations of the poles in question are as follows.

Lemma 6.5.35. *Suppose the lines of integration, in (6.5.3), are shifted from* $\mathrm{Re}(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ *to* $\mathrm{Re}(\alpha_1, \alpha_2, \alpha_3) = (a_1 - 2\delta_1, -a_3 + 2\delta_3, 0)$. *Then:*

(a) *For a fixed s_2 and α_3, any pole crossed in the α_1 variable belongs to the set*

$$\{-s_2 - \alpha_3 - 2\delta_2 \mid \delta_2 \in \mathbb{Z}_{\ge 0},\ \max\{0, r_2 - (r_1 - \delta_1)\} \le \delta_2 \le r_2\}. \tag{6.5.36}$$

(b) *For a fixed s_2 and α_3, any pole crossed in the α_2 variable belongs to the set*

$$\{s_2 - \alpha_3 + 2\delta_2 \mid \delta_2 \in \mathbb{Z}_{\ge 0},\ \max\{0, r_2 - (r_3 - \delta_3)\} \le \delta_2 \le r_2\}. \tag{6.5.37}$$

Proof. As before, no poles will arise from the factors

$$\Gamma\left(\tfrac{2+R+\alpha_j-\alpha_k}{4}\right) \qquad (1 \le j \ne k \le 4)$$

in (6.5.28), if R is sufficiently large.

Nor will any of the terms of the form

$$\frac{\Gamma(\frac{\alpha_k-\alpha_j}{2} - \delta_n)}{\Gamma(\frac{\alpha_j-\alpha_k}{2})\Gamma(\frac{\alpha_k-\alpha_j}{2})}$$

give rise to any poles, for the same reasons as before. The only terms remaining to consider are the factors

$$\Gamma(\tfrac{s_2+\alpha_1+\alpha_3}{2}) \quad \text{and} \quad \Gamma(\tfrac{s_2+\alpha_1+\alpha_4}{2}) = \Gamma(\tfrac{s_2-\alpha_2-\alpha_3}{2}).$$

The former of these factors will give rise to poles when we shift the line of integration in α_1; the latter will do so when we shift the line in α_2.

Consider the first of these factors, $\Gamma(\frac{s_2+\alpha_1+\alpha_3}{2})$. For fixed s_2 and α_3, this factor has poles, as a function of α_1, whenever

$$\alpha_1 = -s_2 - \alpha_3 - 2\delta_2 \quad (\delta_2 \in \mathbb{Z}_{\ge 0}). \tag{6.5.38}$$

But for such an α_1 to lie between the initial line of integration $\mathrm{Re}\,\alpha_1 = 0$ and the terminal line $\mathrm{Re}\,\alpha_1 = a_1 - 2\delta_1$, we must have

$$0 \le \mathrm{Re}\,\alpha_1 = \mathrm{Re}\,(-s_2 - \alpha_3 - 2\delta_2) = a_2 - 2\delta_2 \le a_1 - 2\delta_1. \tag{6.5.39}$$

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But

$$a_2 - 2\delta_2 \le a_1 - 2\delta_1 \implies \delta_2 \ge \frac{a_2 - a_1}{2} + \delta_1 \ge r_2 - r_1 + \delta_1 - 1 + \varepsilon.$$

As δ_2 is a nonnegative integer, we therefore have $\delta_2 \ge \max\{0, r_2 - r_1 + \delta_1\}$. On the other hand,

$$a_2 - 2\delta_2 \ge 0 \implies \delta_2 \le \frac{a_2}{2} \le r_2 + \frac{1}{2} - \varepsilon, \tag{6.5.40}$$

so that $\delta_2 \le r_2$. So part (a) of our lemma is proved.

Part (b) is similar: as a function of α_2, $\Gamma(\frac{s_2-\alpha_2-\alpha_3}{2})$ has poles, for fixed s_2 and α_3, whenever

$$\alpha_2 = s_2 - \alpha_3 + 2\delta_2 \quad (\delta_2 \in \mathbb{Z}_{\ge 0}). \tag{6.5.41}$$

But for such an α_2 to lie between $\operatorname{Re}\alpha_2 = 0$ and $\operatorname{Re}\alpha_2 = -a_3 + 2\delta_3$, we must have

$$-a_3 + 2\delta_3 \le \operatorname{Re}\alpha_2 = \operatorname{Re}(s_2 - \alpha_3 + 2\delta_2) = -a_2 + 2\delta_2 \le 0. \tag{6.5.42}$$

We conclude from (6.5.42) and the fact that δ_3 is a nonnegative integer that

$$\max\{0, r_2 - (r_3 - \delta_3)\} \le \delta_2 \le r_2,$$

so part (b) of our lemma is proved. □

Therefore, to complete our proof of Proposition 6.5.25, it will suffice to show that the residue at each of the above poles in α_1 or α_2 is sufficiently small.

We will consider the poles in α_1 only; those in α_2 may be treated in a very similar fashion. Let us, then, denote such a pole in α_1 by $\widetilde{\alpha}_1$, for some fixed δ_2 and α_3 as described in part (a) of the above lemma. We also write $\widetilde{\alpha}_4 := -\widetilde{\alpha}_1 - \alpha_2 - \alpha_3$, and $\widetilde{\alpha} := (\widetilde{\alpha}_1, \alpha_2, \alpha_3, \widetilde{\alpha}_4)$. Then by (6.5.3) and (6.5.28), the residue at $\widetilde{\alpha}_1$ has the following form:

$$\begin{aligned}
&\operatorname*{Res}_{\alpha_1=\widetilde{\alpha}_1}\left(p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right)\\
&= \frac{(-1)^{\delta_2}}{\delta_2!} \iiint\limits_{\substack{\operatorname{Re}(\alpha_2)=-a_3+2\delta_3\\ \operatorname{Re}(\alpha_3)=0}} e^{\frac{\widetilde{\alpha}_1^2+\alpha_2^2+\alpha_3^2+\widetilde{\alpha}_4^2}{2T^2}} \int\limits_{\operatorname{Re}(s_2)=-a_2} y_1^{\frac{3}{2}-s_2-\alpha_3-2\delta_2+2\delta_1}\, y_2^{2-s_2}\, y_3^{\frac{3}{2}-\alpha_2+2\delta_3}\\
&\cdot \frac{\Gamma(\frac{2+R+\widetilde{\alpha}_1-\alpha_2}{4})\Gamma(\frac{2+R+\alpha_2-\widetilde{\alpha}_1}{4})\Gamma(\frac{\alpha_2-\widetilde{\alpha}_1}{2}-\delta_1)}{\Gamma(\frac{\widetilde{\alpha}_1-\alpha_2}{2})\Gamma(\frac{\alpha_2-\widetilde{\alpha}_1}{2})} \frac{\Gamma(\frac{2+R+\alpha_2-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_2}{4})\Gamma(\frac{\alpha_2-\alpha_3}{2}-\delta_3)}{\Gamma(\frac{\alpha_2-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_2}{2})}\\
&\cdot \frac{\Gamma(\frac{2+R+\widetilde{\alpha}_1-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\widetilde{\alpha}_1}{4})\Gamma(\frac{\alpha_3-\widetilde{\alpha}_1}{2}-\delta_1)}{\Gamma(\frac{\widetilde{\alpha}_1-\alpha_3}{2})\Gamma(\frac{\alpha_3-\widetilde{\alpha}_1}{2})} \frac{\Gamma(\frac{2+R+\widetilde{\alpha}_2-\widetilde{\alpha}_4}{4})\Gamma(\frac{2+R+\widetilde{\alpha}_4-\widetilde{\alpha}_2}{4})\Gamma(\frac{\widetilde{\alpha}_2-\widetilde{\alpha}_4}{2}-\delta_3)}{\Gamma(\frac{\widetilde{\alpha}_2-\widetilde{\alpha}_4}{2})\Gamma(\frac{\widetilde{\alpha}_4-\widetilde{\alpha}_2}{2})}\\
&\cdot \frac{\Gamma(\frac{2+R+\widetilde{\alpha}_1-\widetilde{\alpha}_4}{4})\Gamma(\frac{2+R+\widetilde{\alpha}_4-\widetilde{\alpha}_1}{4})\Gamma(\frac{\widetilde{\alpha}_4-\widetilde{\alpha}_1}{2}-\delta_1)}{\Gamma(\frac{\widetilde{\alpha}_1-\widetilde{\alpha}_4}{2})\Gamma(\frac{\widetilde{\alpha}_4-\widetilde{\alpha}_1}{2})} \frac{\Gamma(\frac{2+R+\widetilde{\alpha}_4-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\widetilde{\alpha}_4}{4})\Gamma(\frac{s_2-\alpha_2-\alpha_3}{2})}{\Gamma(\frac{\widetilde{\alpha}_4-\alpha_3}{2})\Gamma(\frac{\alpha_3-\widetilde{\alpha}_4}{2})}\\
&\cdot \mathcal{F}_R(\alpha)\, g_{\delta_1,\delta_3}(s_2,\widetilde{\alpha})\, ds_2\, d\alpha_3\, d\alpha_2.
\end{aligned} \tag{6.5.43}$$

In order that our bound on (6.5.43) contain the factor $y_1^{3/2+a_1} y_2^{2+a_2} y_3^{3/2+a_3}$, we now move the line of integration in α_3, in (6.5.43), from $\operatorname{Re}\alpha_3 = 0$ to $\operatorname{Re}(3/2 - s_2 - \alpha_3 - 2\delta_2 + 2\delta_1) = 3/2 + a_1$, or equivalently

$$\operatorname{Re}(\alpha_3) = -a_1 + a_2 + 2\delta_1 - 2\delta_2. \tag{6.5.44}$$

In moving this line, we do not cross any poles. This is by the same kinds of arguments as were used above. In particular we note that, if the factor $\Gamma(\frac{s_2-\alpha_2-\alpha_3}{2})$ has a pole, then $s_2 - \alpha_2 - \alpha_3 \in 2\mathbb{Z}$; since

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$s_2 + \widetilde{\alpha}_1 + \alpha_3 = s_2 - \alpha_2 - \widetilde{\alpha}_4 = -2\delta_2$ is also in $2\mathbb{Z}$ (by assumption), we conclude that $\alpha_3 - \widetilde{\alpha}_4 \in 2\mathbb{Z}$, whence the pole from either the term $\Gamma(\frac{\widetilde{\alpha}_4-\alpha_3}{2})$ or the term $\Gamma(\frac{\alpha_3-\widetilde{\alpha}_4}{2})$ in the denominator of (6.5.43) cancels the pole from $\Gamma(\frac{s_2-\alpha_2-\alpha_3}{2})$.

So in estimating (6.5.43), we may replace the line of integration $\mathrm{Re}(\alpha_3) = 0$ with the line given by (6.5.44). The estimation then proceeds as follows. First, the only grouped combination of Gamma functions in (6.5.43) that contributes to the exponential factor in Stirling's formula is the last one. Since

$$\mathrm{Im}\,\widetilde{\alpha}_4 = \mathrm{Im}\,(-\widetilde{\alpha}_1 - \alpha_2 - \alpha_3) = \mathrm{Im}\,(s_2 - \alpha_2) = \xi_2 - \tau_2,$$

these Gamma functions contribute a factor of

$$e^{-\frac{\pi}{2}(|\xi_2-\tau_2-\tau_3|/2-|\xi_2-\tau_2-\tau_3|/2)} = e^0.$$

So our exponential zero set here places no restrictions on our domain of integration in $\xi_2 = \mathrm{Im}(s_2)$.

Next, we write

$$\begin{aligned}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &:= (\mathrm{Re}\,\widetilde{\alpha}_1, \mathrm{Re}\,\alpha_2, \mathrm{Re}\,\alpha_3, \mathrm{Re}\,\widetilde{\alpha}_4)\\ &= (a_1 - 2\delta_1, -a_3 + 2\delta_3, -a_1 + a_2 + 2\delta_1 - 2\delta_2, -a_2 + a_3 + 2\delta_2 - 2\delta_3). \qquad (6.5.45)\end{aligned}$$

Then (6.5.43) yields

$$\begin{aligned}\operatorname*{Res}_{\alpha_1=\widetilde{\alpha}_1}\left(p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right) &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_3}\\ &\cdot \iiint_{\tau_2,\tau_3,\xi_2\in\mathbb{R}} e^{\frac{\widetilde{\alpha}_1^2+\alpha_2^2+\alpha_3^2+\widetilde{\alpha}_4^2}{2T^2}} \left(1+|\tau_2+\tau_3+\xi_2|\right)^{\frac{R+1+\lambda_2-\lambda_1}{2}-\delta_1} \left(1+|2\tau_3+\xi_2|\right)^{\frac{R+1+\lambda_3-\lambda_1}{2}-\delta_1}\\ &\cdot \left(1+|2\xi_2-\tau_2+\tau_3|\right)^{\frac{R+1+\lambda_4-\lambda_1}{2}-\delta_1} \left(1+|\tau_3-\tau_2|\right)^{\frac{R+1+\lambda_2-\lambda_3}{2}-\delta_3} \left(1+|2\tau_2-\xi_2|\right)^{\frac{R+1+\lambda_2-\lambda_4}{2}-\delta_3}\\ &\cdot \left(1+|\xi_2-\tau_2-\tau_3|\right)^{\frac{R+1-a_2-\lambda_2-\lambda_3}{2}}\, d\xi_2\, d\tau_3\, d\tau_2. \qquad (6.5.46)\end{aligned}$$

The factor

$$e^{\frac{\widetilde{\alpha}_1^2+\alpha_2^2+\alpha_3^2+\widetilde{\alpha}_4^2}{2T^2}}$$

in (6.5.46) is of exponential decay in τ_2 if $|\tau_2| \gg T^{1+\varepsilon}$, and similarly for the variables τ_3 and ξ_2. So for our estimate, we need only consider the domain where $|\tau_2|, |\tau_3|, |\xi_2| \ll T^{1+\varepsilon}$. On this domain, each of the other factors in the integrand of (6.5.46) is $\ll T^{c+\varepsilon}$, where c is the exponent on that factor. So (6.5.46) implies

$$\begin{aligned}\operatorname*{Res}_{\alpha_1=\widetilde{\alpha}_1}\left(p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right) &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+2\delta_1+\delta_3}\\ &\cdot \iiint_{|\tau_2|,|\tau_3|,|\xi_2|\ll T^{1+\varepsilon}} T^{\varepsilon+3R+3+\frac{-a_2-3\lambda_1+2\lambda_2-\lambda_3}{2}-3\delta_1-2\delta_3}\, d\xi_2\, d\tau_3\, d\tau_2\\ &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+6-a_1-a_2-a_3+\delta_1+\delta_2+\delta_3}.\end{aligned}$$

So, by (6.5.40) and the assumptions $\delta_j - a_j \le r_j - 1 + 1 - 2r_j \le -r_j$ for $j = 1, 3$, we have

$$\operatorname*{Res}_{\alpha_1=\widetilde{\alpha}_1}\left(p_{T,R}^{13,(\delta_1,\delta_3)}(y;-a_2,\kappa)\right) \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+\frac{13}{2}-r_1-r_2-r_3}.$$

So the sum of the residue terms in (6.5.27) also has a bound of the magnitude described in Proposition 6.5.25, and the proposition is proved. □

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6.6. Bounds for the triple residue terms

There is only one type of triple residue term to consider, namely $p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y)$. This term may be obtained by taking the residue, at $s_3 = \alpha_2 - 2\delta_3$, of the double residue term $p_{T,R}^{12,(\delta_1,\delta_2)}(y;-a_3)$ defined by (6.5.2). Thus

$$p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y) = \iiint\limits_{\mathrm{Re}(\alpha)=0} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, y_1^{\frac{3}{2}+\alpha_1+2\delta_1} y_2^{2+\alpha_1+\alpha_4+2\delta_2} y_3^{\frac{3}{2}-\alpha_2+2\delta_3}\, \mathcal{F}_R(\alpha)\, \Gamma_R(\alpha)$$
$$\cdot\, h_{\delta_1,\delta_2,\delta_3}(\alpha)\Gamma\!\left(\tfrac{\alpha_2-\alpha_1}{2}-\delta_1\right)\Gamma\!\left(\tfrac{\alpha_3-\alpha_1}{2}-\delta_1\right)\Gamma\!\left(\tfrac{\alpha_4-\alpha_1}{2}-\delta_1\right)$$
$$\cdot\, \Gamma\!\left(\tfrac{\alpha_2-\alpha_4}{2}-\delta_2\right)\Gamma\!\left(\tfrac{\alpha_3-\alpha_4}{2}-\delta_2\right)\Gamma\!\left(\tfrac{\alpha_2-\alpha_3}{2}-\delta_3\right) d\alpha, \tag{6.6.1}$$

where $h_{\delta_1,\delta_2,\delta_3}$ is a polynomial of degree at most $2\delta_1 + \delta_2$.

Proposition 6.6.2. *Let r_1, r_2, r_3 be positive integers, and $0 < \varepsilon < 1$. Suppose a_1, a_2, a_3 satisfy the hypotheses of Theorem 4.0.3. If $0 \le \delta_j \le r_j - 1$ for $1 \le j \le 3$, then*

$$\left|p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+6-r_1-r_2-r_3}. \tag{6.6.3}$$

Proof. To obtain the desired bound on $p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y)$, we shift the lines of integration in the α_js to $\mathrm{Re}\,(\alpha) = \kappa$, where $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ is such that the resulting exponents of y_1, y_2, and y_3 have real parts as indicated in the proposition. We do so by choosing

$$\kappa = (a_1 - 2\delta_1, -a_3 + 2\delta_3, -a_2 + a_3 + 2\delta_2 - 2\delta_3) \quad (\text{and } \kappa_4 = -\kappa_1 - \kappa_2 - \kappa_3). \tag{6.6.4}$$

Then

$$p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y) = \iiint\limits_{\mathrm{Re}(\alpha)=\kappa} e^{\frac{\alpha_1^2+\cdots+\alpha_4^2}{2T^2}}\, y_1^{\frac{3}{2}+\alpha_1+2\delta_1} y_2^{2+\alpha_1+\alpha_4+2\delta_2} y_3^{\frac{3}{2}-\alpha_2+2\delta_3} \mathcal{F}_R(\alpha)\, h_{\delta_1,\delta_2,\delta_3}(\alpha)$$
$$\cdot\, \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_2}{4})\Gamma(\frac{2+R+\alpha_2-\alpha_1}{4})\Gamma(\frac{\alpha_2-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_2}{2})\Gamma(\frac{\alpha_2-\alpha_1}{2})}\, \frac{\Gamma(\frac{2+R+\alpha_2-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_2}{4})\Gamma(\frac{\alpha_2-\alpha_3}{2}-\delta_3)}{\Gamma(\frac{\alpha_2-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_2}{2})}$$
$$\cdot\, \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_3}{4})\Gamma(\frac{2+R+\alpha_3-\alpha_1}{4})\Gamma(\frac{\alpha_3-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_3}{2})\Gamma(\frac{\alpha_3-\alpha_1}{2})}\, \frac{\Gamma(\frac{2+R+\alpha_2-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_2}{4})\Gamma(\frac{\alpha_2-\alpha_4}{2}-\delta_2)}{\Gamma(\frac{\alpha_2-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_2}{2})}$$
$$\cdot\, \frac{\Gamma(\frac{2+R+\alpha_1-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_1}{4})\Gamma(\frac{\alpha_4-\alpha_1}{2}-\delta_1)}{\Gamma(\frac{\alpha_1-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_1}{2})}\, \frac{\Gamma(\frac{2+R+\alpha_3-\alpha_4}{4})\Gamma(\frac{2+R+\alpha_4-\alpha_3}{4})\Gamma(\frac{\alpha_3-\alpha_4}{2}-\delta_2)}{\Gamma(\frac{\alpha_3-\alpha_4}{2})\Gamma(\frac{\alpha_4-\alpha_3}{2})}\, d\alpha. \tag{6.6.5}$$

Notice that, in this case, *there are no poles crossed in moving the lines of integration*, and therefore no residue terms to consider. This is for reasons encountered in prior situations: if R is large enough, then none of the terms

$$\Gamma\!\left(\tfrac{2+R+\alpha_k-\alpha_j}{4}\right)\Gamma\!\left(\tfrac{2+R+\alpha_j-\alpha_k}{4}\right)$$

in (6.6.5) will give rise to any poles; moreover, any pole in the numerator of a factor

$$\frac{\Gamma(\frac{\alpha_k-\alpha_j}{2}-\delta_n)}{\Gamma(\frac{\alpha_j-\alpha_k}{2})\Gamma(\frac{\alpha_k-\alpha_j}{2})}$$

will be canceled by a pole in the denominator.

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So we need only bound the right-hand side of (6.6.5), and we do so in the usual way. We get

$$\begin{aligned}\left|p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y)\right| &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+R+2\delta_1+\delta_2}\\ &\cdot \iiint\limits_{|\tau_1|,|\tau_2|,|\tau_3|\le T^{1+\varepsilon}} \big(1+|\tau_1-\tau_2|\big)^{\frac{R+1+\kappa_2-\kappa_1}{2}-\delta_1}\big(1+|\tau_1-\tau_3|\big)^{\frac{R+1+\kappa_3-\kappa_1}{2}-\delta_1}\big(1+|\tau_1-\tau_4|\big)^{\frac{R+1+\kappa_4-\kappa_1}{2}-\delta_1}\\ &\qquad\cdot\big(1+\tau_2-\tau_3\big)^{\frac{R+1+\kappa_2-\kappa_3}{2}-\delta_3}\big(1+|\tau_2-\tau_4|\big)^{\frac{R+1+\kappa_2-\kappa_4}{2}-\delta_2}\big(1+|\tau_3-\tau_4|\big)^{\frac{R+1+\kappa_3-\kappa_4}{2}-\delta_2}\, d\tau\\ &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+2\delta_1+\delta_2} \iiint\limits_{|\tau_1|,|\tau_2|,|\tau_3|\le T^{1+\varepsilon}} T^{3R+3+\frac{-3\kappa_1+3\kappa_2+\kappa_3-\kappa_4}{2}-3\delta_1-2\delta_2-\delta_3}\, d\tau\\ &\ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+6-a_1-a_2-a_3+\delta_1+\delta_2+\delta_3}.\end{aligned}$$

But we're assuming $1\le\delta_j\le r_j-1$ and $-a_j\le 1-2r_j$, whence $\delta_j-a_j\le r_j$, for $1\le j\le 3$. It follows that

$$\left|p_{T,R}^{123,(\delta_1,\delta_2,\delta_3)}(y)\right| \ll y_1^{\frac{3}{2}+a_1} y_2^{2+a_2} y_3^{\frac{3}{2}+a_3}\, T^{\varepsilon+4R+6-r_1-r_2-r_3},$$

which proves our proposition. □

7. Bounding the contribution from the continuous spectrum

Let $2\le n\le 4$. Assuming that $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{C}^n$, we define

$$p_{T,R}^{\sharp,(n)}(\alpha) := \begin{cases} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{2T^2}} \prod\limits_{1\le j\ne k\le n}\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right) & \text{if } n=2,3,\\ e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{2T^2}}\mathcal{F}_R(\alpha) \prod\limits_{1\le j\ne k\le n}\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right) & \text{if } n=4,\end{cases} \tag{7.0.1}$$

where $\mathcal{F}_R(\alpha)$ is as in (3.1.2). In the case of $n=4$, we will sometimes drop n from the notation.

Suppose that ϕ is a Maass cusp form for $\mathrm{GL}(n)$ with Langlands parameter $\alpha(\phi):=\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{C}^n$. Then we define

$$h_{T,R}^{(n)}(\phi) := \frac{\left|p_{T,R}^{\sharp,(n)}(\alpha)\right|^2}{\prod\limits_{1\le j\ne k\le n}\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}. \tag{7.0.2}$$

Theorem 7.0.3 (Weyl Law for GL(2) and GL(3)). *Suppose that $n=2$ or $n=3$. Let $\{\phi_1,\phi_2,\ldots\}$ be an orthogonal basis of Maass cusp forms for $\mathrm{GL}(n)$ ordered by eigenvalue. Then there exists a constant c_n such that*

$$\sum_j \frac{h_{T,R}^{(n)}(\phi_j)}{L(1,\mathrm{Ad}\,\phi_j)} \sim c_n\, T^{\frac{n(n-1)R}{2}+\frac{(n+2)(n-1)}{2}}. \tag{7.0.4}$$

Remark 7.0.5. In the case of $n=3$ this is Theorem 1.3 of [GK13]. The case of $n=2$ is well known, but we remark that it can be proved by the same method as for GL(3) (see [Gol]). The point is that the main term for the left-hand side of (7.0.4) is $\left|p_{T,R}^{\sharp,(n)}\right|^2$, which can be easily estimated using Stirling's estimate for the Gamma function.

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Suppose that $4 = n_1 + \cdots + n_r$ is a partition of 4 and $\Phi = (\phi_1, \ldots, \phi_r)$, where, for $1 \le j \le r$, ϕ_j is a Maass cusp form for $\mathrm{SL}(n_j, \mathbb{Z})$ if $n_j > 1$, while ϕ_j is the constant function (properly normalized) if $n_j = 1$. Let $\mathcal{P} = \mathcal{P}_{n_1,\ldots,n_r}$. Then we define

$$\mathcal{E}_{\mathcal{P},\Phi} := \int\limits_{\mathrm{Re}(s_1)=0} \cdots \int\limits_{\mathrm{Re}(s_{r-1})=0} A_{E_{\mathcal{P},\Phi}}(L,s) \cdot \overline{A_{E_{\mathcal{P},\Phi}}(M,s)} \cdot \left|p^{\#}_{T,R}\big(\alpha_{\mathcal{P},\Phi}(s)\big)\right|^2 \, ds_1 \cdots ds_{r-1},$$

and

$$\mathcal{E}_{\mathcal{P}_{\mathrm{Min}}} := \int\limits_{\mathrm{Re}(s_1)=0} \int\limits_{\mathrm{Re}(s_2)=0} \int\limits_{\mathrm{Re}(s_3)=0} A_{E_{\mathcal{P}_{\mathrm{Min}}}}(L,s)\, \overline{A_{E_{\mathcal{P}_{\mathrm{Min}}}}(M,s)} \cdot \left|p^{\#}_{T,R}\big(\alpha_{\mathcal{P}_{\mathrm{Min}}}(s)\big)\right|^2 \, ds_1\, ds_2\, ds_3.$$

Remark 7.0.6. In the above integrals, $\alpha_{\mathcal{P},\Phi}(s)$, $\alpha_{\mathcal{P}_{\mathrm{Min}}}(s)$ denote the Langlands parameters of the Eisenstein series $E_{\mathcal{P},\Phi}(g,s)$, $E_{\mathcal{P}_{\mathrm{Min}}}(g,s)$, respectively. Also, $A_{E_{\mathcal{P},\Phi}}(L,s)$, $A_{E_{\mathcal{P},\Phi}}(M,s)$ denote the L^{th} and M^{th} Fourier coefficient of $E_{\mathcal{P},\Phi}(g,s)$, and similarly for $E_{\mathcal{P}_{\mathrm{Min}}}(g,s)$.

Thus, if we define

$$\mathcal{E}_{\mathcal{P}} := \sum_{\Phi} c_{L,M,\mathcal{P}} \cdot \mathcal{E}_{\mathcal{P},\Phi},$$

then the contribution to the Kuznetsov trace formula coming from the Eisenstein series (defined in Section 3.7) is given by

$$\mathcal{E} := c_1 \mathcal{E}_{\mathcal{P}_{\mathrm{Min}}} + c_2 \mathcal{E}_{\mathcal{P}_{2,1,1}} + c_3 \mathcal{E}_{\mathcal{P}_{2,2}} + c_4 \mathcal{E}_{\mathcal{P}_{3,1}},$$

for constants $c_1, c_2, c_3, c_4 > 0$.

Theorem 7.0.7. *Suppose the Ramanujan Conjecture (at ∞) for $\mathrm{GL}(n)$ with $n \le 3$: that is, the Langlands parameters are all purely imaginary. Let $L = (\ell, 1, 1)$ and $M = (m, 1, 1)$. Then*

$$|\mathcal{E}_{\mathcal{P}_{Min}}| \ll_\varepsilon (\ell m)^{\varepsilon} \cdot T^{3+8R+\varepsilon}, \qquad |\mathcal{E}_{\mathcal{P}_{2,1,1}}| \ll_\varepsilon (\ell m)^{\frac{7}{64}+\varepsilon} \cdot T^{2+8R+\varepsilon},$$
$$|\mathcal{E}_{\mathcal{P}_{2,2}}| \ll_\varepsilon (\ell m)^{\frac{7}{32}+\varepsilon} \cdot T^{5+8R+\varepsilon}, \qquad |\mathcal{E}_{\mathcal{P}_{3,1}}| \ll_\varepsilon (\ell m)^{\frac{2}{5}+\varepsilon} \cdot T^{6+8R+\varepsilon},$$

as $T \to \infty$ for any fixed $\varepsilon > 0$.

Proof. We require the following standard notation for completed L-functions. Let

$$\zeta^*(w) = \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right) \zeta(w) = \zeta^*(1-w), \qquad (w \in \mathbb{C}).$$

For a Maass cusp form ϕ on $\mathrm{GL}(2)$ with spectral parameter $\frac{1}{2} + v$, define the completed L-function $L^*(s,\phi)$ associated with ϕ by

$$L^*(w,\phi) := \pi^{-w} \Gamma\left(\frac{w+v}{2}\right) \Gamma\left(\frac{w-v}{2}\right) L(w,\phi) = L^*(1-w,\phi), \qquad (w \in \mathbb{C}).$$

If ϕ_1, ϕ_2 are Maass cusp forms on $\mathrm{GL}(2)$ with spectral parameters $\frac{1}{2} + v$ and $\frac{1}{2} + v'$, respectively, then the completed L-function for the Rankin-Selberg convolution $L(w, \phi_1 \times \phi_2)$ is given by

$$L^*(w, \phi_1 \times \phi_2) = \pi^{-2w} \Gamma\left(\frac{w+v+v'}{2}\right) \Gamma\left(\frac{w-v+v'}{2}\right) \cdot \Gamma\left(\frac{w+v-v'}{2}\right) \Gamma\left(\frac{w-v-v'}{2}\right) L(w, \phi_1 \times \phi_2). \quad (7.0.8)$$

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Table 1. *Fourier coefficients of* SL(4, ℤ) *Langlands Eisenstein series.*

Partition Maass form Φ spectral pars.	s variables of $E_{\mathcal{P},\Phi}$ α = Langlands pars.	First coefficient $A_{E_{\mathcal{P},\Phi}}((1,1,1),s)$ (up to a constant factor)	m^{th} Hecke eigenvalue $\lambda_{E_{\mathcal{P},\Phi}}((m,1,1),s)$
4 = 1+1+1+1	$s = \frac{1}{4} + (s_1, s_2, s_3, s_4)$ $\alpha_1 = 3s_1 + 2s_2 + s_3$ $\alpha_2 = -s_1 + 2s_2 + s_3$ $\alpha_3 = -s_1 - 2s_2 + s_3$ $\alpha_4 = -s_1 - 2s_2 - 3s_3$	$\left(\prod_{1\le j<k\le 4} \zeta^*(1+\alpha_j - \alpha_k)\right)^{-1}$	$\sum_{c_1c_2c_3c_4=m} c_1^{\alpha_1} c_2^{\alpha_2} c_3^{\alpha_3} c_4^{\alpha_4}$
$4 = 2+1+1$ ϕ on GL(2) $\frac{1}{2} + \nu$	$s = \left(1 + s_1, -\frac{1}{2} + s_2, s_3\right)$ $\alpha_1 = s_1 + \nu$ $\alpha_2 = s_1 - \nu$ $\alpha_3 = s_2$ $\alpha_4 = -2s_1 - s_2$	$\Big(L(1, \mathrm{Ad}\,\phi)^{\frac{1}{2}} \left\lvert\Gamma\left(\frac{1}{2}+\nu\right)\right\rvert$ $\cdot\zeta^*(1+2s_1+2s_2)$ $\cdot L^*(1+s_1+s_2,\phi)$ $\cdot L^*(1+3s_1+s_2,\phi)\Big)^{-1}$	$\sum_{c_1c_2c_3=m} \lambda_\phi(c_1)c_1^{s_1}$ $\cdot\ c_2^{s_2} c_3^{-2s_1-s_2}$
$4 = 2+2$ $\Phi = (\phi_1, \phi_2)$ on GL(2) × GL(2) $\frac{1}{2}+\nu,\ \ \frac{1}{2}+\nu'$	$s = (1+s_1, -1-s_1)$ $\alpha_1 = s_1 + \nu$ $\alpha_2 = s_1 - \nu$ $\alpha_3 = -s_1 + \nu'$ $\alpha_4 = -s_1 - \nu'$	$\Big(L(1, \mathrm{Ad}\ \phi_1)^{\frac{1}{2}} L(1, \mathrm{Ad}\ \phi_2)^{\frac{1}{2}}$ $\cdot\left\lvert\Gamma\left(\frac{1}{2}+\nu\right)\ \Gamma\left(\frac{1}{2}+\nu'\right)\right\rvert$ $\cdot L^*(1+2s_1, \phi_1\times\phi_2)\Big)^{-1}$	$\sum_{c_1c_2=m} \lambda_{\phi_1}(c_1)$ $\cdot\lambda_{\phi_2}(c_2)\left(\frac{c_1}{c_2}\right)^{s_1}$
$4 = 3+1$ [-6pt] ϕ on GL(3) $\frac{1}{3} + (\nu, \nu')$	$s = \left(\frac{1}{2} + s_1, -\frac{3}{2} - 3s_1\right)$ $\alpha_1 = s_1 + 2\nu + \nu'$ $\alpha_2 = s_1 - \nu + \nu'$ $\alpha_3 = s_1 - \nu - 2\nu'$ $\alpha_4 = -3s_1$	$\Big(L(1, \mathrm{Ad}\ \phi)^{\frac{1}{2}} \cdot \left\lvert\Gamma\left(\frac{1+3\nu}{2}\right)\right\rvert$ $\cdot\left\lvert\Gamma\left(\frac{1+3\nu'}{2}\right)\Gamma\left(\frac{1+3\nu+3\nu'}{2}\right)\right\rvert$ $\cdot L^*(1+4s_1,\phi)\Big)^{-1}$	$\sum_{c_1c_2=m} \lambda_\phi(c_1,1)\, c_1^{s_1} c_2^{-3s_1}$

Finally, for a Maass cusp form ϕ on GL(3) with spectral parameter $\frac{1}{3} + (\nu, \nu')$ define the completed L-function $L^*(w, \phi)$ associated with ϕ by

$$L^*(w,\phi) := \pi^{-\frac{3w}{2}}\Gamma\left(\frac{w+\nu+2\nu'}{2}\right)\Gamma\left(\frac{w+\nu-\nu'}{2}\right)\Gamma\left(\frac{w-2\nu-\nu'}{2}\right)L(w,\phi) = L^*(1-w,\phi).$$

Recall the adjoint L-function of a Maass cusp form ϕ is defined by $L(w, \mathrm{Ad}\ \phi) := \frac{L(w,\phi\times\overline{\phi})}{\zeta(w)}$.

Table 1 (see [GMW]) lists, for each partition, the Maass form Φ (with its associated spectral parameters), the values of s-variables, Langlands parameters, and the Fourier-Whittaker coefficients of the SL(4, ℤ) Eisenstein series $E_{\mathcal{P},\Phi}$.Note that $E_{\mathcal{P}_{1,1,1,1},\Phi} := E_{\mathcal{P}_{\mathrm{Min}}}$.

Remark 7.0.9. The formulas given here for the first coefficient are valid when the form ϕ or ϕ_j is a Maass cusp form.

Following Table 1 of Fourier coefficients of SL(4, ℤ) Langlands Eisenstein series, we now list the integrals arising in the contribution of the continuous spectrum decomposition of the inner product of two Poincaré series given in Proposition 3.7.4. For the rest of the proof, and for each partition of 4, we will give the Langlands parameter $\alpha_{\mathcal{P},\Phi}(s) := (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for $(\mathcal{P}, \Phi)$ and then use Theorem 7.0.3 to obtain the result.

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In each case below we will use the fact that for any $1 \le j, k \le 4$ and $\mathrm{Re}(\alpha_j) = \mathrm{Re}(\alpha_k) = 0$,

$$\frac{\left|\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)\right|^2} \sim c\,\left(1+|\alpha_j-\alpha_k|\right)^R \tag{7.0.10}$$

for some constant c. This follows trivially from Stirling's estimate.

We also will use the bound of Luo-Rudnick-Sarnak (see [LRS99]) for the m^{th}-Fourier coefficient of a $\mathrm{GL}(n)$ $(n \ge 2)$ Maass cusp form ϕ:

$$\lambda_\phi(m, 1, \dots, 1) \ll m^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}. \tag{7.0.11}$$

Note that this is proved in [Gol15] as well. In the special case of GL(2), Kim and Sarnak prove in the appendix of [Kim96] that

$$\lambda_\phi(m) \ll m^{\frac{7}{64}+\varepsilon}. \tag{7.0.12}$$

The integral $\mathcal{E}_{\mathcal{P}_{\mathrm{Min}}}$:

The Langlands parameters for $E_{\mathcal{P}_{Min}}(s)$ with $s = (s_1, s_2, s_3)$ these are given by

$$\alpha_{P_{\mathrm{Min}}}(s) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

$$\alpha_1 = 3s_1 + 2s_2 + s_3,\ \ \alpha_2 = -s_1 + 2s_2 + s_3,\ \ \alpha_3 = -s_1 - 2s_2 + s_3,\ \ \alpha_4 = -s_1 - 2s_2 - 3s_3.$$

Therefore, using Table 1 and the fact that for any $\varepsilon > 0$

$$\lambda_{E_{\mathcal{P}_{\mathrm{Min}}}}\big((m,1,1),s\big) = \sum_{c_1c_2c_3c_4=m} c_1^{\alpha_1}c_2^{\alpha_2}c_3^{\alpha_3}c_4^{\alpha_4} \ll m^\varepsilon \tag{7.0.13}$$

whenever $\mathrm{Re}(\alpha_j) = 0$ $(j = 1, 2, 3, 4)$, we see that

$$\begin{aligned}\mathcal{E}_{\mathcal{P}_{\mathrm{Min}}} &= \int\limits_{\mathrm{Re}(s_1)=0}\int\limits_{\mathrm{Re}(s_2)=0}\int\limits_{\mathrm{Re}(s_3)=0} A_{E_{\mathcal{P}_{\mathrm{Min}}}}(L,s)\,\overline{A_{E_{\mathcal{P}_{\mathrm{Min}}}}(M,s)}\cdot\left|p^{\#}_{T,R}\big(\alpha_{\mathcal{P}_{\mathrm{Min}}}(s)\big)\right|^2\,ds_1ds_2ds_3\\ &\ll \int\limits_{\mathrm{Re}(s_1)=0}\int\limits_{\mathrm{Re}(s_2)=0}\int\limits_{\mathrm{Re}(s_3)=0} \frac{e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{T^2}}\cdot(\ell m)^\varepsilon}{\prod\limits_{1\le j<k\le 4}\left|\zeta(1+\alpha_j-\alpha_k)\right|^2}\cdot\prod_{1\le j<k\le 4}\frac{\left|\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)\right|^2}\\ &\quad\cdot\Big(\big(1+|\alpha_1+\alpha_2-\alpha_3-\alpha_4|\big)\big(1+|\alpha_1+\alpha_3-\alpha_2-\alpha_4|\big)\big(1+|\alpha_1+\alpha_4-\alpha_2-\alpha_3|\big)\Big)^{\frac{2R}{3}}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |ds_1\,ds_2\,ds_3|.\end{aligned}$$

If we make the change of variables

$$\alpha_1 = 3s_1 + 2s_2 + s_3, \quad \alpha_2 = -s_1 + 2s_2 + s_3, \quad \alpha_3 = -s_1 - 2s_2 + s_3,$$

(which implies that $\alpha_4 = -s_1 - 2s_2 - 3s_3$) in the above integral, it follows from the Jacobian transformation that $ds_1\,ds_2\,ds_3$ maps to $\frac{d\alpha_1\,d\alpha_2\,d\alpha_3}{32}$.

Now, we have the Vinogradov (see [Vin58]) bound

$$\frac{1}{|\zeta(1+it)|} \ll \big(1+|t|\big)^\varepsilon, \qquad (t\in\mathbb{R}), \tag{7.0.14}$$

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which together with the above coordinate change imply that

$$E_{\mathcal{P}_{\mathrm{Min}}} \ll (\ell m)^{\varepsilon} \iiint\limits_{\substack{\mathrm{Re}(\alpha)=0\\ |\alpha|\le T}} \prod_{1\le j<k\le 4} \left(1+|\alpha_j-\alpha_k|\right)^{R+\varepsilon} \Big(\left(1+|\alpha_1+\alpha_2-\alpha_3-\alpha_4|\right)$$

$$\cdot \left(1+|\alpha_1+\alpha_3-\alpha_2-\alpha_4|\right)\left(1+|\alpha_1+\alpha_4-\alpha_2-\alpha_3|\right)\Big)^{\frac{2R}{3}} |d\alpha_1\, d\alpha_2\, d\alpha_3|.$$

Next, make the change of variables $\alpha_1 \to \alpha_1 T, \quad \alpha_2 \to \alpha_2 T, \quad \alpha_3 \to \alpha_3 T$. It easily follows that

$$E_{\mathcal{P}_{\mathrm{Min}}} \ll (\ell m)^{\varepsilon}\, T^{8R+3+\varepsilon}.$$

<u>The integral $\mathcal{E}_{\mathcal{P}_{2,1,1},\Phi}$:</u>

We take ϕ to be a GL(2) Maass cusp form with spectral parameter $\frac{1}{2}+\nu$, where $\nu \in \mathbb{C}$ is pure imaginary. The Langlands parameters $\alpha_{\mathcal{P}_{2,1,1},\Phi}(s)$ for $E_{\mathcal{P}_{2,1,1},\Phi}(s)$ with $s=(s_1,s_2,s_3)$ are given by $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ where

$$\alpha_1 = s_1+\nu, \quad \alpha_2 = s_1-\nu, \quad \alpha_3 = s_2, \quad \alpha_4 = -2s_1-s_2.$$

It follows that

$$\mathcal{E}_{\mathcal{P}_{2,1,1},\Phi} := \int\limits_{\mathrm{Re}(s_1)=0}\ \int\limits_{\mathrm{Re}(s_2)=0} A_{E_{\mathcal{P}_{2,1,1},\Phi}}(L,s)\cdot \overline{A_{E_{\mathcal{P}_{2,1,1},\Phi}}(M,s)}\cdot \left|p^{\#}_{T,R}(\alpha)\right|^2\, ds_1 ds_2$$

$$= \int\limits_{\mathrm{Re}(s_1)=0}\ \int\limits_{\mathrm{Re}(s_2)=0} e^{\frac{(s_1+\nu)^2+(s_1-\nu)^2+s_2^2+(2s_1+s_2)^2}{T^2}} \cdot \lambda_{E_{\mathcal{P}_{2,1,1},\Phi}}(L,s)\, \overline{\lambda_{E_{\mathcal{P}_{2,1,1},\Phi}}(M,\ s)}$$

$$\cdot \Big(\left(1+|4s_1|\right)\left(1+|2s_1+2s_2+2\nu|\right)\left(1+|2s_1+2s_2-2\nu|\right)\Big)^{\frac{2R}{3}}$$

$$\cdot \frac{\left|\Gamma\left(\frac{2+R+s_1-s_2-\nu}{4}\right)\Gamma\left(\frac{2+R+s_1-s_2+\nu}{4}\right)\Gamma\left(\frac{2+R+3s_1+s_2-\nu}{4}\right)\Gamma\left(\frac{2+R+3s_1+s_2+\nu}{4}\right)\right|^4}{\left|L^*(1+s_1-s_2,\phi)L^*(1+3s_1+s_2,\phi)\right|^2}$$

$$\cdot \frac{\left|\Gamma\left(\frac{2+R+2\nu}{4}\right)\Gamma\left(\frac{2+R+2s_1+2s_2}{4}\right)\right|^4}{L(1,\mathrm{Ad}\ \phi)\left|\Gamma\left(\frac{1+2\nu}{2}\right)\right|^2\left|\zeta^*(1+2s_1+2s_2)\right|^2}\, ds_1\, ds_2,$$

from which we obtain the bound

$$\mathcal{E}_{\mathcal{P}_{2,1,1},\Phi} \ll \frac{e^{\frac{2\nu^2}{T^2}}\left|\Gamma\left(\frac{2+R+2\nu}{4}\right)\right|^4 T^{2R}}{L(1,\mathrm{Ad}\,\phi)\left|\Gamma\left(\frac{1+2\nu}{2}\right)\right|^2} \int\limits_{\mathrm{Re}(s_1)=0}\ \int\limits_{\mathrm{Re}(s_2)=0} e^{\frac{2s_1^2+s_2^2+(2s_1+s_2)^2}{T^2}}$$

$$\cdot \frac{\left|\Gamma\left(\frac{2+R+s_1-s_2-\nu}{4}\right)\Gamma\left(\frac{2+R+s_1-s_2+\nu}{4}\right)\Gamma\left(\frac{2+R+3s_1+s_2-\nu}{4}\right)\Gamma\left(\frac{2+R+3s_1+s_2+\nu}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_1-s_2-\nu}{2}\right)\Gamma\left(\frac{1+s_1-s_2+\nu}{2}\right)\Gamma\left(\frac{1+3s_1+s_2-\nu}{2}\right)\Gamma\left(\frac{1+3s_1+s_2+\nu}{2}\right)\right|^2}$$

$$\cdot \frac{\lambda_{E_{\mathcal{P}_{2,1,1},\Phi}}(L,s)\,\overline{\lambda_{E_{\mathcal{P}_{2,1,1},\Phi}}(M,s)}\left|\Gamma\left(\frac{2+R+2s_1+2s_2}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+2s_2-2s_1}{2}\right)\zeta(1+2s_1+2s_2)L(1+s_1-s_2,\phi)L(1+3s_1+s_2,\phi)\right|^2}\, |ds_1 ds_2|.$$

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Here

$$\lambda_{E_{\mathcal{P}_{2,1,1},\Phi}}\Big((m,1,1),\ s\Big) = \sum_{c_1c_2c_3=m} \lambda_\phi(c_1)\cdot c_1^{s_1}c_2^{s_2}c_3^{-2s_1-s_2} \ll m^{\frac{7}{64}+\varepsilon},$$

by the bound for $\lambda_\phi(c)$ given in (7.0.12). It follows from [HL94], [HR95], [GLS04] that

$$L(1+it,\phi) \gg (1+|\nu|+|t|)^{-\varepsilon}.$$

It then follows from the above bound, Stirling's estimate for the Gamma function and (7.0.14), that

$$\begin{aligned}\mathcal{E}_{\mathcal{P}_{2,1,1},\Phi} &\ll (\ell m)^{\frac{7}{64}+\varepsilon}\cdot T^{2+7R+\varepsilon}\cdot \frac{e^{\frac{2\nu^2}{T^2}}\left|\Gamma\left(\frac{2+R+2\nu}{4}\right)\right|^4}{L(1,\mathrm{Ad}\,\phi_j)\left|\Gamma\left(\frac{1+2\nu}{2}\right)\right|^2}\\ &= (\ell m)^{\frac{7}{64}+\varepsilon}\cdot T^{2+8R+\varepsilon}\cdot\frac{h^{(2)}_{T,R}(\phi_j)}{L(1,\mathrm{Ad}\,\phi_j)}.\end{aligned}$$

To bound $\mathcal{E}_{\mathcal{P}_{2,1,1}}$, we simply sum $\mathcal{E}_{\mathcal{P}_{2,1,1},\Phi}$ over all Maass cusp forms Φ for $SL(2,\mathbb{Z})$ using the Weyl law for $GL(2)$ given in Theorem 7.0.3. The stated result follows.

The integral $\mathcal{E}_{\mathcal{P}_{2,2},\Phi}$: Here, we take $\Phi = (\phi_1,\phi_2)$ to be Maass cusp forms for $GL(2)$ with spectral parameters $\frac12+\nu,\ \frac12+\nu'$, respectively. The Langlands parameters $\alpha_{\mathcal{P}_{2,2},\Phi}$ for $E_{\mathcal{P}_{2,2},\Phi}(s)$ with $s=(s_1,s_2)$ are given by

$$\alpha_1 = s_1+\nu,\quad \alpha_2 = s_1-\nu,\quad \alpha_3=-s_1+\nu',\quad \alpha_4=-s_1-\nu'.$$

It follows that

$$p^{\sharp,(4)}_{T,R}(\alpha) = e^{\frac{2s_1^2+\nu^2+\nu'^2}{T^2}}\cdot\Gamma\left(\frac{2+R+2\nu}{4}\right)\Gamma\left(\frac{2+R+2\nu'}{4}\right)\prod_{\delta,\delta'\in\{\pm1\}}\left|\Gamma\left(\frac{2+R+2s_1+\delta\nu+\delta'\nu'}{4}\right)\right|^2$$
$$\cdot\Big(\big(1+4|s_1|\big)\big(1+2|\nu+\nu'|\big)\big(1+2|\nu-\nu'|\big)\Big)^{\frac{R}{3}}.$$

Using this and the fact that

$$L^*(1+2s_1,\phi_1\times\phi_2) = \pi^{-2(1+2s_1)}L(1+2s_1,\phi_1\times\phi_2)\prod_{\delta,\delta'\in\{\pm1\}}\left|\Gamma\left(\frac{1+2s_1+\delta\nu+\delta'\nu'}{2}\right)\right|,$$

we see that

$$\mathcal{E}_{\mathcal{P}_{2,2},\Phi} \ll T^{2R+\varepsilon}\frac{h^{(2)}_{T,R}(\phi_1)}{L(1,\mathrm{Ad}\,\phi_1)}\frac{h^{(2)}_{T,R}(\phi_2)}{L(1,\mathrm{Ad}\,\phi_2)}\int\limits_{\mathrm{Re}(s_1)=0}\frac{\left|\lambda_{E_{\mathcal{P}_{2,2},\Phi}}(L,s)\,\overline{\lambda_{E_{\mathcal{P}_{2,2},\Phi}}(M,\ s)}\right|}{|L(1+2s_1,\phi_1\times\phi_2)|^2}$$
$$\cdot\, e^{\frac{4s_1^2}{T^2}}\prod_{\delta,\delta'\in\{\pm1\}}\frac{\left|\Gamma\left(\frac{2+R+2s_1+\delta\nu+\delta'\nu'}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+2s_1+\delta\nu+\delta'\nu'}{2}\right)\right|^2}\,|ds_1|,$$

and by the bounds for $\lambda_{\phi_1}(c),\lambda_{\phi_2}(c)$ given in (7.0.12),

$$\lambda_{E_{\mathcal{P}_{2,2},\Phi}}\Big((m,1,1),s\Big) = \sum_{c_1c_2=m}\lambda_{\phi_1}(c_1)\,\lambda_{\phi_2}(c_2)\cdot\left(\frac{c_1}{c_2}\right)^{s_1} \ll m^{\frac{7}{32}+\varepsilon}.$$

It follows from [HR95], [Mor85] that

$$\frac{1}{L(1+2s_1,\phi_1\times\phi_2)} \ll \big(1+|s_1|+|\nu|+|\nu'|\big)^{\varepsilon}.$$

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Note that the bound for the case that $|\mathrm{Im}(s_1)|$ is small involves the non-existence of Siegel zeros that is proved in [HR95] while the case when $|\mathrm{Im}(s_1)|$ is large was first proved in [Mor85]. See also [GL18], [HB19].

Applying these bounds, the Theorem 7.0.3 bound, together with Stirling's bound to estimate the integral in s_1, we find, as claimed, that

$$|\mathcal{E}_{\mathcal{P}_{2,2}}| \ll (\ell m)^{\frac{7}{32}+\varepsilon} \cdot T^{\varepsilon+6R+1} \sum_{(\phi_1,\phi_2)} \frac{h^{(2)}_{T,R}(\phi_1)}{L(1,\mathrm{Ad}\,\phi_1)} \frac{h^{(2)}_{T,R}(\phi_2)}{L(1,\mathrm{Ad}\,\phi_2)} \ll (\ell m)^{\frac{7}{32}+\varepsilon} \cdot T^{\varepsilon+8R+3}.$$

The integral $\mathcal{E}_{\mathcal{P}_{3,1},\Phi}$:

Let $\beta = (\beta_1, \beta_2, \beta_3)$ and $\frac{1}{3}+(\nu,\nu')$ denote the Langlands and spectral parameters, respectively, associated with a Maass cusp form ϕ on $\mathrm{GL}(3)$. Here

$$\beta_1 = 2\nu + \nu', \quad \beta_2 = -\nu + \nu', \quad \beta_3 = -\nu - 2\nu'.$$

The Langlands parameters $\alpha_{\mathcal{P}_{3,1},\Phi}(s)$ for $E_{\mathcal{P}_{3,1},\Phi}(s)$ with $s = (s_1, -3s_1)$ are given by $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where

$$\alpha_1 = s_1 + \beta_1, \quad \alpha_2 = s_1 + \beta_2, \quad \alpha_3 = s_1 + \beta_3, \quad \alpha_4 = -3s_1.$$

Note that in this case, since

$$\sum_{j=1}^{4} \alpha_j^2 = 9s_1^2 + \sum_{j=1}^{3} (s_1+\beta_j)^2 = 12 s_1^2 + \sum_{j=1}^{3} \beta_j^2$$

and $s_1, \beta_1, \beta_2, \beta_3$ are purely imaginary, we have

$$p^{\sharp}_{T,R}(\alpha) = p^{\sharp,(3)}_{T,R}(\beta) \cdot e^{\frac{6s_1^2}{T^2}} \cdot \prod_{k=1}^{3} \left|\Gamma\left(\tfrac{2+R+4s_1-\beta_k}{4}\right)\right|^2 \cdot \Big(\big(1 + |\beta_1 - \beta_2 - \beta_3 - 4s_1|\big) \\ \cdot \big(1+|\beta_1+\beta_2-\beta_3+4s_1|\big)\big(1+|\beta_1-\beta_2+\beta_3+4s_1|\big)\Big)^{\frac{R}{3}}$$

where $p^{\sharp,(3)}_{T,R}(\beta)$ is defined by (7.0.1).

This allows us to write

$$\mathcal{E}_{\mathcal{P}_{3,1},\Phi} := \int\limits_{\mathrm{Re}(s_1)=0} A_{E_{\mathcal{P}_{3,1},\Phi}}(L,s) \cdot \overline{A_{E_{\mathcal{P}_{3,1},\Phi}}(M,s)} \cdot \left|p^{\sharp}_{T,R}(\alpha)\right|^2 \, ds_1 \\ = \int\limits_{\mathrm{Re}(s_1)=0} \frac{\lambda_{E_{\mathcal{P}_{3,1},\Phi}}(L,s) \cdot \overline{\lambda_{E_{\mathcal{P}_{3,1},\Phi}}(M,s)}}{L(1,\mathrm{Ad}\,\phi)\cdot |L^*(1+4s_1,\phi)|^2} \cdot h^{(3)}_{T,R}(\beta) \cdot e^{\frac{12s_1}{T^2}} \\ \cdot \prod_{k=1}^{3} \left|\Gamma\left(\tfrac{2+R+4s_1+\beta_k}{4}\right)\right|^4 \cdot \Big(\big(1 + |\beta_1 - \beta_2 - \beta_3 - 4s_1|\big) \\ \cdot \big(1+|\beta_1+\beta_2-\beta_3+4s_1|\big)\big(1+|\beta_1-\beta_2+\beta_3+4s_1|\big)\Big)^{\frac{2R}{3}} \, ds_1$$

where, using (7.0.11),

$$\sum_{c_1c_2=m} \lambda_\phi(c_1,1) \cdot c_1^{s_1} c_2^{-3s_1} \ll m^{\frac{2}{5}+\varepsilon}. \tag{7.0.15}$$

In the above

$$L^*(1+4s_1,\phi) = \pi^{-\frac{3+12s_1}{2}} L(1+4s_1,\phi) \prod_{j=1}^{3} \Gamma\left(\tfrac{1+4s_1+\beta_j}{2}\right).$$

It follows from [Mor85] and [HR95] that for every $\varepsilon > 0$

$$L(1+4s_1,\phi) \;\gg_\varepsilon\; \frac{1}{\big(1+|s_1|+|\nu|+|\nu'|\big)^{\varepsilon}}, \tag{7.0.16}$$

where the implied constant in the $\gg_\varepsilon$ symbol is effective unless ϕ is a self-dual Maass cusp form that is not a symmetric square lift from $\mathrm{GL}(2)$. Note that the bound for the case that $|\mathrm{Im}(s_1)|$ is small involves the non-existence of Siegel zeros that is proved in [HR95] while the case when $|\mathrm{Im}(s_1)|$ is large was first proved in [Mor85]. See also [Sar04].

Let $\{\phi_1,\phi_2,\ldots\}$ be the Maass cusp forms for $\mathrm{GL}(3)$ ordered by eigenvalue, and set $\mathcal{L}_j := L(1,\mathrm{Ad}\,\phi_j)$. It follows from (7.0.15) and (7.0.16) that

$$\begin{aligned}\sum_j \mathcal{E}_{\mathcal{P}_{3,1},\phi_j} &= \sum_j \frac{h^{\sharp,(3)}_{T,R}(\beta^{(j)})}{\mathcal{L}_j} \int\limits_{\mathrm{Re}(s_1)=0} \frac{\lambda_{E_{\mathcal{P}_{3,1},\phi_j}}(L,s)\cdot\overline{\lambda_{E_{\mathcal{P}_{3,1},\phi_j}}(M,s)}}{|L(1+4s_1,\phi_j)|^2}\, e^{\frac{12s_1^2}{T^2}} \\ &\quad\cdot \prod_{k=1}^{3} \frac{\left|\Gamma\big(\frac{2+R+4s_1+\beta_k^{(j)}}{4}\big)\right|^4}{\left|\Gamma\big(\frac{1+4s_1+\beta_k^{(j)}}{2}\big)\right|^2} \cdot \Big(\big(1+|\beta_1-\beta_2-\beta_3-4s_1|\big) \\ &\quad\cdot \big(1+|\beta_1+\beta_2-\beta_3+4s_1|\big)\big(1+|\beta_1-\beta_2+\beta_3+4s_1|\big)\Big)^{\frac{2R}{3}}\, ds_1 \\ &\ll (\ell m)^{\frac{2}{5}+\varepsilon}\cdot T^{2R} \sum_j \frac{h^{(3)}_{T,R}(\beta^{(j)})}{\mathcal{L}_j} \int\limits_{\mathrm{Re}(s_1)=0} \frac{e^{\frac{12s_1^2}{T^2}}}{L(1+s_1,\phi_j)} \prod_{k=1}^{3} \frac{\left|\Gamma\big(\frac{2+R+4s_1+\beta_k^{(j)}}{4}\big)\right|^4}{\left|\Gamma\big(\frac{1+4s_1+\beta_k^{(j)}}{2}\big)\right|^2}\, |ds_1|.\end{aligned}$$

Using Stirling's estimate for the Gamma functions here, it easy to see that

$$\prod_{k=1}^{3} \frac{\left|\Gamma\big(\frac{2+R+4s_1+\beta_k^{(j)}}{4}\big)\right|^4}{\left|\Gamma\big(\frac{1+4s_1+\beta_k^{(j)}}{2}\big)\right|^2} \ll \prod_{k=1}^{3} \big(1+|4s_1+2\beta_k^{(j)}|\big)^R.$$

Combining this with the previous bounds and Theorem 7.0.3, we find, as claimed, that

$$\mathcal{E}_{\mathcal{P}_{3,1}} \ll \sum_j \left|\mathcal{E}_{\mathcal{P}_{3,1},\phi_j}\right| \ll (\ell m)^{\frac{2}{5}+\varepsilon}\cdot T^{6+8R+\varepsilon}. \qquad \square$$

Remark 7.0.17. As outlined in [Blo13], it should be possible to obtain bounds for the more general case of $L=(\ell_1,\ell_2,\ell_3)$ and $M=(m_1,m_2,m_3)$ using the relations for the $\mathrm{GL}(4)$ Hecke operators.

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Appendix A. Integral bounds

Lemma A.1. *Suppose that e, f are real numbers. Then*

$$\int_{x=0}^{T} \frac{dx}{(1+T-x)^e (1+x)^f} \ll (1+T)^{-\min\{e,f,e+f-1\}+\varepsilon}$$

for any $T \geq 0$ and $\varepsilon > 0$, and the implied constant does not depend on T.

Proof. We consider the integrals

$$\int_{x=0}^{T/2} (1+T-x)^{-e}(1+x)^{-f}\ dx \quad \text{and} \quad \int_{x=T/2}^{T} (1+T-x)^{-e}(1+x)^{-f}\ dx$$

individually. Since

$$\int_{x=1}^{T/2} x^{-f}\ dx \ll \begin{cases} T^{-f+1}+1 & \text{if } f \neq 1, \\ \log T + 1 & \text{if } f = 1, \end{cases}$$

it follows in the case of $f \neq 1$ that

$$\int_{x=0}^{T/2} (1+T-x)^{-e}(1+x)^{-f}\ dx \ll (1+T)^{-e}\left(1+(1+T)^{-f+1}\right).$$

In like fashion, we find that if $e \neq 1$,

$$\int_{x=T/2}^{T} (1+T-x)^{-e}(1+x)^{-f}\ dx \ll (1+T)^{-e-f+1} + (1+T)^{-f}.$$

Putting this together proves the result. In the case that $e = 1$ or $f = 1$, the logarithm contributes T^{ε} as claimed. ◻

Lemma A.2. *Assume $B_1 \leq B_2 \leq \cdots \leq B_k$. Then for any $\varepsilon > 0$*

$$\int_{x=B_1}^{B_k} \prod_{i=1}^{k} (1+|x-B_j|)^{-e_j}\ dx \ll (1+B_k-B_1)^{\varepsilon}$$
$$\cdot \sum_{j=1}^{k-1} (1+B_{j+1}-B_j)^{-\min\{e_j,e_{j+1},e_j+e_{j+1}-1\}} \prod_{i\neq j,j+1} (1+|B_j^*(i)-B_i|)^{-e_i},$$

where

$$B_j^*(i) := \begin{cases} B_j & \text{if } i < j \text{ and } e_i > 0, \\ B_{j+1} & \text{if } i < j \text{ and } e_i < 0, \\ B_{j+1} & \text{if } i > j+1 \text{ and } e_i > 0, \\ B_j & \text{if } i > j+1 \text{ and } e_i < 0. \end{cases}$$

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Proof. First,

$$\int_{x=B_1}^{B_k} = \sum_{j=1}^{k-1} \int_{x=B_j}^{B_{j+1}}.$$

For every $j = 1, 2, \ldots, k-1$, we have

$$\int_{B_j}^{B_{j+1}} \prod_{i=1}^{k} \left(1+|x-B_i|\right)^{-e_i} dx \ll \int_{B_j}^{B_{j+1}} \left(1+x-B_j\right)^{-e_j} \left(1+B_{j+1}-x\right)^{-e_{j+1}}$$
$$\cdot \prod_{i=1}^{j-1} \left(1+x-B_i\right)^{-e_i} \prod_{\ell=j+2}^{k} \left(1+B_i-x\right)^{-e_\ell}\, dx.$$

For each of the terms with $i < j$ and any $B_j \le x \le B_{j+1}$,

$$\left(1+x-B_i\right)^{-e_i} \ll \begin{cases} \left(1+B_j-B_i\right)^{-e_i} & \text{if } e_i > 0, \\ \left(1+B_{j+1}-B_i\right)^{-e_i} & \text{otherwise.} \end{cases}$$

A similar bound holds for the terms with $\ell > j+1$. So in order to complete the proof, we need the bound

$$\int_{B_j}^{B_{j+1}} \left(1+x-B_j\right)^{-e_j} \left(1+B_{j+1}-x\right)^{-e_{j+1}}\, dx$$
$$\ll \left(1+B_k-B_1\right)^{\varepsilon} \left(1+B_{j+1}-B_j\right)^{-\min\left\{e_j, e_{j+1}, e_j+e_{j+1}-1\right\}},$$

which follows from Lemma A.1 by a simple change of variables. □

Lemma A.3. *Assume $B_1 \le B_2 \le \cdots \le B_k$. Suppose that $1 \le j_{min} < j_{max} \le k$. Then for any $\varepsilon > 0$,*

$$\int_{x=B_{j_{\min}}}^{B_{j_{\max}}} \prod_{i=1}^{k} \left(1+|x-B_j|\right)^{-e_j}\, dx \ll \left(1+B_{j_{\max}}-B_{\min}\right)^{\varepsilon}$$
$$\cdot \sum_{j=j_{\min}}^{j_{\max}-1} \left(1+B_{j+1}-B_j\right)^{-\min\left\{e_j, e_{j+1}, e_j+e_{j+1}-1\right\}} \prod_{i\ne j, j+1} \left(1+|B_j^*(i)-B_i|\right)^{-e_i},$$

where

$$B_j^*(i) := \begin{cases} B_j & \text{if } i < j \text{ and } e_i > 0, \\ B_{j+1} & \text{if } i < j \text{ and } e_i < 0, \\ B_{j+1} & \text{if } i > j+1 \text{ and } e_i > 0, \\ B_j & \text{if } i > j+1 \text{ and } e_i < 0. \end{cases}$$

Proof. First,

$$\int_{x=B_{j_{\min}}}^{B_{j_{\max}}} = \sum_{j=j_{\min}}^{j_{\max}-1} \int_{x=B_j}^{B_{j+1}}.$$

For every $j = 1, 2, \ldots, k-1$, we have

$$\int_{B_j}^{B_{j+1}} \prod_{i=1}^{k} (1 + |x - B_i|)^{-e_i} dx \ll \int_{B_j}^{B_{j+1}} (1 + x - B_j)^{-e_j} (1 + B_{j+1} - x)^{-e_{j+1}} \cdot \prod_{i=1}^{j-1} (1 + x - B_i)^{-e_i} \cdot \prod_{\ell=j+2}^{k} (1 + B_i - x)^{-e_\ell} \, dx.$$

For each of the terms with $i < j$ and any $B_j \le x \le B_{j+1}$,

$$(1 + x - B_i)^{-e_i} \ll \begin{cases} (1 + B_j - B_i)^{-e_i} & \text{if } e_i > 0, \\ (1 + B_{j+1} - B_i)^{-e_i} & \text{otherwise.} \end{cases}$$

A similar bound holds for the terms with $\ell > j+1$. So in order to complete the proof, we need the bound

$$\int_{B_j}^{B_{j+1}} (1 + x - B_j)^{-e_j} (1 + B_{j+1} - x)^{-e_{j+1}} \, dx \ll (1 + B_k - B_1)^{\varepsilon} \cdot (1 + B_{j+1} - B_j)^{-\min\{e_j, e_{j+1}, e_j + e_{j+1} - 1\}},$$

which follows from Lemma A.1 by a simple change of variables. □

Appendix B. Kloosterman sums on GL(4) by Bingrong Huang

B.1. Introduction

The classical Kloosterman sum is given by

$$S(m, n; c) = \sum_{d \pmod c}^{*} e\left(\frac{md + n\bar{d}}{c}\right),$$

where $d\bar{d} \equiv 1 \pmod c$ and $e(x) = e^{2\pi i x}$, which arises when one computes the Fourier expansion of the GL(2) Poincaré series. Weil [Wei48] obtained the algebreo-geometric estimate

$$|S(m, n; c)| \le \gcd(m, n, c)^{1/2} c^{1/2} \tau(c),$$

where $\tau(\cdot)$ is the divisor function. Bump, Friedberg and Goldfeld [BFG88] introduced Poincaré series for $GL(n)$, $n \ge 2$, and showed in the case $n = 3$ that certain ‘GL(3) Kloosterman sums’ arise in the Fourier expansion. Friedberg [Fri87] and Stevens [Ste87] extended this result to all n, studying $GL(n)$ Poincaré series and their related $GL(n)$ Kloosterman sums, from the classical and adelic points of view, respectively. Friedberg, following the work of Larsen ($n = 3$) [BFG88], obtained upper bounds for $GL(n)$ in certain cases. Stevens [Ste87] gave a nontrivial estimate for the GL(3) Kloosterman sum corresponding to the long element of the Weyl group. By their results, we get nontrivial upper bounds for all GL(3) Kloosterman sums.

In this appendix, we consider all GL(4) Kloosterman sums. We will write $\mathbb{Q}_p$ for the completion of $\mathbb{Q}$ at a place p and write $\mathbb{A}$ for the adeles of $\mathbb{Q}$. Let $G = GL(4)$. Let W be the Weyl group of G. Let $U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ be the standard unipotent group, and let

$$U_w = (w^{-1} \cdot U \cdot w) \cap U, \quad \bar{U}_w = (w^{-1} \cdot {}^t U \cdot w) \cap U, \quad w \in W.$$

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Table B.1. *Main results for* GL(4) *Kloosterman sums.*

Weyl element	Compatibility conditions	Upper bounds of the Kloosterman sum
$w_1 = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$	$m_1 = n_1, m_2 = n_2,$ $m_3 = n_3;$ $c_1 = c_2 = c_3 = 1$	$\delta_{c_1,1}\delta_{c_2,1}\delta_{c_3,1}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}$
$w_2 = \begin{pmatrix} &&& -1 \\ 1 &&& \\ & 1 && \\ && 1 & \end{pmatrix}$	$n_1 = \frac{c_1c_3m_2}{c_2^2}, n_2 = \frac{c_2m_3}{c_1^2};$ $c_1 \vert c_2 \vert c_3$	• Friedberg [Fri87, Theorem C]; • $\tau(c_1c_2c_3)^{\kappa_2}(m_1, c_3/c_2)^{1/4}(m_2, c_2/c_1)^{1/2}$ $\cdot(m_3, n_3, c_1)^{3/4}(c_1c_2c_3)^{3/4}$
$w_3 = \begin{pmatrix} & 1 && \\ && 1 & \\ &&& 1 \\ -1 &&& \end{pmatrix}$	$n_3 = \frac{c_1c_3m_2}{c_2^2}, n_2 = \frac{c_2m_1}{c_3^2};$ $c_3 \vert c_2 \vert c_1$	• Similar to Friedberg [Fri87, Theorem C]; • $\tau(c_1c_2c_3)^{\kappa_2}(m_3, c_1/c_2)^{1/4}(m_2, c_2/c_3)^{1/2}$ $\cdot(m_1, n_1, c_3)^{3/4}(c_1c_2c_3)^{3/4}$
$w_4 = \begin{pmatrix} && 1 & \\ &&& 1 \\ 1 &&& \\ & 1 && \end{pmatrix}$	$n_1 = \frac{m_3c_2}{c_1^2}, n_3 = \frac{m_1c_2}{c_3^2};$ $c_1 \vert c_2,\ c_3 \vert c_2$	$c_1c_2c_3$
$w_5 = \begin{pmatrix} &&& -1 \\ & 1 && \\ && 1 & \\ 1 &&& \end{pmatrix}$	$n_2 = \frac{c_1c_3m_2}{c_2^2}$	$c_1c_2c_3$
$w_6 = \begin{pmatrix} && 1 & \\ &&& 1 \\ & 1 && \\ -1 &&& \end{pmatrix}$	$n_3 = \frac{c_2m_1}{c_3^2};$ $c_3 \vert c_2$	$c_1c_2c_3$
$w_7 = \begin{pmatrix} &&& -1 \\ && 1 & \\ 1 &&& \\ & 1 && \end{pmatrix}$	$n_1 = \frac{c_2m_3}{c_1^2};$ $c_1 \vert c_2$	$c_1c_2c_3$
$w_8 = \begin{pmatrix} &&& 1 \\ && 1 & \\ & 1 && \\ 1 &&& \end{pmatrix}$		• $\tau(c_1c_2c_3)^{\kappa_8}(m_1n_3, [c_1, c_2, c_3])^{1/2}$ $\cdot(m_2n_2, [c_1, c_2, c_3])^{1/2}(m_3n_1, [c_1, c_2, c_3])^{1/2}$ $\cdot \min\left\{[c_1, c_3]^{1/2}(c_1, c_3)c_2\ ,\ c_1c_3(c_1, c_3)c_2^{1/2}\right\}$ • $\tau(c_1c_2c_3)^{\kappa_8}(m_1n_3, [c_1, c_2, c_3])^{1/2}$ $\cdot(m_2n_2, [c_1, c_2, c_3])^{1/2}$ $\cdot(m_3n_1, [c_1, c_2, c_3])^{1/2}(c_1c_2c_3)^{9/10}$

Let $c_1, \ldots, c_3$ be non-zero integers, and set

$$c = \mathrm{diag}(1/c_3, c_3/c_2, c_2/c_1, c_1).$$

Following Stevens [Ste87, §2], we define

$$C(cw) := U(\mathbb{Q}_p)cwU(\mathbb{Q}_p) \cap G(\mathbb{Z}_p), \qquad X(cw) := U(\mathbb{Z}_p)\backslash C(cw)/\bar{U}_w(\mathbb{Z}_p).$$

By the Bruhat decomposition we have natural maps

$$u : X(cw) \to U(\mathbb{Z}_p)\backslash U(\mathbb{Q}_p), \qquad u' : X(cw) \to \bar{U}_w(\mathbb{Q}_p)/\bar{U}_w(\mathbb{Z}_p).$$

defined by the relation $x = u(x)cwu'(x)$ for $x \in X(cw)$. Let $\psi : U(\mathbb{A})/U(\mathbb{Q}) \mapsto \mathbb{C}^*$ be a character of $U(\mathbb{A})$ that is trivial on $U(\mathbb{Q})$. Every such character has the form $\psi = \psi_N$ for some $N = (n_1, n_2, n_3) \in \mathbb{Q}^3$,

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where ψ_N is given by

$$\psi_N \begin{pmatrix} 1 & x_1 & * & * \\ 0 & 1 & x_2 & * \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \xi(n_1x_1 + n_2x_2 + n_3x_3)$$

and $\xi : \mathbb{A} \to \mathbb{C}^*$ is the standard additive character. We can write $\psi = \prod_p \psi_p$, where ψ_p is a character of $U(\mathbb{Q}_p)$ that is trivial on $U(\mathbb{Z}_p)$. The *local Kloosterman sum* is defined by

$$Kl_p(\psi_p, \psi'_p; c, w) = \sum_{x \in X(cw)} \psi_p(u(x)) \cdot \psi'_p(u'(x)).$$

The *global Kloosterman sum* is defined by $Kl(\psi, \psi'; c, w) = \prod_p Kl_p(\psi_p, \psi'_p, c, w)$. Our main results for $Kl(\psi_M, \psi_N; c, w)$ are in Table B.1.

It was shown in Friedberg [Fri87, §1] that the Kloosterman sums are non-zero only if $w \in W$ is of the form $w = \begin{pmatrix} & & & I_{k_1} \\ & & I_{k_2} & \\ & \iddots & & \\ I_{k_r} & & & \end{pmatrix}$, where the I_k are $k \times k$ identity matrices and $k_1 + \cdots + k_r = n$ (may have some minus sign to make its determinant 1).

For the case $w = w_1$, we have $Kl(\psi_M, \psi_N; c, w_1) = \delta_{c_1,1}\delta_{c_2,1}\delta_{c_3,1}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}$, where $\delta_{m,n} = 1$ if $m = n$, and $\delta_{m,n} = 0$ otherwise.

For the case $w = w_2$ or w_3, Friedberg [Fri87] gave some very nice bounds for GL(n) Kloosterman sums attached to $w = \left(\begin{smallmatrix} & \pm 1 \\ I_{n-1} & \end{smallmatrix} \right)$. For $n = 3$, this is due to Larsen, see [BFG88, Appendix]. Then Friedberg [Fri87, §4] generalized it to all n. In some applications, we may need to give a bound with power saving in terms of all c_1, c_2, c_3. One can modify Friedberg's proof to give such a bound in the case $n = 4$. Note that the main situation is when $c_j = p^{ja}$, $a \geq 1$, $1 \leq j \leq 3$, in which case $(c_1c_2c_3)^{3/4}$ agrees with $c_j^{9/2j}$ in [Fri87]. In the proof, we need Deligne's deep theorems from algebraic geometry [Del77]. For the case $w = w_3$, one can use the involution operator $\iota : g \mapsto w_8{}^t g^{-1} w_8$ to get the result.

By [DR98, Theorem 0.3 (i)], we have the 'trivial' bound

$$Kl(\psi, \psi'; c, w) \leq \#X(cw) \leq c_1c_2c_3. \tag{B.1}$$

We use this for $w = w_j$, with $4 \leq j \leq 7$. In fact, this kind of bound holds for a general Kloosterman sum.

B.2. Stevens' approach

In this section, we follow Stevens' approach [Ste87] to bound the GL(4) long element Kloosterman sums. For $w \in W$, we define $w(j)$, $j \in \{1, 2, 3, 4\}$ by the formula

$$w \cdot e_j = \pm e_{w(j)},$$

where e_1, e_2, e_3, e_4 is the standard basis of column vectors. Let $\nu_1, \nu_2, \nu_3, \nu'_1, \nu'_2, \nu'_3 \in \mathbb{Z}_p$ and define the characters ψ, ψ' of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$ by

$$\psi \begin{pmatrix} 1 & u_1 & * & * \\ & 1 & u_2 & * \\ & & 1 & u_3 \\ & & & 1 \end{pmatrix} = \xi(\nu_1u_1 + \nu_2u_2 + \nu_3u_3), \quad \psi' \begin{pmatrix} 1 & u_1 & * & * \\ & 1 & u_2 & * \\ & & 1 & u_3 \\ & & & 1 \end{pmatrix} = \xi(\nu'_1u_1 + \nu'_2u_2 + \nu'_3u_3). \tag{B.2}$$

Fix

$$c = \mathrm{diag}(p^{-t}, p^{t-r}, p^{r-s}, p^s). \tag{B.3}$$

We will use the same notation as in Stevens [Ste87, §4]. And we need Definition 4.9 and Theorem 4.10 in [Ste87]. Note that n in [Ste87] will be our cw.

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Our main result in this appendix is the following theorem.

Theorem B.4. *Let $Kl_p(\psi,\psi';c,w_8)$ be the local Kloosterman sum attached to the long element w_8. Let ψ,ψ' be as in (B.2), $\ell = \max(r,s,t)$, $\varrho = \max(t,s)$, $\sigma = \min(t,s)$, and*

$$C_8 = 64(|\nu_1\nu_3'|_p^{-1}, p^{\ell})^{1/2}(|\nu_2\nu_2'|_p^{-1}, p^{\ell})^{1/2}(|\nu_3\nu_1'|_p^{-1}, p^{\ell})^{1/2}(\varrho+1)(r+1)^2(\sigma+1)^2.$$

Then

$$|Kl_p(\psi,\psi';c,w_8)| \le C_8 \min(p^{r+\sigma+\varrho/2}, p^{\varrho+2\sigma+r/2}). \tag{B.5}$$

In particular, we have $|Kl_p(\psi,\psi';c,w_8)| \le C_8 p^{9(t+r+s)/10}$.

Suppose we are given $\alpha,\beta,\gamma \in \mathbb{Z}_p^\times$, and nonnegative integers a,b,c,d,e,f [3] satisfying

$$a \le s, \quad d \le s, \quad e = s, \quad f \le r, \quad b+c \le \max(t,f); \tag{B.6}$$

$$\begin{cases} x = -\gamma p^{r-f} \in \mathbb{Z}_p, \\ y = p^r(\beta p^{-s} - \gamma p^{-a-f}) \in \mathbb{Z}_p, \\ z = p^t(\gamma p^{-f} - p^{-b-c}) \in \mathbb{Z}_p; \end{cases} \tag{B.7}$$

$$\begin{cases} \lambda = p^r(\beta p^{-b-s} - \alpha\gamma p^{-d-f}) \in \mathbb{Z}_p^\times, \\ \mu = p^t(\gamma p^{-a-f} + \alpha p^{-d-c} - p^{-a-b-c} - \beta p^{-s}) \in \mathbb{Z}_p^\times. \end{cases} \tag{B.8}$$

Hence, by $\lambda \in \mathbb{Z}_p^\times$, we have

$$b \le r, \quad a+f \le \max(r,s). \tag{B.9}$$

Then there is an element $x_{a,b,c}^{d,f,\alpha,\beta,\gamma} \in X(cw_8)$ for which

$$u'(x_{a,b,c}^{d,f,\alpha,\beta,\gamma}) = \begin{pmatrix} 1 & p^{-a} & \alpha p^{-d} & \beta p^{-s} \\ & 1 & p^{-b} & \gamma p^{-f} \\ & & 1 & p^{-c} \\ & & & 1 \end{pmatrix} \pmod{U(\mathbb{Z}_p)}. \tag{B.10}$$

Indeed, we have the matrix identity

$$\begin{pmatrix} \mu^{-1} & & & \\ z\lambda^{-1} & \mu\lambda^{-1} & & \\ x\beta^{-1} & y\beta^{-1} & \lambda\beta^{-1} & \\ p^s & p^{s-a} & \alpha p^{s-d} & \beta \end{pmatrix} = \begin{pmatrix} 1 & u_1 & U_4 & u_5 \\ & 1 & u_2 & u_6 \\ & & 1 & u_3 \\ & & & 1 \end{pmatrix} cw_8 \begin{pmatrix} 1 & p^{-a} & \alpha p^{-d} & \beta p^{-s} \\ & 1 & p^{-b} & \gamma p^{-f} \\ & & 1 & p^{-c} \\ & & & 1 \end{pmatrix}, \tag{B.11}$$

where

$$\begin{aligned} u_1 &= \mu^{-1}p^{r-t}(p^{-a-b} - \alpha p^{-d}), & U_4 &= -\mu^{-1}p^{s-r-a}, \\ u_2 &= \lambda^{-1}p^{t-r}(\alpha p^{s-c-d} - \beta), & u_5 &= \mu^{-1}p^{-s}, \\ u_3 &= -\beta^{-1}\gamma p^{r-s-f}, & u_6 &= \lambda^{-1}p^{t-s}(\gamma p^{-f} - p^{-b-c}). \end{aligned} \tag{B.12}$$

Write (B.11) as $g = ucw_8u'$. Then we have

$$g^{\iota} = w_8{}^t g^{-1} w_8 = \begin{pmatrix} \beta^{-1} & & & \\ -\alpha\lambda^{-1}p^{s-d} & \beta\lambda^{-1} & & \\ (\alpha p^{r-d} - p^{r-a-b})\mu^{-1} & -y\mu^{-1} & \lambda\mu^{-1} & \\ p^t & -p^{t-c} & -z & \mu \end{pmatrix},$$

[3] Note that here we use c in two meanings, one for a matrix, and another for a nonnegative integers. However, one can easily determine what it means in the context.

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and its Bruhat decomposition is

$$\begin{pmatrix} 1 & -u_1 & * & * \\ & 1 & -u_2 & * \\ & & 1 & -u_3 \\ & & & 1 \end{pmatrix} \begin{pmatrix} p^{-s} & & & \\ & p^{s-r} & & \\ & & p^{r-t} & \\ & & & p^t \end{pmatrix} w_8 \begin{pmatrix} 1 & -p^{-c} & * & * \\ & 1 & -p^{-b} & * \\ & & 1 & -p^{-a} \\ & & & 1 \end{pmatrix}.$$

Since $g \in X(cw_8)$, we have $g^\iota \in X((cw_8)^\iota) \subseteq G(\mathbb{Z}_p)$, hence

$$c \le t, \quad a + b \le \max(r, d), \quad \alpha p^{r-d} - p^{r-a-b} \in \mathbb{Z}_p. \tag{B.13}$$

Let ψ, ψ' be characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. For a, b, c, and $d, f, \alpha, \beta, \gamma$ satisfying (B.6)–(B.9) and (B.13), let $X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw_8) = T(\mathbb{Z}_p) * x_{a,b,c}^{d,f,\alpha,\beta,\gamma}$ be the orbit through $x_{a,b,c}^{d,f,\alpha,\beta,\gamma}$, and let

$$S_{a,b,c}^{d,f,\alpha,\beta,\gamma}(\psi, \psi'; c, w_8) = \sum_{x \in X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw_8)} \psi(u(x))\psi'(u'(x))$$

be the Kloosterman sum restricted to this orbit. For a, b, c satisfying (B.6), (B.9) and (B.13), let $X_{a,b,c}(cw_8) = \bigcup_{d,f,\alpha,\beta,\gamma} X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw_8)$, where d, f run over nonnegative integers, and α, β, γ run over the elements of $\mathbb{Z}_p^\times$ satisfying (B.6)–(B.8). Let

$$S_{a,b,c}(\psi, \psi'; c, w_8) = \sum_{x \in X_{a,b,c}(cw_8)} \psi(u(x))\psi'(u'(x)).$$

Lemma B.14. *We have* $X(cw_8) = \coprod_{a,b,c} X_{a,b,c}(cw_8)$*, where* $a, b, c \ge 0$ *run over integers satisfying* (B.6)*,* (B.9)*, and* (B.13)*.*

Proof. See Stevens [Ste87, Lemmas 5.2 and 5.7]. □

Lemma B.15. *Let* $\ell = \max(s, r, t)$*, and* $a \le s,\ b \le r,\ c \le t$ *be nonnegative integers. Then*

$$\begin{aligned} &|S_{a,b,c}(\psi, \psi'; c, w_8)| \\ &\quad \le 64(|\nu_1\nu_3'|_p^{-1}, p^\ell)^{1/2}(|\nu_2\nu_2'|_p^{-1}, p^\ell)^{1/2}(|\nu_3\nu_1'|_p^{-1}, p^\ell)^{1/2} p^{-\frac{a+b+c}{2}} \#(X_{a,b,c}(cw_8)). \end{aligned}$$

Proof. The involution ι sends $X_{a,b,c}(cw_8)$ to $X_{c,b,a}((cw_8)^\iota)$. Composing ψ and ψ' with ι has the effect of replacing (ν_1, ν_2, ν_3) by $(-\nu_3, -\nu_2, -\nu_1)$ and (ν_1', ν_2', ν_3') by $(-\nu_3', -\nu_2', -\nu_1')$. Applying ι to cw_8 reverses the roles of t and s. Thus we may assume $t \ge s$ without loss of generality.

Let $\ell = \max(r, t)$. The conditions (B.6)–(B.9) and (B.13) imply that the matrix entries of $u(x)$ and $u'(x)$ lie in $p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p$ for every $x \in X(cw_8)$. Indeed, by Lemma B.14, it is enough to verify this for $x = x_{a,b,c}^{d,f,\alpha,\beta,\gamma}$. Note that $\mu = p^{-s}p^t(\alpha p^{s-d-c} - \beta) + p^{-a}p^t(\gamma p^{-f} - p^{-b-c}) = p^{-s}\lambda u_2 p^r + p^{-a}z \in \mathbb{Z}_p^\times$. We have $u_2 \in p^{-r}\mathbb{Z}_p$. The claim is now easily verified.

Now let $\mathcal{S}$ be a finite subset of $\mathbb{Z}_{\ge 0}^2 \times (\mathbb{Z}_p^\times)^3$ such that $X_{a,b,c}(cw_8)$ is the disjoint union of the $X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw_8)$ with $(d, f, \alpha, \beta, \gamma) \in \mathcal{S}$. Then as in [Ste87, Th. 4.10] we have

$$S_{a,b,c}(\psi, \psi'; c, w_8) = p^{-3\ell}(1 - p^{-1})^{-3} \sum_{(d,f,\alpha,\beta,\gamma) \in \mathcal{S}} \#\left(X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw)\right) S_{w_8}\left(\theta_{a,b,c}^{d,f,\alpha,\beta,\gamma}; \ell\right), \tag{B.16}$$

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where S_{w_8} is defined in [Ste87, Def. 4.9], and $\theta_{a,b,c}^{d,f,\alpha,\beta,\gamma} : A_{w_8}(\ell) \to \mathbb{C}^\times$ is the character given by

$$\begin{aligned}\theta_{a,b,c}^{d,f,\alpha,\beta,\gamma}(\underline{\lambda}\times\underline{\lambda}') &= e\left(\nu_1u_1\lambda_1+\nu_2u_2\lambda_2+\nu_3u_3\lambda_3+\nu_1'p^{-a}\lambda_1'+\nu_2'p^{-b}\lambda_2'+\nu_3'p^{-c}\lambda_3'\right)\\ &= e\left(\frac{(-\nu_1\mu^{-1}p^{\ell+r-t}(p^{-a-b}-\alpha p^{-d}))\lambda_1+(\nu_2\lambda^{-1}p^{\ell+t-r}(\alpha p^{s-c-d}-\beta))\lambda_2}{p^\ell}\right.\\ &\qquad\left.+\frac{(\nu_3\beta^{-1}\gamma p^{\ell+r-s-f})\lambda_3+\nu_1'p^{\ell-a}\lambda_1'+\nu_2'p^{\ell-b}\lambda_2'+\nu_3'p^{\ell-c}\lambda_3'}{p^\ell}\right).\end{aligned}$$

By Example 4.12 in Stevens [Ste87], we have

$$\begin{aligned}S_{w_8}(\theta_{a,b,c}^{d,f,\alpha,\beta,\gamma};\ell) &= S_2(\nu_1\mu^{-1}p^{\ell+r-t}(p^{-a-b}-\alpha p^{-d}),\nu_3'p^{\ell-c};p^\ell)\\ &\quad\cdot S_2(\nu_2\lambda^{-1}p^{\ell+t-r}(\alpha p^{s-c-d}-\beta),\nu_2'p^{\ell-b};p^\ell)\\ &\quad\cdot S_2(-\nu_3\beta^{-1}\gamma p^{\ell+r-s-f},\nu_1'p^{\ell-a};p^\ell),\end{aligned} \tag{B.17}$$

where S_2 is the classical GL(2)-Kloosterman sum. By Weil [Wei48], we have the inequality

$$|S_2(m,n;p^\ell)| \le 2(\gcd(|m|_p^{-1},|n|_p^{-1},p^\ell))^{1/2}p^{\ell/2}, \tag{B.18}$$

for $m,n\in\mathbb{Z}_p$. In order to apply this bound, we first note

$$\begin{aligned}\gcd(|\nu_3p^{\ell+r-s-f}|_p^{-1},|\nu_1'p^{\ell-a}|_p^{-1},p^\ell) &\le \gcd(|\nu_3\nu_1'|_p^{-1},p^\ell)p^{\ell-a},\\ \gcd(|\nu_2p^{\ell+t-r}(\alpha p^{s-c-d}-\beta)|_p^{-1},|\nu_2'p^{\ell-b}|_p^{-1},p^\ell) &\le \gcd(|\nu_2\nu_2'|_p^{-1},p^\ell)p^{\ell-b},\\ \gcd(|\nu_1p^{\ell+r-t}(p^{-a-b}-\alpha p^{-d})|_p^{-1},|\nu_3'p^{\ell-c}|_p^{-1},p^\ell) &\le \gcd(|\nu_1\nu_3'|_p^{-1},p^\ell)p^{\ell-c}.\end{aligned}$$

Hence we have

$$|S_{w_8}(\theta_{a,b,c}^{d,f,\alpha,\beta,\gamma};\ell)| \le 8(|\nu_1\nu_3'|_p^{-1},p^\ell)^{1/2}(|\nu_2\nu_2'|_p^{-1},p^\ell)^{1/2}(|\nu_3\nu_1'|_p^{-1},p^\ell)^{1/2}p^{3\ell-\frac{a+b+c}{2}}.$$

This inequality, together with (B.16), gives

$$\begin{aligned}|S_{a,b,c}(\psi,\psi';c,w_8)| &\le 8(|\nu_1\nu_3'|_p^{-1},p^\ell)^{1/2}(|\nu_2\nu_2'|_p^{-1},p^\ell)^{1/2}(|\nu_3\nu_1'|_p^{-1},p^\ell)^{1/2}\\ &\quad\cdot(1-p^{-1})^{-3}p^{-\frac{a+b+c}{2}}\sum_{(d,f,\alpha,\beta,\gamma)\in\mathcal{S}}\#\left(X_{a,b,c}^{d,f,\alpha,\beta,\gamma}(cw_8)\right).\end{aligned} \tag{B.19}$$

The sum appearing on the right-hand side is equal to $\#(X_{a,b,c}(cw_8))$. Since $p\ge 2$, we have $(1-p^{-1})^{-3}\le 8$, by (B.19), we prove the lemma. □

Proof of Theorem B.4. By the involution ι, we can assume $t\ge s$ without loss of generality. Let

$$C = 64(|\nu_1\nu_3'|_p^{-1},p^\ell)^{1/2}(|\nu_2\nu_2'|_p^{-1},p^\ell)^{1/2}(|\nu_3\nu_1'|_p^{-1},p^\ell)^{1/2}(r+1)(s+1).$$

At first, we deal with the case $t\ge r$.

- If $a+b+c\le t$ and $d+f\le r$, then $\#(d,f)\le(s+1)(r+1)$, $\#(\alpha,\gamma,\beta)\le p^{d+s+f}$, so

  $$\#(X_{a,b,c}(cw_8)) \le (r+1)(s+1)p^{a+b+c+d+f+s} \le (r+1)(s+1)p^{r+s+a+b+c}.$$

  Hence by Lemma B.15, we have $|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{r+s+t/2}$.

- If $a+b+c \le t$ and $d+f > r$, then we assume that $d+f = r+k$, $k \ge 1$. Note that $d \le s$, $f \le r$, we have $k \le s$. By (B.8), we have $b+s = d+f = r+k$. Since $\lambda \in \mathbb{Z}_p^\times$, we have $\#\{(\alpha,\gamma,\beta)\} \le p^{d+f+(s-k)} = p^{r+s}$. Hence

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{r+s+\frac{a+b+c}{2}} \le Cp^{r+s+t/2}.$$

- If $a+b+c > t$ and $d+f \le r$, then by (B.8) and a similar argument as above, we have $\#\{(\alpha,\gamma,\beta)\} \le p^{(d-m)+f+s}$. Hence

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{d-m+f+s+\frac{a+b+c}{2}} \le Cp^{r+s+t/2}.$$

- If $a+b+c > t$ and $d+f > r$, then we have $\#\{(\alpha,\gamma,\beta)\} \le p^{(d-m)+f+(s-k)}$. Hence

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{d-m+f+s-k+\frac{a+b+c}{2}} \le Cp^{r+s+t/2}.$$

Note that in this case, we always have $r+s+t/2 \le t+2s+r/2$. Theorem B.4 now follows from the equality $Kl_p(\psi,\psi';c,w_8) = \sum_{a\le s, b\le r, c\le t} S_{a,b,c}(\psi,\psi';c,w_8)$.

Now we handle the case $r > t$. By a similar argument as above, we obtain

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{r+s+t/2}.$$

If t is small, this bound is not good enough. So we need to bound this in other way.

- If $f > t$, then by (B.7), we have $b+c = f$, and $a+f \le r$. By (B.8), we have $\#(\alpha,\gamma) \le p^{d+f-(a+f-t)}$. If $d+f \le r$, then we have

$$\begin{aligned}|S_{a,b,c}(\psi,\psi';c,w_8)| &\le Cp^{d+f-(a+f-t)+s+\frac{a+b+c}{2}} \le Cp^{t+s+d+\frac{b+c}{2}}\\ &\le Cp^{t+s+\frac{d}{2}+\frac{d+f}{2}} \le Cp^{t+3s/2+r/2}.\end{aligned}$$

- If $f > t$ and $d+f > r$, then by writing $d+f = r+k$, $1 \le k \le s$, we have

$$\begin{aligned}|S_{a,b,c}(\psi,\psi';c,w_8)| &\le Cp^{d+f-(a+f-t)+s-k+\frac{a+b+c}{2}}\\ &\le Cp^{t+s+\frac{d}{2}+\frac{d+f}{2}-k} \le Cp^{t+3s/2+r/2}.\end{aligned}$$

- If $f \le t$ and $a+b+c > r$. Since $\mu \in \mathbb{Z}_p^\times$, we have $\#(\alpha,\gamma) \le p^{d+f-(a+b+c-t)}$. Then by the same argument on the size of $d+f$, we have

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{r+t+s-\frac{a+b+c}{2}} \le Cp^{t+s+r/2}.$$

- If $f \le t$, and $a+b+c \le r$, then we have

$$|S_{a,b,c}(\psi,\psi';c,w_8)| \le Cp^{d+f+s+\frac{a+b+c}{2}} \le Cp^{t+2s+r/2}.$$

This proves (B.5).

We now give a proof of the second claim. If $r+\sigma+\varrho/2 \le \varrho+2\sigma+r/2$ – that is, $r \le \varrho+2\sigma$ – then $\sigma+r \le 4\varrho$, so $r+\sigma+\varrho/2 \le 9(\varrho+r+\sigma)/10$. If $r+\sigma+\varrho/2 > \varrho+2\sigma+r/2$ – that is, $r > \varrho+2\sigma$ – then $9\sigma < 3r$ and $\varrho+11\sigma < 4r$, so $\varrho+2\sigma+r/2 < 9(\varrho+r+\sigma)/10$. This proves $\min(p^{r+\sigma+\varrho/2}, p^{\varrho+2\sigma+r/2}) \le p^{9(t+r+s)/10}$, as claimed, and hence Theorem B.4. □

Remark B.20. The result is not optimal. To improve the bound in some cases, one may use the stationary phase formulas as Dabrowski and Fisher did for GL(3), see [DF97].

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Remark B.21. Stevens' method can be used to bound other Kloosterman sums as well. It's not too hard to prove bounds similar to (B.1). But to improve these 'trivial' bounds, one may need new ideas.

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