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LOCAL INTEGRABILITY OF CR-HYPERSURFACES

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By characterising CR-hypersurfaces of dimension m having m strongly independent CR-functions, we find the necessary condition for a CR-hypersurface to be locally integrable and show that the same condition is also sufficient if the CR-hypersurface is strongly pseudoconvex and has m strongly independent CR-functions.

1. PRELIMINARIES

Let V be a CR-hypersurface of dimension m ($\geqslant 1$) in an open set Ω in $\mathbf{R}^{2m+1}$. That is, V is a subbundle of $\mathbf{C} \otimes T\Omega$ of fibre dimension m such that

$$[V, V] \subset V \tag{1.1}$$

$$V \cap \overline{V} = \{0\} \ (\overline{V} \text{is "the complex conjugate" of } V). \tag{1.2}$$

We call V locally integrable at p in Ω if in some neighborhood U of p in Ω there are $m+1$ C^∞-functions $\{u_j\}_1^{m+1}$ such that

(1.3) u_j are CR-functions, that is, $Lu_j = 0$ for any smooth section L of V on U;

$$du_1 \wedge \cdots \wedge du_{m+1} \neq 0. \tag{1.4}$$

Let V' be the orthogonal of V in $\mathbf{C} \otimes T^*\Omega$. Then the condition (1.2) is equivalent to $V' + \overline{V}' = \mathbf{C} \otimes T^*\Omega$. Hence, if V is locally integrable, then there are m CR-functions $\{u_j\}_1^m$ such that

$$du_1 \wedge \cdots \wedge du_m \wedge d\overline{u}_1 \wedge \cdots \wedge d\overline{u}_m \neq 0. \tag{1.5}$$

We call m such CR-functions to be strongly independent. Not every CR-hypersurface is locally integrable (see for example [**3, 5, 8, 9**]) (even if it has m strongly independent CR-functions). In [**8**], Nirenberg constructed a single C^∞-vector field in $\mathbf{R}^3$

$$L = \frac{\partial}{\partial \overline{z}} + iz\frac{\partial}{\partial t} + iz\phi(x, y, t)\frac{\partial}{\partial t}, \qquad (\phi \text{ is real-valued})$$

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satisfying for any C^∞-function $\psi(x, y, t)$ with $L\psi = 0$, $\partial\psi/\partial\overline{z}$ and $\partial\psi/\partial t$ vanish of infinite order at the origin. Thus the vector field L spans a CR-hypersurface of dimension 1 having a strongly independent CR-function z, which is also strongly pseudoconvex (that is, L, $\overline{L}$, and $[L, \overline{L}]$ are linearly independent) but not locally integrable at the origin. Motivated by the above example, Hanges [2] found a necessary condition for a CR-hypersurface of dimension 1 to be locally integrable and showed that the same condition is also sufficient in some restricted cases. Here, we shall extend the results in [2] to the CR-hypersurfaces of any dimension $m \geqslant 1$.

Once we characterise the CR-hypersurfaces having m strongly independent CR-functions in terms of local coordinates (see for example, Proposition 2.1), the extension from $m = 1$ to any $m \geqslant 1$ is rather straightforward.

2. Main results

From now on, V always means a CR-hypersurface of dimension $m \geqslant 1$ in Ω.

Proposition 2.1. *Locally V has m strongly independent CR-functions if and only if Ω can be covered by local charts $(U; x, y, t)$ in which V is spanned by m smooth vector fields*

$$L_j = \frac{\partial}{\partial \overline{z}_j} + a_j(x, y, t)\frac{\partial}{\partial t}, \qquad z_j = x_j + iy_j, \quad j = 1, \cdots, m. \tag{2.1}$$

Proof: The sufficiency is clear since $z_1, \cdots, z_m$ are strongly independent CR-functions of V. Conversely, let $u_1, \cdots, u_m$ be strongly independent CR-functions about a point, say 0, in Ω; then condition (1.5) is equivalent to

$$dReu_1 \wedge \cdots \wedge dReu_m \wedge dImu_1 \wedge \cdots \wedge dImu_m \neq 0. \tag{2.2}$$

Hence we can take a local coordinate system (x, y, t) about 0 where

$$x_1 = Reu_1, \cdots, \ x_m = Reu_m, y_1 = Imu_1, \cdots, y_m = Imu_m. \tag{2.3}$$

For any local basis $B = \{L_j\}_1^m$ of V about 0, we have

$$L_j = \sum_{k=1}^{m} 2L_j(Reu_k)\frac{\partial}{\partial \overline{z}_k} + L_j(t)\frac{\partial}{\partial t}, \ z_j = x_j + iy_j, \ j = 1, \cdots, m. \tag{2.4}$$

Note that the matrix $A = 2[L_j(Reu_k)]_{j,k}$ is nonsingular at 0 and so in some neighbourhood of 0. Indeed, if otherwise, the vector fields $\{L_j - L_j(t)\partial/\partial t\}_1^m$ are linearly dependent at 0 and so $\partial/\partial t$ must be in $V \cap \overline{V}$ at 0, which contradicts to (1.2). Then $A^{-1}B$ is another local basis of V in a neighbourhood of 0 having the form (2.1). ∎

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Note that when V is spanned by L_j in (2.1), condition (1.1) is equivalent to

$$[L_j, L_k] = 0, \text{ that is, } L_j a_k - L_k a_j = 0, \ j, k = 1, \cdots, m. \tag{2.5}$$

Proposition 2.2. *V is locally integrable at p in Ω if and only if there is a local chart $(U; x, y, t)$ about p in which V is spanned by vector fields L_j in (2.1) and a CR-function $u \in C^\infty(U)$ of the form*

$$u(x, y, t) = t + i\phi(x, y, t) \tag{2.6}$$

where $\phi \in C^\infty(U)$ is real-valued and $\phi(p) = 0$ and $d\phi(p) = 0$.

Proof: The necessity is trivial since $\{z_1, \cdots, z_m, u\}$ are $m+1$ independent CR-functions of V on U. Conversely, by Proposition 2.1, locally, V is spanned by L_j in (2.1). Furthermore, we may assume $a_j(p) = 0$ via suitable linear change of coordinates fixing x and y. Then $\{z_j\}_1^m$ are m independent CR-functions of V on U. Since V is locally integrable, there is a CR-function $u \in C^\infty(U)$ (possibly after shrinking U) such that

$$dz_1 \wedge \cdots \wedge dz_m \wedge du \neq 0. \tag{2.7}$$

Then $(\partial u/\partial t)(p) \neq 0$, since otherwise du is a linear combination of $\{dz_j\}_1^m$ which contradicts (2.7). If we set $v = Au + \sum_{j=1}^m B_j z_j$ where $A = (\partial u/\partial t)(p)^{-1}$ and $B_j = -(\partial u/\partial z_j)(p)\{(\partial u/\partial t)(p)\}^{-1}$, then v is a CR-function satisfying $dv(p) = dt$. Hence $v(x, y, t) = t + W(x, y, t)$ with $dW(p) = 0$ so that $\partial Rev/\partial t(p) \neq 0$. Then in a new coordinate system (x, y, t') with $t' = t + ReW$ about p, we have $v = t' + i\phi(x, y, t')$ and $d\phi(p) = 0$. We may assume $\phi(p) = 0$. ∎

For later use, we note that a_j and ϕ are related by

$$i\frac{\partial \phi}{\partial \bar{z}_j} + a_j\left(1 + i\frac{\partial \phi}{\partial t}\right) = 0, j = 1, \cdots, m. \tag{2.8}$$

We say that a smooth function $a(x, y, t)$ defined near a point p in $\mathbf{R}^{2m+1}$ is microanalytic at $(p, +1)$ (relative to t) (respectively $(p, -1)$) if there exist a neighbourhood $\mathcal{O}$ of $(p, 0)$ in $\mathbf{R}^{2m+2}$ and $A(x, y, t, s) \in C^\infty(\mathcal{O})$ such that

$$\frac{\partial A}{\partial \bar{w}} = 0 \text{ in } \mathcal{O}^+ (\text{ respectively } \mathcal{O}^-); \tag{2.9}$$

$$A(x, y, t, 0) = a(x, y, t) \text{ in } \mathcal{O}^0 = \{(x, y, t) \in \mathbf{R}^{2m+1} : (x, y, t, 0) \in \mathcal{O}\}, \tag{2.10}$$

where $w = t + is$ and $\mathcal{O}^+(\mathcal{O}^-) = \{(x, y, t, s) \in \mathcal{O} : s > 0 \ (s < 0)\}$.

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Theorem 2.1. (Necessity) *If V is locally integrable in Ω, then for any p in Ω there is a local chart $(U; x, y, t)$ about p in which V is spanned by the vector fields L_j in (2.1), where all a_j are microanalytic at $(p, +1)$.*

We shall omit the proof since it is essentially the same as that of Theorem 1 in [**2**].

For a local basis $\{L_j\}$ of V and any real vector field $L_0 \notin V + \overline{V}$ about a point p in Ω, define complex numbers $C_{j,k}(p)$ by

$$-\frac{i}{2}[L_j, \overline{L}_k](p) - C_{j,k}(p)L_0(p) \in V_p + \overline{V}_p. \tag{2.11}$$

We call the $m \times m$ matrix $L(p) = [C_{j,k}(p)]_{j,k}$ the Levi matrix of V at p. When it is nonsingular, the absolute value of its signature is an invariant of V. We call V strongly pseudoconvex at p if $L(p)$ is positive or negative definite.

Theorem 2.2. (Sufficiency) *Let V be a CR-hypersurface of dimension m having m strongly independent CR-functions so that V has a local basis $\{L_j\}_1^m$ in (2.1). Then V is locally integrable at p in Ω if the Levi matrix $L(p)$ of the system $\partial/\partial t, L_1, \cdots, L_m$ is positive (respectively negative) definite at p and the coefficients a_j of L_j are microanalytic at $(p, +1)$ (respectively $(p, -1)$). In this case, there is a CR-function u of the form (2.6) which is also microanalytic at $(p, +1)$ (respectively $(p, -1)$).*

We need the following lemma which can be easily proved by Proposition 2.2 (see for example, [**2**]).

Lemma. *Let V be the same as in Theorem 2.2. Then V is locally integrable at p in Ω if and only if there exist an open neighbourhood U of p and $v \in C^\infty(U)$ satisfying*

$$L_j v = -\frac{\partial a_j}{\partial t}, \quad j = 1, \cdots, m. \tag{2.12}$$

Proof of Theorem 2.2: We always assume $p = 0$. Since a_j and so $f_j := -\partial a_j/\partial t$ are microanalytic at $(0, +1)$, we have an open neighbourhood $\mathcal{O}$ of 0 in $\mathbf{R}^{2m+2}$ and $A_j, F_j \in C^\infty(\mathcal{O})$ such that

$$\frac{\partial A_j}{\partial \overline{w}} = \frac{\partial F_j}{\partial \overline{w}} = 0 \qquad \text{in } \mathcal{O}^+, \tag{2.13}$$

$$A_j(x, y, t, 0) = a_j(x, y, t),\ F_j(x, y, t, 0) = f_j(x, y, t) \quad \text{in } \mathcal{O}^0. \tag{2.14}$$

On $\mathcal{O}^+$, define $L_0^+ = \partial/\partial\overline{w}$ and $L_j^+ = \partial/\partial\overline{z}_j + A_j\partial/\partial w$, $j = 1, \cdots, m$. Then they are linearly independent and commute with each other. The latter comes directly from (2.5), (2.13) and Schwarz's reflection principle. Hence $\{L_j^+\}_0^m$ define a (almost) complex structure on $\mathcal{O}^+$ by Newlander-Nirenberg theorem [7].

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Now in order to solve the system (2.12), it suffices to solve

$$L_0^+ u = 0,\ L_j^+ u = F_j,\ j = 1, \cdots, m \tag{2.15}$$

smoothly up to $\{s = 0\}$.

Let $\alpha_0, \cdots, \alpha_m$ be 1-forms which are dual to $L_0^+, \cdots, L_m^+$ (for example, $\alpha_0 = d\overline{w} - \sum_1^m \overline{A}_j dz_j$ and $\alpha_j = d\overline{z}_j$, $j = 1, \cdots, m$). Then (2.15) becomes

$$\overline{\partial} u = \sum_{j=1}^{m} F_j \alpha_j \tag{2.16}$$

where the operator $\overline{\partial}$ is defined on $v \in C^\infty(\mathcal{O}^+)$ by $\overline{\partial} v = \sum_{j=0}^{m} (L_j^+ v)\alpha_j$.

Using (2.3), (2.13) and Schwarz's reflection principle, we can see that the 1-form $\sum_{j=1}^{m} F_j\alpha_j$ is $\overline{\partial}$-closed, that is, $\overline{\partial}\left(\sum_{j=1}^{m} F_j\alpha_j\right) = 0$.

As in [2], we can find a strongly pseudoconvex domain in $\mathcal{O}^+$ whose boundary intersects with $\{s = 0\}$ near 0. Hence, by the well known fact about the $\overline{\partial}$-problem (see for example, [1]), we can solve the equation (2.16) (or equivalently (2.15)) smoothly up to $\{s = 0\}$. ∎

Remark. In fact, if $m \geqslant 4$, then Theorem 2.2 gives only a simple proof of a special case of Kuranishi's theorem [6] saying that any strongly pseudoconvex CR-hypersurface of dimension $\geqslant 4$ is locally integrable. However, for $m < 4$ (at least for $m = 1$ by Nirenberg's example in [8]) not every strongly pseudoconvex CR-hypersurface of dimension m is locally integrable.

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