Adv. Appl. Prob. **20**, 686–689 (1988)
Printed in N. Ireland

A NOTE ON THE COMPONENT LIFETIME ESTIMATION OF A MULTISTATE MONOTONE SYSTEM THROUGH THE SYSTEM LIFETIME

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Abstract

In this paper we consider the observed lifetime of a multistate monotone system and 'the critical lower set' which causes the system deterioration. Then, under suitable conditions, we identify the component lifetime distribution using a Newton–Kantorovic iterative procedure as in Meilijson (1981).

CRITICAL LOWER SET; DETERIORATING SYSTEM

1. Introduction

Given the joint distribution of the lifetime T of a binary coherent system and the set K of components which cause the death of the machine, Meilijson (1981) identifies the component life distributions, which are assumed to be non-atomic and possess the same essential extrema. He uses a Newton–Kantorovic iterative method, under an assumption of the rank of the incidence matrix.

In this paper we propose to study the same problem for a multistate monotone system.

2. Preliminaries and notation

First we consider the stochastic behavior of the system and its components. Let $\{X_i(t), t \geqq 0\}$ be a right-continuous non-increasing stochastic process with values in $S = \{0, 1, \cdots, m\}$, representing the statistical behavior of component $i, i = 1, \cdots, n$. The multistate monotone system is represented by $\{\phi(\boldsymbol{X}(t)),\ t \geqq 0\}$ where $\phi: S^n \to S$ is a non-decreasing function and $\boldsymbol{X}(t) = (X_1(t), \cdots, X_n(t))$. Also assume that $\phi(0, \cdots, 0) = 0$ and $\phi(m, \ldots, m) = m$.

Block and Savits (1982) decomposed $\phi(\boldsymbol{X}(t))$ into binary structures:

$$\psi_k(\beta(\boldsymbol{X}(t))) = \min_{z \in L_k} \max_{(i,j) \in L_k(z)} \beta_{ij}(X(t)), \qquad k = 0, \cdots, m-1,$$

where $\beta(\boldsymbol{X}(t))$ is the vector $(\beta_{ij}(\boldsymbol{X}(t)),\ 1 \leqq i \leqq n,\ 0 \leqq j \leqq m-1)$ lexicographically ordered and $\beta_{ij}(\boldsymbol{X}(t)) = 1$ if $X_i(t) > j$ and 0 otherwise. L_k is the set of all critical lower vectors for level k and $L_k(z) = \{(i, z_i) : z_i \neq m\}$.

The binary structure $\psi_k(\beta(\boldsymbol{X}(t)))$ characterizes the level k of the multistate monotone system ϕ in the sense that $\phi(\boldsymbol{X}(t)) \leqq k$ if and only if $\psi_k(\beta(\boldsymbol{X}(t))) = 0$.

We define $T_{ij} = \inf\{t \geqq 0 : X_i(t) \leqq j\}$; $i = 1, \cdots, n$; $j = 0, \cdots, m-1$ and $T_k = \inf\{t \geqq 0 : \phi(\boldsymbol{X}(t)) \leqq k\}$, $k = 0, \cdots, m-1$. It is easy to prove that

$$T_k = \inf\{t \geqq 0 : \psi_k(\beta(\boldsymbol{X}(t))) = 0\} = \min_{z \in L_k} \max_{(i,j) \in L_k(z)} T_{ij}.$$

Received 30 October 1987; revision received 6 May 1988.

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Supported in part by CNPq–Brasil.

That means, to observe the first time that the system deteriorates to at most level k is the same as observing the lifetime of the binary system $\psi_k(\beta(\boldsymbol{X}(t)))$.

3. The main result

In our interpretation the observer allows the system to deteriorate until reaching the next level lower than $k+1$ and then makes an autopsy which reveals the 'critical lower set' that caused the allowed deterioration.

Let $F_{ij}(\cdot)$ be the distribution function of T_{ij}, $i=1,\cdots,n$; $j=0,\cdots,m-1$ and let M be the incidence matrix of order $\sum_{k=0}^{m-1}\#(L_k)\times mn$ whose rows are determined by $L_k(z)$, $z\in L_k$, $k=0,\cdots,m-1$, in the following manner:

$$M_{ij}=1 \quad \text{if} \quad (i,j)\in L_k(z) \quad \text{and} \quad M_{ij}=0 \quad \text{if} \quad (i,j)\notin L_k(z).$$

We assume that:

(i) The rank of M is mn;

(ii) All F_{ij}'s are non-atomic and possess the same essential infimum and the same essential supremum.

Under these hypotheses we can consider the identification problem. After the observation and diagnosis of T_k suppose that we have

$$\{T_k=t,\ L_k(z)\}=\left\{\max_{(i,j)\notin L_k(z)} T_{ij}=t,\ \min_{(i,j)\in L_k(z)} T_{ij}>t\right\}.$$

Then we can calculate the distribution function:

$$G_{L_k(z)}(x)=P\{T_k\leqq x,\ L_k(z)\}=\int_0^x \prod_{(i,j)\notin L_k(z)}(1-F_{ij}(t))\,d\left[\prod_{(i,j)\in L_k(z)}F_{ij}(t)\right].$$

Differentiating (Radon–Nikodym) both sides and dividing by the integrand and then integrating, we obtain:

$$\prod_{(i,j)\in L_k(z)}F_{ij}(x)=\int_0^x\left[\prod_{(i,j)\notin L_k(z)}(1-F_{ij}(t))\right]^{-1}dG_{L_k(z)}(t). \tag{1}$$

It is important to note that for both of the above products and for each i, $i=1,\cdots,n$, there is only one j such that the pair (i,j) is a term of the product. With this interpretation, we define the matrix $\bar{M}=[\bar{M}_{ij}]$ of order $\sum_{k=0}^{m-1}\#(L_k)\times mn$, where each row corresponds to a critical lower set $L_k(z)$. For this row $\bar{M}_{ij}=1$ if (i,j) is a term of the product $\Pi_{(i,j)\notin L_k(z)}(1-F_{ij}(t))^{-1}$ and $\bar{M}_{ij}=0$, otherwise.

Furthermore, we denote:

$$F(\cdot)=(F_{10}(\cdot),\cdots,F_{1m-1}(\cdot),F_{20}(\cdot),\cdots,F_{2m-1}(\cdot),\cdots,F_{nm-1}(\cdot))'$$

and

$$G(\cdot)=(G_{L_k(z)}(\cdot),\ z\in L_k,\ k=0,\cdots,m-1)',\quad \text{lexicographically ordered.}$$

Also we consider that, if h is a function of one variable and V is a vector, $h(V)$ is the vector whose ijth coordinate is $h(V_{ij})$ and the product of two column vectors is their coordinatewise multiplication.

With these considerations the logarithm of the equation (1) is the row of the matrix equation corresponding to $L_k(z)$.

$$M\log F(x)=\log\int_0^x\exp\{-\bar{M}\log(\mathbf{1}-F(t))\}\,dG(t).$$

Now, under the assumption that the rank of M is mn, we can calculate $T=(M'M)^{-1}M'$ to

https://doi.org/10.2307/1427044 Published online by Cambridge University Press

obtain

$$F(x)=\exp\left\{T\log\int_0^x \exp\{-\bar{M}\log(\mathbf{1}-F(t))\}\,dG(t)\right\}$$

which is an implicit equation for F.

Meilijson (1981) put this equation in the form:

$$\exp\{-V(x)\}+\exp\left\{T\log\int_0^x \exp\{\bar{M}V(t)\}\,dG(t)\right\}-1=0$$

where $V_{ij}(x)=-\log(1-F_{ij}(x))$, and considers V as an element of the Banach space $B[0,x_0]$ of R^{nm}-valued, Borel-measurable bounded functions on $[0,x_0]$ with the supremum norm $\sup_x \max|V_{ij}(x)|$. He shows, using a Newton–Kantorovic iterative method, that this equation possesses a unique root. This root is the vector of lifetime distribution of the components.

4. Examples

We are going to use Barlow and Wu's (1978) structure function to produce an example. It is easy to show that for this structure,

$$L_k=\{(m-k)\boldsymbol{x}+\boldsymbol{k}:\boldsymbol{x}\in K_0\}$$

and

$$L_k((m-k)\boldsymbol{x}+\boldsymbol{k})=\{(i,k):i\in K_0(\boldsymbol{x})\}$$

where K_0 is the set of all minimal cut vectors of a binary coherent system (as in Barlow and Proschan (1981)) and $K_0(\boldsymbol{x})$ is the minimal cut set corresponding to the minimal cut vector $\boldsymbol{x}$.

Now let us consider the $(n-r+1)$ out-of-n system. The system fails if and only if r of the r components fail. So $\#(L_k)=\binom{n}{r}$ and $\sum_{k=0}^{m-1}\#(L_k)=m\binom{n}{r}$. Furthermore, each row of the incidence matrix M is a mn vector with r 1's and $(mn-r)$ 0's. So, a diagonal element of $M'M$ is equal to $m\binom{mn-1}{r-1}$ and all of the off-diagonal terms equal $m\binom{mn-2}{r-2}$. We conclude that $M'M$ is non-singular and that the implicit equations for F become

$$F_{ij}(x)=\left(\frac{\prod_{L_k(z):(i,j)\in L_k(z)} H_{L_k(z)}}{\left[\prod_{L_k(z):(i,j)\notin L_k(z)} H_{L_k(z)}\right]^{\frac{r(r-1)}{(r(mn+1-r)-1)}}}\right)^{\frac{1}{m\binom{mn-1}{r-1}}}$$

where

$$H_{L_k(z)}=\int_0^x\left[\prod_{(i,j)\notin L_k(z)}(1-F_{ij}(t))\right]^{-1}dG_{L_k(z)}(t).$$

Acknowledgement

I thank the editor and the referee for various comments and suggestions on a previous draft of this letter.

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