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NON-NEGATIVE DEFORMATIONS OF WEIGHTED HOMOGENEOUS SINGULARITIES

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Abstract. We consider a weighted homogeneous germ of complex analytic variety $(X, 0) \subset (\mathbb{C}^n, 0)$ and a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. We derive necessary and sufficient conditions for some deformations to have non-negative degree (i.e., for any additional term in the deformation, the weighted degree is not smaller) in terms of an adapted version of the relative Milnor number. We study the cases where $(X, 0)$ is an isolated hypersurface singularity and the invariant is the Bruce-Roberts number of f with respect to $(X, 0)$, and where $(X, 0)$ is an isolated complete intersection or a curve singularity and the invariant is the Milnor number of the germ $f : (X, 0) \to \mathbb{C}$. In the last part, we give some formulas for the invariants in terms of the weights and the degrees of the polynomials.

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1. Introduction. The Milnor number is an important invariant in Singularity Theory. When it is defined, it reflects interesting geometric properties of the germ. For instance, if $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is an isolated singularity function germ, then its generic fibre has the homotopy type of a bouquet of spheres of real dimension $n - 1$. The number of such spheres, denoted by $\mu(f)$, is the Milnor number of f (see [**15**]).

Lê-Ramanujam [**12**] and Timourian [**23**] show that a family of hypersurfaces in $(\mathbb{C}^n, 0)$ (with $n \neq 3$) has constant Milnor number if and only if all the hypersurfaces in the family have the same topological type. In other words, the Milnor number is a topological invariant whose constancy in a family controls the topological triviality. Because of these facts it is important to know whether a family has constant Milnor number. In [**24**], Varchenko shows the following result.

THEOREM 1.1. *If $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is a weighted homogeneous polynomial with an isolated singularity and if $f_t(x) = f(x) + \sum_{i=1}^{k} \sigma_i(t)\alpha_i(x)$ is a deformation of f with $\sigma_i(0) = 0$, then the Milnor number of f_t, $\mu(f_t)$, is constant if and only if the weighted degree of α_i is not smaller than the weighted degree of f, for all $i = 1, \ldots, k$.*

A deformation like the one in Theorem 1.1 (the weighted degree of α_i is not smaller than the weighted degree of f, for all i such that σ_i is not zero) is called a *non-negative deformation*.

In this paper, we want to find necessary and sufficient conditions for a deformation to be non-negative in other cases where the Milnor number (or another similar invariant) is well defined. In the case of varieties with isolated singularity, the Milnor number is known to be well defined for a space curve singularity [**5**], an isolated complete intersection singularity – ICIS [**11**], or an isolated determinantal singularity – IDS [**19**]. In the case of functions, on one hand, we have the Bruce-Roberts number of a function $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ with respect to a variety $(X, 0) \subset (\mathbb{C}^n, 0)$ [**3**]. On the other hand, we can also consider the Milnor number of a function $f : (X, 0) \to \mathbb{C}$, where $(X, 0)$ is either a curve [**9**, **17**] or an IDS [**19**]. Moreover, we also provide some formulas for these invariants in terms of the weights and the degrees of the polynomials.

2. Weighted homogeneous functions and varieties.

We use this section to present the basic definitions we will use in the work. Let us denote by $\mathcal{O}_n$ the local ring of germs of complex analytic functions $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. We fix positive integer numbers $w_1, \dots, w_n \in \mathbb{N}^*$ such that $\gcd(w_1, \dots, w_n) = 1$. Assume that $f \in \mathcal{O}_n$ is expressed as a convergent power series:

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

We say that f is weighted homogeneous if there exists d such that $\alpha_1 w_1 + \dots + \alpha_n w_n = d$ for all α such that $a_\alpha \neq 0$. In such a case, we say that d is the weighted degree of f and we write, $wt(f) = d$. Equivalently, f is a polynomial such that

$$f(t^{w_1} x_1, \dots, t^{w_n} x_n) = t^d f(x_1, \dots, x_n), \quad \forall x \in \mathbb{C}^n, \ \forall t \in \mathbb{C}.$$

We say that a variety germ $(X, 0) \subset (\mathbb{C}^n, 0)$ is weighted homogeneous if it is the zero set of an ideal $I \subset \mathcal{O}_n$ which can be generated by weighted homogeneous germs $\phi_1, \dots, \phi_p$ (with respect to the same weights, but with possibly different degrees). Now, this is equivalent to say that

$$(t^{w_1} x_1, \dots, t^{w_n} x_n) \in X, \quad \forall x \in X, \ \forall t \in \mathbb{C}.$$

In particular, we take the convention that $(\mathbb{C}^n, 0)$ is itself weighted homogeneous. Associated with $(X, 0)$, we can consider in a natural way the ring

$$\mathbb{C}[x_1, \dots, x_n] / \langle \phi_1, \dots, \phi_p \rangle$$

as a graded ring with respect to the weights $(w_1, \dots, w_n)$ (see, for instance, [**14**, p. 93]).

Definition 2.1. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be weighted homogeneous as above. We say that a polynomial germ $f : (X, 0) \to \mathbb{C}$ is weighted homogeneous if it is homogeneous in the graded ring $\mathbb{C}[x_1, \dots, x_n] / \langle \phi_1, \dots, \phi_p \rangle$. In particular, there exists $d \in \mathbb{N}$ such that

$$f(t^{w_1} x_1, \dots, t^{w_n} x_n) = t^d f(x_1, \dots, x_n), \quad \forall x \in X, \ \forall t \in \mathbb{C}.$$

In this case, we call d the weighted degree of f and write $d = \mathrm{wt}(f)$.

Any non-zero polynomial germ $f : (X, 0) \to \mathbb{C}$ can be written in a unique way as

$$f = f_d + f_{d+1} + \cdots + f_l,$$

with $f_d \neq 0$, where each f_i is weighted homogeneous of degree i. We call f_d the initial part of f and denote it by $\mathrm{in}(f)$.

We say that f is semi-weighted homogeneous if $\mathrm{in}(f)$ has isolated singularity (that is, there exist a representative X of $(X, 0)$ such that $X \setminus \{0\}$ is non-singular and $\mathrm{in}(f)$ is regular on $X \setminus \{0\}$).

In this paper, by a deformation of a map germ $f : (X, 0) \to (Y, 0)$, we mean another map germ $F : (\mathbb{C} \times X) \to (Y, 0)$ such that $F(0, x) = f(x)$, for all $x \in X$. We also assume that F is origin preserving, that is, $F(t, 0) = 0$ for all t, so we have a one-parameter family of map germs $f_t : (X, 0) \to (Y, 0)$ given by $f_t(x) = F(t, x)$. By abuse of notation, sometimes it is usual to denote the deformation F just by f_t. On the other hand, we can also consider the associated unfolding $\tilde{F} : (\mathbb{C} \times X) \to (\mathbb{C} \times Y, 0)$ given by $\tilde{F}(t, x) = (t, f_t(x))$.

In the particular case of a polynomial function $f : (X, 0) \to (\mathbb{C}, 0)$, any polynomial deformation f_t can be written as

$$f_t(x) = f(x) + \sum_{i=1}^{k} \sigma_i(t)\alpha_i(x), \tag{1}$$

for some $\alpha_i : (X, 0) \to (\mathbb{C}, 0)$ and $\sigma_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ $i = 1, \ldots, k$.

DEFINITION 2.2. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous variety germ. We say that a polynomial deformation f_t of $f : (X, 0) \to (\mathbb{C}, 0)$ as in (1) is non-negative if $\mathrm{wt}(\mathrm{in}(\alpha_i)) \geq \mathrm{wt}(\mathrm{in}(f))$, for $i = 1, \ldots, k$ such that $\alpha_i \neq 0$.

3. The Bruce-Roberts number. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an analytic variety germ. Bruce and Roberts introduce in **[3]** a generalization of $\mu(f)$ which we call *Bruce-Roberts number* of f with respect to $(X, 0)$. It is defined by

$$\mu_{\mathrm{BR}}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\theta_X)},$$

where θ_X is the $\mathcal{O}_n$-module of vector fields in $(\mathbb{C}^n, 0)$ which are tangent to $(X, 0)$ and df denotes the differential map of f. We remark that if $X = \emptyset$, then $\mu_{\mathrm{BR}}(f, X) = \mu(f)$.

Like the Milnor number of f, this number shows some geometric properties of f and X. For instance, if we consider the group $\mathcal{R}_X$ of biholomorphisms of $(\mathbb{C}^n, 0)$ which preserve X, then f is finitely determined with respect to the action of $\mathcal{R}_X$ on $\mathcal{O}_n$ if and only if $\mu_{\mathrm{BR}}(f, X)$ is finite and, in this case, the co-dimension of the orbit of f under this action is equal to $\mu_{\mathrm{BR}}(f, X)$ (see **[3]**).

In **[18]**, we show the following result, relating the Milnor and Bruce-Roberts numbers.

THEOREM 3.1. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous hypersurface with an isolated singularity and let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be an $\mathcal{R}_X$-finitely determined function germ. Then,*

$$\mu_{\mathrm{BR}}(f, X) = \mu(f) + \mu(X \cap f^{-1}(0), 0).$$

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The goal of this section is to provide a result like Theorem 1.1 for the Bruce-Roberts number. In order to do this, we will need the following related property. We recall that a deformation f_t of a function germ $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is called C^0-$\mathcal{R}_X$-trivial if there is a homeomorphism $\Psi:(\mathbb{C}\times\mathbb{C}^n)\to(\mathbb{C}\times\mathbb{C}^n)$ which is an unfolding of the identity map in $\mathbb{C}^n$ such that $f_t\circ\psi_t=f$ and ψ_t preserves $(X,0)$ (i.e., $\psi_t(X)=X$ for all $t\in\mathbb{C}$).

THEOREM 3.2 [**6**, **21**]. *Let $(X,0)$ be a weighted homogeneous variety with an isolated singularity and let f be a weighted homogeneous $\mathcal{R}_X$-finitely determined function. If f_t is a non-negative deformation of f, then f_t is C^0-$\mathcal{R}_X$-trivial.*

REMARK 3.3. In [**6**, **21**], it is used the term "consistent with $(X,0)$" to refer to the fact that f is weighted homogenous with the same weights of $(X,0)$, but here this is not necessary since we fix the weights from the beginning.

An interesting open problem is to know whether the Bruce-Roberts number is or is not a topological invariant. A partial answer is given in [**8**]: If $(X,0)$ is a hypersurface whose logarithmic characteristic variety, LC(X), is Cohen-Macaulay and if f_t is a C^0-$\mathcal{R}_X$-trivial deformation of f, then $\mu_{\mathrm{BR}}(f_t,X)$ is constant (see [**22**] for the definition of LC(X)). In [**18**], we see that if $(X,0)$ is a weighted homogeneous hypersurface with an isolated singularity, then LC(X) is Cohen-Macaulay. Therefore, we have:

COROLLARY 3.4. *Let $(X,0)$ be a weighted homogeneous hypersurface with an isolated singularity and let $f:(\mathbb{C}^n,0)\to\mathbb{C}$ be an $\mathcal{R}_X$-finitely determined function germ. If f_t is a C^0-$\mathcal{R}_X$-trivial deformation of f, then $\mu_{\mathrm{BR}}(f_t,X)$ is constant.*

In [**1**], it is shown that if $(X,0)$ is a weighted homogeneous hypersurface with an isolated singularity and if f and g are C^0-$\mathcal{R}_X$-equivalent function germs then $\mu_{\mathrm{BR}}(f,X)=\mu_{\mathrm{BR}}(g,X)$. The following example shows that the converse of this result is not true.

EXAMPLE 3.5. Let $(X,0)=(\phi^{-1}(0),0)\subset(\mathbb{C}^2,0)$, where $\phi(x,y)=x^2+y^3$. Let $f,g:(\mathbb{C}^2,0)\to\mathbb{C}$ defined by $f(x,y)=x^3+y^3$ and $g(x,y)=x^2+y^5$.

We have

$$\mu_{\mathrm{BR}}(f,X)=\mu_{\mathrm{BR}}(g,X)=9,$$

but f and g are not C^0-$\mathcal{R}_X$-equivalent (in fact, they are not C^0-$\mathcal{R}$-equivalent, since the plane curve $f^{-1}(0)$ has three branches while $g^{-1}(0)$ has only one branch).

However, inspired by Theorem 1.1, it is natural to ask if the Bruce-Roberts number is invariant under non-negative deformations. We present a partial answer in the following result.

THEOREM 3.6. *Let $(X,0)\subset(\mathbb{C}^n,0)$ be a weighted homogeneous hypersurface with isolated singularity, let $f:(\mathbb{C}^n,0)\to\mathbb{C}$ be a weighted homogeneous $\mathcal{R}_X$-finitely determined function and let f_t be a deformation of f as in (1). The following statements are equivalent:*

(1) *the family f_t is C^0-$\mathcal{R}_X$-trivial;*
(2) *f_t is a non-negative deformation;*
(3) *$\mu_{\mathrm{BR}}(f_t,X)$ is constant.*

Proof. The implication (2) ⇒ (1) is given in Theorem 3.2, so we see (1) ⇒ (2). If f_t is C^0-$\mathcal{R}_X$-trivial, then it is C^0-$\mathcal{R}$-trivial and thus, $\mu(f_t)$ is constant (since f_t has an isolated singularity). By Theorem 1.1, f_t is a non-negative deformation.

The implication (1) ⇒ (3) is given in Corollary 3.4, so it is enough to show (3) ⇒ (2). If $\mu_{\mathrm{BR}}(f_t, X)$ is constant for t small enough, then, from Theorem 3.1 and from the fact that $\mu(f_t)$ and $\mu(X \cap f_t^{-1}(0), 0)$ are upper semi-continuous, these numbers are constant. Therefore, the result follows from Theorem 1.1. □

We see that the implication (3) ⇒ (2) is also true even if f is weighted homogeneous with different weights of $(X, 0)$ (that is, it is not "consistent with $(X, 0)$" in the terminology of [**6**, **21**]). However, the converse is not true in general in that case, as the following example shows.

EXAMPLE 3.7. Let $(X, 0)$ be the homogeneous surface germ defined by the zeros of $\phi = x^2 + y^2 + z^2$. Let $f : (\mathbb{C}^3, 0) \to \mathbb{C}$ be the function germ defined by

$$f(x, y, z) = z^5 + y^7 x + x^{15},$$

then f is weighted homogeneous. Finally, we consider the non-negative deformation f_t of f given by

$$f_t(x, y, z) = f(x, y, z) + ty^6 z.$$

We have $\mu_{\mathrm{BR}}(f, X) = 429$, but $\mu_{\mathrm{BR}}(f_t, X) = 425$ for $t \neq 0$ small enough.

4. The Milnor number of a function on a curve and on an ICIS. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a reduced curve germ. We denote its local ring by $\mathcal{O}_{X,0}$. We also denote by $\Omega_{X,0}$ the module of holomorphic one-forms on $(X, 0)$ and by $\omega_{X,0}$ the dualizing module of Grothendieck (see [**4**, Chapter 3]); that is,

$$\omega_{X,0} := \mathrm{Ext}^{n-1}_{\mathcal{O}_n}(\mathcal{O}_{X,0}, \Omega^n_{\mathbb{C}^n,0}),$$

where $\Omega^n_{\mathbb{C}^n,0}$ is the module of holomorphic n-forms on $(\mathbb{C}^n, 0)$. We consider a normalization of the curve $n : (\overline{X}, \overline{0}) \to (X, 0)$, which induces the so-called class map $c_{X,0} : \Omega_{X,0} \to \omega_{X,0}$ as the composition

$$\Omega_{X,0} \to \Omega_{\overline{X},\overline{0}} \cong \omega_{\overline{X},\overline{0}} \to \omega_{X,0}.$$

The composition of the differential operator $\mathrm{d} : \mathcal{O}_{X,0} \to \Omega_{X,0}$ with the class map gives a map which we also denote by $\mathrm{d} : \mathcal{O}_{X,0} \to \omega_{X,0}$. The Milnor number of the reduced curve $(X, 0)$ is defined in [**5**] as

$$\mu(X, 0) = \dim_{\mathbb{C}} \frac{\omega_{X,0}}{\mathrm{d}\mathcal{O}_{X,0}}.$$

Let $f : (X, 0) \to \mathbb{C}$ be a finite function germ on the curve X. The Milnor number $\mu(f)$ has been introduced in [**9**] for curves in $\mathbb{C}^3$ and later in [**17**] for the general case as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\omega_{X,0}}{\mathrm{d}f \wedge \mathcal{O}_{X,0}}, \tag{2}$$

where $\mathrm{d}f\wedge : \mathcal{O}_{X,0} \to \omega_{X,0}$ denotes the composition of $\mathrm{d}f\wedge : \mathcal{O}_{X,0} \to \Omega_{X,0}$ with the class map $c_{X,0} : \Omega_{X,0} \to \omega_{X,0}$. In [**20**], it is shown that

$$\mu(f) = \mu(X, 0) + \deg(f) - 1. \tag{3}$$

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The following result presents a theorem like Theorem 1.1 for the Milnor number of $f : (X, 0) \to \mathbb{C}$.

THEOREM 4.1. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous curve and let $f : (X, 0) \to \mathbb{C}$ be a finite map germ. Let f_t be a deformation of f as in (1). Then, $\mu(f_t)$ is constant if and only if $f_t|_{X_j}$ is a non-negative deformation of $f|_{X_j}$, for each branch X_j of X.*

Proof. By (3), we have

$$\mu(f_t) = \mu(X, 0) + \deg(f_t) - 1.$$

Hence, $\mu(f_t)$ is constant if and only if $\deg(f_t)$ is constant. Assume that $(X, 0)$ has r branches $X_1, \dots, X_r$. For each $j = 1, \dots, r$ we choose a point $(a_{1j}, \dots, a_{nj}) \in X_j$ not zero (in such a way that $f(a_{1j}, \dots, a_{nj})$ is also different from zero by the finiteness of f). We define $\gamma_j : \mathbb{C} \to \mathbb{C}^n$, $\gamma_j(s) = (a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n})$, where $(w_1, \dots, w_n)$ are the weights of $(X, 0)$. Since $\gamma_j(\mathbb{C}) \subset X_j$ and γ_j is injective ($\gcd(w_1, \dots, w_n) = 1$), we deduce that γ_j in fact parametrizes X_j.

We see that each branch X_j is also weighted homogeneous and denote $d_j = \mathrm{wt}(\mathrm{in}(f|_{X_j}))$, the degree of the initial part of each $f|_{X_j}$. Then, we have

$$\begin{aligned}
\deg(f_t|_{X_j}) &= m_0(f_t(\gamma_j(s))) \\
&= m_0(f(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n}) + \sum_{i=1}^{k} \sigma_i(t)\alpha_i(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n})) \\
&= m_0(f_{d_j}(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n}) + f_{d_j+1}(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n}) + \dots + \\
&\quad f_{l_j}(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n}) + \sum_{i=1}^{k} \alpha_i(t)\alpha_i(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n})) \\
&= m_0(s^{d_j} f_{d_j}(a_{1j}, \dots, a_{nj}) + s^{d_j+1} f_{d_j+1}(a_{1j}, \dots, a_{nj}) + \dots + \\
&\quad s^{l_j} f_{l_j}(a_{1j}, \dots, a_{nj}) + \sum_{i=1}^{k} \sigma_i(t)\alpha_i(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n})),
\end{aligned}$$

where $m_0(p)$ denotes the multiplicity (i.e., the order) of a function germ p.

We have $\deg(f|_{X_j}) = d_j$ and moreover, $\deg(f|_{X_j}) = \deg(f_t|_{X_j})$ if and only if

$$m_0(\alpha_i(a_{1j}s^{w_1}, \dots, a_{nj}s^{w_n})) \geq d_j,$$

for all i such that $\alpha_i \neq 0$, that is, if and only if $\mathrm{wt}(\mathrm{in}(\alpha_i)) \geq d_j$, for all i such that $\alpha_i \neq 0$.

Furthermore,

$$\deg(f_t) = \sum_{j=1}^{r} \deg(f_t|_{X_j})$$

and $\deg(f_t|_{X_j})$ is upper semi-continuous. Therefore, $\deg(f_t)$ is constant if and only if $\deg(f_t|_{X_j})$ is constant for every j. That is, $\mu(f_t)$ is constant if and only if $f_t|_{X_j}$ is a non-negative deformation of $f|_{X_j}$, for each $j = 1, \dots, r$. □

It follows from the results of **[20]** that $\mu(f_t)$ is constant if and only if the family f_t is good and topologically trivial (see **[20]** for the precise definition of both concepts).

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We remark that if a deformation of a map germ $f : (X, 0) \to \mathbb{C}$ is non-negative it is not necessarily non-negative in each branch, as we can see in the following example.

EXAMPLE 4.2. Let $(X, 0)$ be the curve in $\mathbb{C}^2$ defined as the zeros of the map $\phi(x, y) = xy$ and let $f : (X, 0) \to \mathbb{C}$ be the map germ $f(x, y) = x^3 + y^5$. We define the deformation $f_t(x) = x^3 + y^5 + ty^4$ and we note that, considering X weighted homogeneous with weights $(1, 1)$, f_t is a non-negative deformation of f, but $f_t|_{x=0}$ is not. In this case, we have, $\mu(f) = 8$ but $\mu(f_t) = 7$ for $t \neq 0$.

If, in the previous theorem, $f : (X, 0) \to \mathbb{C}$ is a semi-weighted homogeneous map germ, and $f_t(x) = f_d(x) + t[f_{d+1}(x) + \cdots + f_l(x)]$, we have that f_t is a non-negative deformation of f_d. Hence, we conclude the following result.

COROLLARY 4.3. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous curve and let $f : (X, 0) \to \mathbb{C}$ be a semi-weighted homogeneous germ. Then, $\mu(f) = \mu(in(f))$.*

Another case for which the Milnor number of a germ $f : (X, 0) \to \mathbb{C}$ with an isolated singularity is defined is when $(X, 0)$ is an ICIS. Assume $X = \phi^{-1}(0)$, where $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a holomorphic map germ. Then, we call Milnor number of f to the number:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi, J(\phi, \bar{f}) \rangle},$$

where $\bar{f} : (\mathbb{C}^n, 0) \to \mathbb{C}$ is any holomorphic extension of f and $J(\phi, \bar{f})$ is the ideal generated by the maximal minors of the Jacobian matrix of $(\phi, \bar{f})$. The Lê-Greuel Formula [**13**] can be expressed as

$$\mu(f) = \mu(X, 0) + \mu(f^{-1}(0), 0).$$

In the following theorem, we present a result analogous to Theorem 1.1 for this case.

THEOREM 4.4. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous ICIS and let $f : (X, 0) \to \mathbb{C}$ be a weighted homogeneous germ with an isolated singularity and let f_t be a deformation of f as in (1). If f_t is non-negative, then $\mu(f_t)$ is constant.*

Proof. By the Lê-Greuel formula, we have

$$\mu(f_t) = \mu(X, 0) + \mu(f_t^{-1}(0), 0).$$

We only need to show that $\mu(f_t^{-1}(0), 0)$ is constant. Moreover, if f_t is a non-negative deformation of f, then we can choose polynomial extensions $\bar{f}_t : (\mathbb{C}^n, 0) \to \mathbb{C}$ of f_t and $\bar{f} : (\mathbb{C}^n, 0) \to \mathbb{C}$ of f such that $\bar{f}_t$ is a non-negative deformation of $\bar{f}$.

On the other hand, if $(X, 0)$ is defined by $X = \phi^{-1}(0)$ with $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ then $f_t^{-1}(0)$ is an ICIS defined by $f_t^{-1}(0) = (\phi, \bar{f}_t)^{-1}(0)$. Therefore, it is enough to show that for a non-negative deformation of a map germ which defines an ICIS, the family of ICIS has constant Milnor number.

Let $\psi = (\psi_1, \ldots, \psi_p) : (\mathbb{C}^n, 0) \to \mathbb{C}^p$ be a weighted homogeneous map germ defining an ICIS $(V, 0) = (\psi^{-1}(0), 0)$ and let $\psi_t = (\psi_{1t}, \ldots, \psi_{pt})$ be a non-negative deformation of ψ where $\psi_{jt}(x) = \psi_j(x) + \sum_{i=1}^{l_j} \sigma_i^j(t)\alpha_i^j(x)$. We denote $V_t = \psi_t^{-1}(0)$. We will proceed by finite induction over p. If $p = 1$, the result is equivalent to

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Theorem 1.1. Assume that the result is true for $p-1$. By the Lê-Greuel formula,

$$\mu(V_t) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_t} - \mu(Y_t, 0),$$

where I_t is the ideal in $\mathcal{O}_n$ defined by $I_t = \langle \psi_{1t}, \ldots, \psi_{p-1t}, J(\psi_{1t}, \ldots, \psi_{pt})\rangle$ and Y_t is given by the zeros of $(\psi_{1t}, \ldots, \psi_{p-1,t})$. Since ψ_t is a non-negative deformation of ψ, we have that $\mathrm{wt}(\alpha_i^j) \geq \mathrm{wt}(\psi_j)$, for all $j = 1, \ldots, p$.

By the induction hypothesis, $\mu(Y_t, 0)$ is constant. Moreover, it is not difficult to show that each function in $\{\psi_{1t}, \ldots, \psi_{p-1t}, J(\psi_{1t}, \ldots, \psi_{pt})\}$ is a non-negative deformation of its corresponding initial term. Since I_0 is weighted homogeneous and $\dim_{\mathbb{C}} \mathcal{O}_n/I_0 < \infty$ we have $v(I_0) = \{0\}$, where $v(I_0)$ denotes the set of zeros of the ideal I_0. Then, it follows from the results of [**6**] that $v(I_t) = \{0\}$ also for any t.

On the other hand, since $\mathcal{O}_{n+1}/I_t$ is Cohen-Macaulay, we can apply the principle of conservation of number

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_0} = \sum_{x \in v(I_t)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{I_{t,x}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_t}.$$

Therefore, $\mu(V_t, 0) = \mu(V, 0)$, for any t.

□

REMARK 4.5. From the proof of the previous theorem, we have that if $(X, 0)$ is the ICIS defined by the zero set of the weighted homogeneous map germ $\phi : (\mathbb{C}^n, 0) \to \mathbb{C}^p$, ϕ_t is a non-negative deformation of ϕ and $(X_t, 0)$ is the ICIS defined by ϕ_t, then the Milnor number of $(X_t, 0)$ is constant. However, it is not clear for us what would be a non-negative deformations of a weighted homogeneous ICIS because it would depend on the generators of the ideal defining the ICIS.

5. Some remarks. In this section, the germs are weighted homogeneous with weights $(w_1, \ldots, w_n)$. In [**16**], it is shown that if $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is weighted homogeneous germ of degree d with an isolated singularity, then its Milnor number is

$$\mu(f) = \left(\frac{d - w_1}{w_1}\right) \ldots \left(\frac{d - w_n}{w_n}\right).$$

In [**18**], we show a similar result for the Bruce-Roberts number. If $X = \phi^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity with $\mathrm{wt}(\phi) = d_2$ and $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is weighted homogeneous $\mathcal{R}_X$-finitely determined germ with $\mathrm{wt}(f) = d_1$, then

$$\mu_{\mathrm{BR}}(f, X) = \frac{d_1}{w_1, \ldots, w_n} \sum_{j=1}^{n} (d_1 - w_1) \ldots (d_1 - w_{j-1})(d_2 - w_{j+1}) \ldots (d_2 - w_n).$$

In the following remark, we exhibit an analogous formula for the Milnor number of a function on a curve.

REMARK 5.1. Let $(X, 0)$ be a curve defined by the zeros of a weighted homogeneous map $\phi = (\phi_1, \ldots, \phi_{n-1}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ with $\mathrm{wt}(\phi_i) = d_i$. Let $f : (X, 0) \to \mathbb{C}$ be a

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finite weighted homogeneous function with $\mathrm{wt}(f) = d_n$. Then,

$$\mu(f) = \frac{d_1 \dots d_{n-1}(d_1 + d_2 + \dots + d_n - w_1 - w_2 - \dots - w_n)}{w_1 \dots w_n}.$$

In fact, since $(X, 0)$ is an ICIS, we have

$$\mu(f) = \mu(X, 0) + \mu(f^{-1}(0), 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi_1, \dots, \phi_{n-1}, J(\bar{f}, \phi) \rangle}.$$

where $J(\bar{f}, \phi)$ is weighted homogenous of degree $d_1 + d_2 + \dots + d_n - w_1 - w_2 - \dots - w_n$. Thus, the result follows from Bezout theorem.

By using the above remark, we can write the Milnor number of an one-dimensional ICIS in terms of the weights and the degrees. We remark that this result was already obtained in different ways in **[2]** and in **[10]**.

REMARK 5.2. Let $(X, 0)$ be a curve defined by the zeros of a weighted homogeneous map $\phi = (\phi_1, \dots, \phi_{n-1}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ $\mathrm{wt}(\phi_i) = d_i$. Then,

$$\mu(X, 0) = \frac{d_1 \dots d_{n-1}(d_1 + d_2 + \dots + d_{n-1} - w_1 - w_2 - \dots - w_n)}{w_1 \dots w_n} + 1.$$

In fact, take any weighted homogeneous finite function $f : (X, 0) \to (\mathbb{C}, 0)$. By (3), we have

$$\mu(X, 0) = \mu(f) - \deg(f) + 1.$$

To compute the degree of f, we use again Bezout theorem:

$$\deg(f) = \dim_C \frac{\mathcal{O}_{X,0}}{\langle f \rangle} = \dim_C \frac{\mathcal{O}_n}{\langle \phi_1, \dots, \phi_{n-1}, f \rangle} = \frac{d_1 \dots d_{n-1} d_n}{w_1 \dots w_n}.$$

Hence,

$$\begin{aligned} \mu(X, 0) &= \frac{d_1 \dots d_{n-1}(d_1 + d_2 + \dots + d_{n-1} + d_n - w_1 - w_2 - \dots - w_n) - d_1 \dots d_{n-1} d_n}{w_1 \dots w_n} + 1 \\ &= \frac{d_1 \dots d_{n-1}(d_1 + d_2 + \dots + d_{n-1} - w_1 - w_2 - \dots - w_n)}{w_1 \dots w_n} + 1. \end{aligned}$$

EXAMPLE 5.3. Let $(X, 0) = (\phi^{-1}(0), 0)$, with $\phi : (\mathbb{C}^3, 0) \to \mathbb{C}^2$ defined by $\phi(x, y, z) = (x^2 - y^3 - z^2, xy^3 - z^3)$, and let $f : (X, 0) \to \mathbb{C}$ be the function $f(x, y, z) = xz - y^3$. We have that f and ϕ are weighted homogeneous with weights $(3, 2, 3)$ and degrees $(6, 9)$ and (6), respectively.

Then,

$$\mu(f) = \frac{6 \cdot 9 \cdot (6 + 9 + 6 - 3 - 2 - 3)}{3 \cdot 2 \cdot 3} = 39,$$

and

$$\mu(X, 0) = \frac{6 \cdot 6 \cdot (6 + 9 - 3 - 2 - 3)}{3 \cdot 2 \cdot 3} + 1 = 22.$$

REMARK 5.4. If $(X, 0)$ is a weighted homogeneous curve with r branches and $f : (X, 0) \to \mathbb{C}$ is weighted homogeneous on each branch of X with weighted degree d_j, then from the proof of the Theorem 4.1, $\deg(f) = d_1 + \cdots + d_r$, hence

$$\mu(f) = \mu(X) + d_1 + \cdots + d_r - 1.$$

Inspired by the results of this section it is natural to ask if the Milnor number of a weighted homogeneous curve depends only on its weights and degrees. In a forthcoming paper, we will consider this question for weighted homogeneous IDS.

REMARK 5.5. Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a polynomial function with isolated singularity. We write $f = f_d + \cdots + f_l$, where f_i is weighted homogeneous of degree i. In **[7]**, it is proved that

$$\mu(f) \geq \frac{(d - w_1) \ldots (d - w_n)}{w_1 \ldots w_n}$$

and the equality holds if and only if f is semi-weighted homogeneous.

Here, we show an analogous result for a finite polynomial function $f : (X, 0) \to \mathbb{C}$ on a weighted homogenous curve $(X, 0) \subset (\mathbb{C}^n, 0)$. We write $f = f_d + \cdots + f_l$, where each f_i is weighted homogeneous of degree i. We have

$$\mu(f) \geq \mu(X, 0) + dr - 1,$$

where r is the number of branches of X. The equality holds if and only if f restricted to each branch of X is semi-weighted homogeneous.

In fact, let X_j, $j = 1, \ldots, r$ be the branches of X. We parametrize each X_j by $\gamma_j(s) = (a_{1j}s^{w_1}, \ldots, a_{nj}s^{w_j})$ as in the proof of the Theorem 4.1. Then, for each j,

$$\begin{aligned}
\deg(f|_{X_j}) &= m_0(f(\gamma_j(s))) \\
&= m_0(f(a_{1j}s^{w_1}, \ldots, a_{nj}s^{w_n})) \\
&= m_0(f_d(a_{1j}s^{w_1}, \ldots, a_{nj}s^{w_n}) + \cdots + f_l(a_{1j}s^{w_1}, \ldots, a_{nj}s^{w_n})) \\
&= m_0(s^d f_d(a_{1j}, \ldots, a_{nj}) + \cdots + s^k f_l(a_{1j}, \ldots, a_{nj}) \\
&\geq d,
\end{aligned}$$

and it is equal to d if and only if f_d restricted to each branch is finite.

Hence,

$$\begin{aligned}
\mu(f) &= \mu(X, 0) + \deg(f) - 1 \\
&= \mu(X, 0) + \sum_{j=1}^{r} \deg(f|_{X_j}) - 1 \\
&\geq \mu(X, 0) + dr - 1
\end{aligned}$$

and the equality holds if and only if f restricted to each branch is semi-weighted homogeneous.

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