VERY-HIGH-ECCENTRICITY LIBRATIONS AT SOME HIGHER ORDER RESONANCES

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Abstract. Motions near the 3:1, 4:1 and 5:2 resonances with Jupiter are studied by means of numerical integrations of a semi-analytically averaged Sun-Jupiter-asteroid planar problem. In order to have a model including the very-high-eccentricity regions of the phase space, we adopted a set of local expansions of the disturbing potential, adequate to perform the numerical exploration of regions in the phase space with eccentricities higher than 0.9 (Ferraz-Mello and Klafke, 1991). Individual solutions and qualitative results thus obtained are completely reproduced by numerical integration of the complete equations by filtering off the short-period components of these solutions.

1. INTRODUCTION

Most of the studies in asteroidal motion employs the classical expansion of the disturbing forces introduced almost two hundred years ago, by Laplace. This expansion is a power series in the eccentricities and inclinations and its convergence, thoroughly discussed by several authors in the beginning of this century (see Sundman, 1916, Hagihara, 1971), has not been often taken into account. A recent reevaluation of the criteria of Sundman and Silva by the authors, strictly confirm the results of Sundman for planar expansion. The convergence of the series is, generally, limited to small values of eccentricities. In fact, recent works (*e.g.* Lemaître and Henrard, 1988) have shown that high-eccentricity librations can not be studied with the classical symmetric approaches.

To avoid the difficulties concerning the poor convergence of the classical expansion of the disturbing function, we must use local expansions, providing a better representation of that function in the vicinity of any point in the phase space, including in the very-high-eccentricity regions.

2. THE MODEL

In this paper we use a model developed by Ferraz-Mello and Klafke(1991) based on the Ferraz-Mello and Sato(1989) asymmetrical expansion. We consider a planar elliptic restricted three body problem and adopt a set of non-singular variables related to the usual equinoxial Keplerian elements a, e, λ and ϖ, as follows,

$$\begin{aligned} k &= e\cos\sigma \\ h &= e\sin\sigma \\ \sigma &= \varphi - \varpi \\ \sigma_1 &= \varphi - \varpi_1 \\ Q &= \tfrac{\lambda-\lambda_1}{q}. \end{aligned} \tag{1}$$

The angles $\varphi = \frac{(p+q)}{q}\lambda_1 - \frac{p}{q}\lambda$, σ and Q are, respectively, the longitude of conjunction associated with the q^{th}-order resonance $(p+q):p$, the critical angle and

S. Ferraz-Mello (ed.), *Chaos, Resonance and Collective Dynamical Phenomena in the Solar System,* 153–158.

the short period angle. The averaging is done with respect to Q. The subscript 1 refers to Jupiter's elements.

The averaged variational equations in there elements are

$$\frac{da}{dt} = \frac{2r}{na}[h\frac{\partial F}{\partial k} - k\frac{\partial F}{\partial h} - \frac{\partial F}{\partial \sigma_1}] \tag{2}$$

$$\frac{d\sigma_1}{dt} = (r+1)n_1 - rn + \frac{2r}{na}\frac{\partial F}{\partial a} - \frac{r\beta}{na^2(1+\beta)}[k\frac{\partial F}{\partial k} + h\frac{\partial F}{\partial h}] \tag{3}$$

$$\frac{dk}{dt} = \frac{\beta}{na^2}\frac{\partial F}{\partial h} - h\frac{d\sigma_1}{dt} - \frac{k\beta}{2a(1+\beta)}\frac{da}{dt} \tag{4}$$

$$\frac{dh}{dt} = -\frac{\beta}{na^2}\frac{\partial F}{\partial k} + k\frac{d\sigma_1}{dt} - \frac{h\beta}{2a(1+\beta)}\frac{da}{dt} \tag{5}$$

where $r = \frac{p}{q}$, $n = \mu^{\frac{1}{2}}/a^{\frac{3}{2}}$ is the osculating mean motion of the minor planet, μ is the gravitational constant in astronomical units and $\beta = \sqrt{1-e^2}$. The orbital elements of Jupiter are assumed to be constant.

The system has the energy integral

$$\mathcal{E} = -\frac{\mu}{2a} - \frac{p+q}{p}n_1 na^2 - F. \tag{6}$$

The function F is the averaged disturbing function. It is expanded about a generic center (k_0, h_0), up to the second power of the differences $\delta k = k - k_0$ and $\delta h = h - h_0$(Ferraz-Mello and Sato, 1989).

$$F = \frac{\mu m_1}{a_1}[A_0 + A_1\delta k + A_2\delta h + \frac{1}{2}A_3\delta k^2 + \frac{1}{2}A_4\delta h^2 + A_5\delta k\delta h$$

$$+(A_6 + A_8\delta k + A_{10}\delta h)e_1\cos\sigma_1 + (A_7 + A_9\delta k + A_{11}\delta h)e_1\sin\sigma_1 \tag{7}$$

$$+\frac{1}{2}A_{12}e_1^2 + \frac{1}{2}A_{13}e_1^2\cos 2\sigma_1 + \frac{1}{2}A_{14}e_1^2\sin 2\sigma_1].$$

The coefficients A_j are obtained numerically and depend on the semi-major axis a and on the center (k_0, h_0).

The averaged equations (2-5) are integrated numerically and, to prevent any convergence problem, our model consider a set of local expansions in the $z = e \cdot \exp i\sigma$ plane, suitable to perform the numerical exploration of regions in the phase space with eccentricity higher than 0.9.

The coefficients A_j and its derivatives with respect to a were calculated in a net of 41x41 values of z ($|z| < 1$), for a fixed value of the semi-major axis and are stored in a double precision matrix to be used later in the numerical integrations. The dependence of the coefficients with a is given by a linear approximation.

At every moment, in the numerical integration, we choose in the 41x41 net, the expansion center which is closest of the solution, and use the correponding coefficients.

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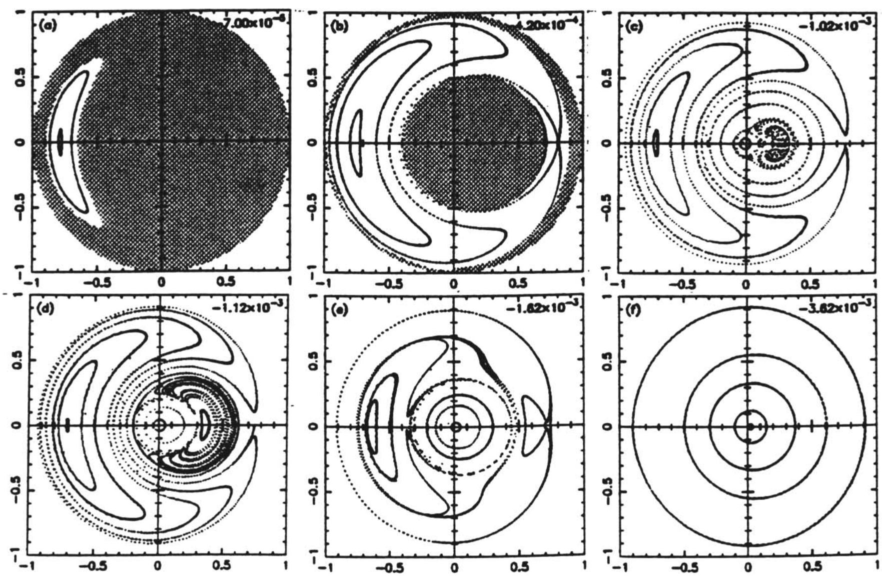

Fig. 1. Surfaces of sections of the 3:1 resonance in the Z-plane at some relevant energy levels. Every trajectory has been integrated in a time interval of $5 \times 10^4 - 3 \times 10^5$ years. The shadded area represent a region where has no real root in the energy equation.

3. RESULTS

In this section, we present some results of the numerical integrations with the proposed model for the 3:1, 4:1 and 5:2 mean-motion resonances with Jupiter.

The surfaces of sections of these resonances show, at high eccentricities, regions of regular motion around stable periodic orbits, characterized by librations of the perihelion (secular resonance) and of the critical angle σ (period resonance), as well as chaotic regions emanating from unstable periodic orbits. These periodic orbits belongs to families starting from the high-eccentricity corotation solutions (see Ferraz-Mello, Tsuchida and Klafke, in these proceedings). All these centers correspond to maxima of the energy.

3.1. The 3:1 Resonance

In the 3:1 resonance, librations of the critical angle σ occurs about either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Figures 1(a-f) show surfaces of section in the plane $Z = e \cdot \exp i(\varpi - \varpi_1)$ defined by $\sigma_0 = \frac{\pi}{2}$ and $\dot{\sigma} < 0$ for several energy levels in this resonance.

The energy integral has a global maximum at $\mathcal{E}_{max} = -23.69038$, which corresponds to a stable corotation center at $e = 0.812$, and there is no motion above this limit. For values slightly below this maximum the motions are regular invariant curves around the equilibrium solution. Figure 1a shows such solutions for a level $\Delta\mathcal{E} = \mathcal{E} - \mathcal{E}_{max} = -7 \times 10^{-5}$ below the global maximum. The shaded area repre-

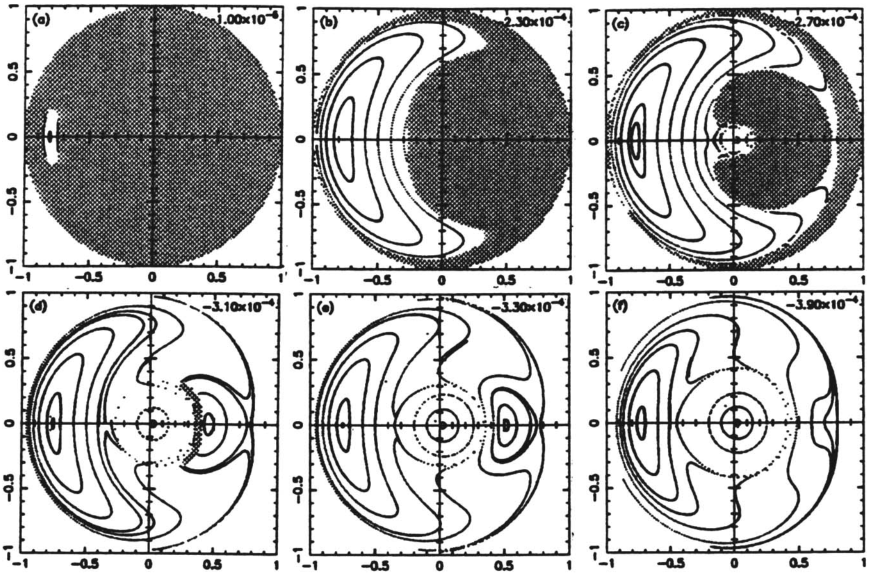

Fig. 2. Surfaces of section of the 4:1 resonance. The axes are the same as in the previous figures.

sents the region where equation (6) has no real root. Figure 1b corresponds to the energy level $\Delta\mathcal{E} = -4.2 \times 10^{-4}$, which lies in the vicinity of an unstable corotation center at $e = 0.788$ and $\varpi - \varpi_1 = 0$. The next two figures (1c-1d) correspond to the levels $\Delta\mathcal{E} = -1.02 \times 10^{-3}$ and $\Delta\mathcal{E} = -1.12 \times 10^{-3}$. The inner parts of these figures present a large chaotic region confined by invariant curves as first pointed out by Wisdom(1985) (see also Henrard and Caranicolas, 1990).

Our model shows that for values of the energy lower than $\Delta\mathcal{E} = -1.02 \times 10^{-3}$ the invariant tori shown in figures (1c-1d) disappear allowing the stable and unstable manifold starting from saddles at low and high eccentricities to entangle (heteroclinic points) and the chaotic regions emanating from different saddles mix (figure 1e). In this situation, orbits starting at relative low eccentricities ($e \approx 0.25$) may reach very high eccentricities ($e \approx 0.9$), after $10^5 - 10^6$ years.

3.2. The 4:1 Resonance

In the 4:1 resonance, the energy integral has a global maximum at $\mathcal{E}_{max} = -28.69797$ which corresponds to a stable corotation center at $e = 0.8067$. The librations of σ occurs about either 0, $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Surfaces of section, with $\sigma_0 = 0$ and $\dot{\sigma} < 0$, are displayed in figures (2a-2f) starting from a level $\Delta\mathcal{E} = -1 \times 10^{-5}$ below $\mathcal{E}_{max}$.

As in figures 1 the shaded area correspond to regions where equation (6) has no real roots. In such regions, the critical angle never crosses the selected value of σ_0. As a consequence, there are open curves in the surface of section starting from the

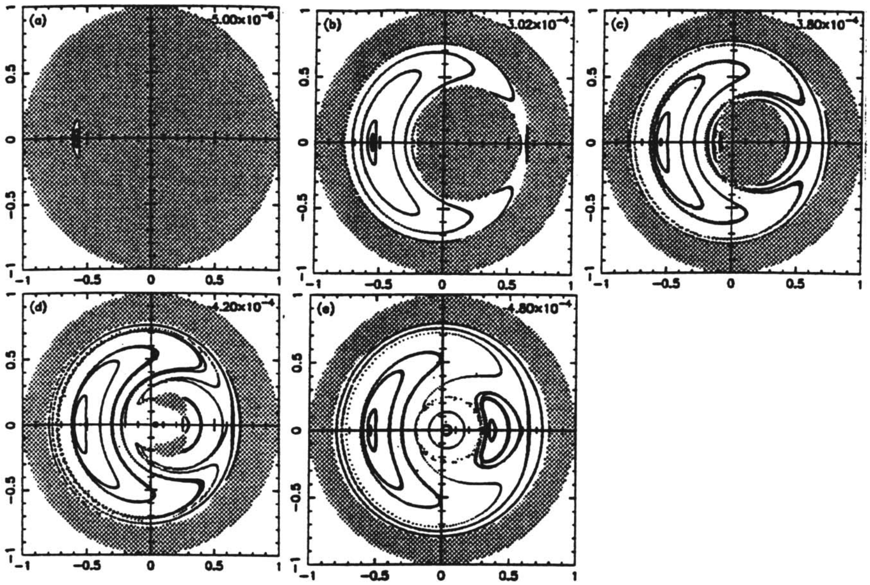

Fig. 3. Surfaces of sections to 5:2 resonance. Because of the singularity in the disturbing potential at $e \approx 0.8$, the phase spaces showed in these figures are smaller than in the preceding cases. The axes are the same as in figures 1 and 2.

boundary of this region.

A very interesting feature occurs for $\Delta\mathcal{E} = -2.7 \times 10^{-4}$ (figure 2c). Very close to the origin there is a stable corotation center. Immediately afterwards there is a chaotic zone associated with a saddle near $e \approx 0.15$ and $\varpi - \varpi_1 = \pi$. A trajectory starting from this zone may reach very high eccentricities ($e \approx 0.999$) after 3×10^5 years.

The next three figures (2d-2f) show the structure in the subsequent levels. The structure changes continuously until to exhibit the same aspect of the figure 1f.

3.3. The 5:2 Resonance

The evolutionary structure of the phase space in the 5:2 resonance is quite similar to the previous ones. Figures (3a-3e) show this structure. σ_0 and $\dot{\sigma}$ are the same of the 4:1 resonance, but in the present case, the energy limit is $\mathcal{E}_{max} = -20.98052$. Because of the energy variation associated with some border effect in the cellular representation of the asymmetric disturbing function (see Ferraz-Mello and Klafke, 1991) the investigation was stopped at $\Delta\mathcal{E} = -4.8 \times 10^{-4}$ bellow the energy maximum.

https://doi.org/10.1017/S0074180900091063 Published online by Cambridge University Press

4. COMMENTS

The qualitative results shows in the previous section were fully reproduced by numerical integration of the complete three-body equations. The averaged solutions were obtained from filtering off the short period terms of the complete solution, using a harmonic filter. Despite of some difficulties related to the energy variations, which imply that the energy surfaces were not flat, but deformed upwards for $\varpi - \varpi_1 = \pi$ and downwards for $\varpi - \varpi_1 = 0$, the qualitative results with our model are in good agreement with the full numerical integrations and have the advantage of being almost twenty times faster.

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