STABLE TREES

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Abstract. We show that a tree is stable if and only if its automorphism group contains a transposition. The method is constructive and so enables a stabilising sequence to be found.

1. Introduction

Throughout all graps are finite, undirected, and without loops or multiple edges. All of the basic graph concepts can be found in [1] and all group concepts and notation can be found in [5] except that, as in [1], we use $G + H$ for the sum of two permutation groups G, H, and $G[H]$ for the composition (wreath product) of G around H.

A graph $\mathscr{G}$ on a vertex set V is said to be *semi-stable* (at v) if there exists a vertex v such that $\Gamma(\mathscr{G}_v) = \Gamma(\mathscr{G})_v$ (see [4]) where $\Gamma(\mathscr{G})$ is the automorphism group of $\mathscr{G}$. Further $\mathscr{G}$ is *stable* if when $|V| = n$ there exists a sequence $v_1, v_2, \cdots, v_n$ of the vertices of $\mathscr{G}$ (a stabilising sequence) such that $\Gamma(\mathscr{G}_{v_1 v_2 \cdots v_k}) = \Gamma(\mathscr{G})_{v_1 v_2 \cdots v_k}$ for $k = 1, 2, \cdots, n$.

The concept of stability was introduced in [3] and in [4] it was shown that if a graph is stable then it is necessary that its automorphism group contain a transposition. We show here that if $\mathscr{T}$ is a tree, then for $\mathscr{T}$ to be stable the possession of a transposition by $\Gamma(\mathscr{T})$ is also sufficient.

2. Bunches

If in a tree $\mathscr{T}$ there are r vertices of degree one (end vertices), $r \geqq 2$, adjacent to a common vertex v, then $\mathscr{T}$ is said to have a *bunch* of size r (an r-bunch) at v. In such a case $\mathscr{T}$ can be considered as $\mathscr{T}' . \mathscr{K}_{1,r}$ ([1, p.23]) where $\mathscr{T}'$ is the subtree of $\mathscr{T}$ with the vertices of the r-bunch removed.

LEMMA 1. *If* $|V| > 2$ *and* $\Gamma(\mathscr{T})$ *contains a transposition then* $\mathscr{T}$ *has a bunch.*

PROOF. If $|V| = 3$ the result is true since $\mathscr{T} = \mathscr{K}_{1,2}$ which has a 2-bunch and if $|V| = 4$ then $\mathscr{T} = \mathscr{K}_{1,3}$ and this has a 3-bunch.

Let $|V| > 4$, let $(u_1 v_1) \in \Gamma(\mathscr{T})$, and let $u_1 u_2$, $v_1 v_2$ be edges in $\mathscr{T}$. Then $u_1 u_2^{(u_1 v_1)}$

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$= v_1u_2$, and $v_1v_2^{(u_1v_1)} = u_1v_2$, so v_1 and u_2 are adjacent in $\mathscr{T}$ as are u_1 and v_2. Hence $u_2 = v_2$ since otherwise $\mathscr{T}$ contains the cycle u_1, v_2, v_1, u_2.

Similarly if u_1u_3 and v_1v_3 are further edges of $\mathscr{T}$ then $u_3 = v_3$. But this means that u_1, u_3, v_1, v_3 is a cycle in $\mathscr{T}$ and so $u_2 = u_3$.

Hence if $(u_1v_1) \in \Gamma(\mathscr{T})$, u_1 and v_1 must have degree one and be adjacent to a common vertex of $\mathscr{T}$, and so they are part of a bunch of $\mathscr{T}$.

The main result of the paper is clearly true for $|V| = 2$ and for $|V| > 2$ it is enough to prove that any tree with a bunch is stable. We do this in stages first noting that Heffernan [2] has shown that all rooted trees are semi-stable at an end vertex. It follows immediately therefore that all *rooted* trees are stable, regardless of their automorphism group.

Before continuing we note three lemmas. A proof of Lemma 3 can be found in [3].

LEMMA 2. *For graphs $\mathscr{G}$ and $\mathscr{H}$ with v a vertex of $\mathscr{G}$ and $\mathscr{G} \neq \mathscr{H}$ and $\mathscr{G}_v \neq \mathscr{H}$ then*

$$\Gamma(\mathscr{G} \cup \mathscr{H})_v = [\Gamma(\mathscr{G}) + \Gamma(\mathscr{H})]_v = \Gamma(\mathscr{G})_v + \Gamma(\mathscr{H})$$

LEMMA 3. *For v a vertex of some copy of $\mathscr{G}$ in the graph $n\mathscr{G}$*

$$\Gamma(n\mathscr{G})_v = S_n(\Gamma(\mathscr{G})]_v = S_{n-1}[\Gamma(\mathscr{G})] + \Gamma(\mathscr{G})_v$$

LEMMA 4. *For the tree $\mathscr{T}$ if v is fixed by $\Gamma(\mathscr{T})$ then $\Gamma(\mathscr{T}) = \Gamma(\underline{\mathscr{T}})$ where $\underline{\mathscr{T}}$ is the tree $\mathscr{T}$ rooted at v.*

If $\mathscr{B}_i$ are the branches of $\mathscr{T}$ at v then Lemma 4 enables us to treat $\Gamma(\mathscr{T})$ as $\Gamma(\cup\underline{\mathscr{B}}_i)$ where $\underline{\mathscr{B}}_i$ denotes $\mathscr{B}_i$ rooted at v. We can now dispose of the trees with only one bunch.

THEOREM 1. *A tree with only one bunch is stable.*

Proof. If $\mathscr{T}$ possesses a single bunch attached to v then v is fixed by $\Gamma(\mathscr{T})$. In view of Lemma 4 it follows that if $\mathscr{T} = \mathscr{T}'.\mathscr{K}_{1,r}$ then $\Gamma(\mathscr{T}) = \Gamma(\underline{\mathscr{T}}') + \Gamma(\mathscr{K}_{1,r})$. But we know that $\underline{\mathscr{T}}'$ is stable so the vertices of the stabilising sequence of $\underline{\mathscr{T}}'$ can be removed in order until the situation of Figure 1(a) is reached. The vertices removed will form part of the stabilising sequence of $\mathscr{T}$

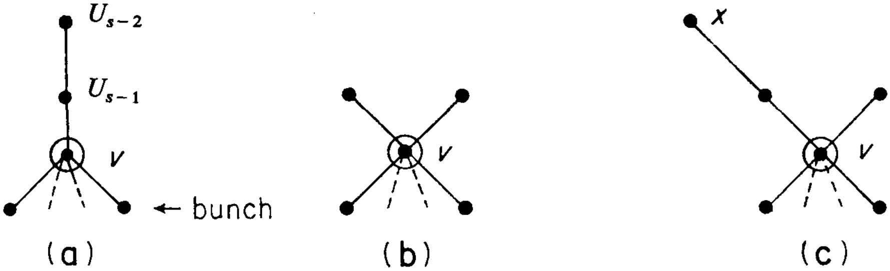

Figure 1

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because
$$\begin{aligned}\Gamma(\mathscr{T}_w) &= \Gamma(\underline{\mathscr{T}}'_w) + \Gamma(\underline{\mathscr{K}}_{1,r}) \\ &= \Gamma(\underline{\mathscr{T}}')_w + \Gamma(\underline{\mathscr{K}}_{1,r}) && \text{as } \underline{\mathscr{T}}' \text{ is semi-stable at } w \\ &= \Gamma(\mathscr{T})_w && \text{by Lemma 2.}\end{aligned}$$

The graph of Figure 1(a) is semi-stable at u_{s-1} but not at u_{s-2} so in the stabilising sequence for $\mathscr{T}$ we next remove u_{s-1} and then u_{s-2}. $\mathscr{T}$ is now reduced to $\mathscr{K}_{1,r}$ and this is stable by [3] with v and then the vertices of degree one as a stabilising sequence. Hence $\mathscr{T}$ is stable.

The situation of the type shown in Figure 1(b) does not arise since (b) would have been derived from (c), and the rooted tree obtained from (c) by removing the vertices of the bunch is not semi-stable at x. Further at no stage in the removal of $\underline{\mathscr{T}}'$ could another bunch occur as this would lead to the introduction of an automorphism not present in the previous stage and the semi-stability would have been violated.

3. A stability algorithm

A tree $\mathscr{T}$ is said to have been *pruned* if its stabilising sequence is $\{v_1, v_2, \cdots, v_n\}$ and if the vertices $v_1, v_2, \cdots, v_p$, with $p \leqq n$, have been removed in order. We now show that a tree with any number of bunches can be pruned to a tree with one bunch so that Theorem 1 can be applied.

Now every tree has a centre consisting of one or two vertices. We deal with the central and bicentral cases separately, noting that the concept of lightest branch at a vertex will be of use throughout. A *lightest branch* is one with smallest diameter and a minimum number of vertices.

It is assumed throughout that $\mathscr{T}$ has more than one bunch—the algorithm stops as soon as $\mathscr{T}$ is pruned to one bunch.

A. Central Case. (i) Let c be the centre of $\mathscr{T}$ and let c have degree greater than 2. Let $\mathscr{B}_i$, $i = 1, 2, \cdots, s$, be the lightest branches at c which have no bunches, and $\mathscr{C}_j$, $j = 1, 2, \cdots, t$, the lightest branches at c which have bunches. Let $\mathscr{T}'$ be the sub-tree of $\mathscr{T}$ with no vertices in common with the $\mathscr{B}_i$ or the $\mathscr{C}_j$ except c. Then as c is fixed by $\Gamma(\mathscr{T})$ we have

$$\Gamma(\mathscr{T}) = \Gamma\left(\bigcup_{i=1}^{s} \underline{\mathscr{B}}_i\right) + \Gamma\left(\bigcup_{i=1}^{t} \underline{\mathscr{C}}_i\right) + \Gamma(\underline{\mathscr{T}}').$$

In view of Lemmas 2 and 3 if $\underline{\mathscr{B}}_1$ is semi-stable at v then so is $\mathscr{T}$ and as $\underline{\mathscr{B}}_1$ is a rooted tree (rooted at c) then it is semi-stable at an end vertex. But

$$\Gamma(\mathscr{T}_v) = \Gamma[(\underline{\mathscr{B}}_1)_v] + \Gamma\left(\bigcup_{i=2}^{s} \underline{\mathscr{B}}_i\right) + \Gamma\left(\bigcup_{i=1}^{t} \underline{\mathscr{C}}_i\right) + \Gamma(\underline{\mathscr{T}}')$$

and since $(\underline{\mathscr{B}}_1)_v$ is semi-stable then so is $\mathscr{T}_v$. Hence $\underline{\mathscr{B}}_1$ can be pruned down to c, and this is also a pruning of $\mathscr{T}$. No extra automorphism can occur in the relevant

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subgroups of $\Gamma(\mathscr{T})$ as $(\mathscr{B}_1)_v$ and its sub-branches are too light to map into any other branches, and any extra automorphisms occurring mapping these lighter sub-branches into themselves would defy the semi-stability.

The remaining $\underline{\mathscr{B}}_i$ can now be pruned in a similar fashion until either c has degree 2 or all the $\underline{\mathscr{B}}_i$ have been pruned. Next the $\underline{\mathscr{C}}_i$ are pruned in the same manner till c has degree 2, or all the $\underline{\mathscr{C}}_i$ have been pruned, or only one bunch remains on the tree.

If in our pruned tree at this stage c has degree greater than 2 and it has more than one bunch we return to the start of A(i). We can eventually then prune $\mathscr{T}$ to a tree with one bunch (in which case we are finished) or a tree in which c has degree 2.

By alternating our pruning with no bunch branches followed by bunch branches we ensure that c is the centre of every tree pruned from $\mathscr{T}$ to date. This is essential to ensure that c is always fixed by every automorphism group considered, and so no extra automorphism arises to upset the semi-stability at any stage.

(ii) But what if c has degree 2 in $\mathscr{T}$? In such a case either c has a branch without any bunches or both branches have bunches.

(a) In the former case move along the branch which has bunches, away from c, until a vertex c' of degree greater than 2 is reached. Since c' is unique on this branch it is a fixed point of $\Gamma(\mathscr{T})$ and we can treat $\Gamma(\mathscr{T})$ as the group of certain trees rooted at c'. We now return to A(i) and proceed with c replaced by c'. Pruning is stopped when the degree of c' is 2 (assuming more than one bunch is left on the tree at this stage). At this stage c' has one branch with bunches and one without because the algorithm at A(i) prunes lightest branches first and the "heaviest" branch must go unpruned. This branch is the one containing c and has no bunches.

When c' has degree 2 then we move to the next vertex away from c with degree greater than 2 and return to A(ii) (a) of the algorithm. In this way $\mathscr{T}$ is pruned eventually to a tree with only one bunch.

(b) If now c has bunches on both branches then either the branches are identical or they are not. Assume both branches are $\mathscr{B}$ and let $\underline{\mathscr{B}}$ be semi-stable at v. Then $\Gamma(\mathscr{T}) = S_2[\Gamma(\underline{\mathscr{B}})]$ and $\mathscr{T}$ is also semi-stable at v as a result of Lemmas 2, 3 and 5.

LEMMA 5. *Let c be the centre of $\mathscr{T}$ with degree 2, and let c possess two equal branches. Then the removal of an endvertex of $\mathscr{T}$ leaves c fixed unless $\mathscr{T} = \mathscr{P}_n$ (n odd).*

PROOF. In $\mathscr{T}$ c is of degree 2. In $\mathscr{T}_v$ c is not fixed so a distance argument shows that $\mathscr{T}_v$ is bicentral with c and c' the centres.

If in $\mathscr{T}_v$ c is of degree 1 then so is c' and $\mathscr{T} = \mathscr{P}_3$.

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If in $\mathscr{T}_v$ c is of degree 2 then so is c'. As a result for c_1 adjacent to c, c_1 is of degree 2 in $\mathscr{T}$. We continue inductively to show that all vertices in $\mathscr{T}_v$ must be of degree 2 except the vertex adjacent to v, and the vertex corresponding to v on the branch of $\mathscr{T}$ not including v. Hence $\mathscr{T} = \mathscr{P}_n$ and n must be odd because c has two equal branches.

So we now find that $\mathscr{T}$ can be pruned at v because the only tree where this is not possible contains no bunches.

Having removed v, $\mathscr{T}$ is now pruned to a tree with c as centre and c has either a branch without a bunch — in which case $\mathscr{T}_v$ has only one bunch, or two unequal branches with bunches. It is enough to consider the last case to complete the algorithm at stage (b).

We here move out from c to the first vertex on the lightest branch of degree greater than 2, and use the methods of A(ii) (a). It may be that after pruning all bunches from the lightest branch at c we may still have a number of bunches left on the tree. If this is so we consider the pruned tree to determine whether it is central or bicentral and proceed accordingly.

B. Bicentral Case. If the centres are c_1 and c_2 either the branches at c_1 are identical with those at c_2 or they are not. In the former case we need the following Lemma.

LEMMA 6. *Let $\mathscr{T}$ be bicentral with an identical set of branches at each centre. Then the removal of an end vertex v from $\mathscr{T}$ leaves the centre closest to v fixed in $\Gamma(\mathscr{T}_v)$ unless $\mathscr{T} = \mathscr{P}_n$ (n even).*

PROOF. Let c_1 and c_2 (the centres of $\mathscr{T}$) be of degree d in $\mathscr{T}$ and let the distance between c_2 and v be smaller than that between c_1 and v. Further let h be an automorphism of $\mathscr{T}_v$ sending c_2 into c_{11}, a vertex adjacent to c_1. Then c_{11} has degree d in $\mathscr{T}_v$. (Unless c_2 is adjacent to v in $\mathscr{T}$.) Now in $\mathscr{T}$, c_{11} maps into a vertex c_{21} adjacent to c_2 via g say, and so c_{21} is of degree d. Eventually by switching from $\mathscr{T}_v$ to $\mathscr{T}$, and using h and g only, we obtain a chain of adjacent vertices of degree d. Either this chain goes on idefinitely or reaches c_{2i} the vertex adjacent to $v = c_{2,i+1}$ in $\mathscr{T}$. (c_{2i} may be c_2.) In $\mathscr{T}_v$ c_{2i} has degree $d-1$. Hence there exists a $c_{1,i+1}$ of degree $d-1$ adjacent to c_{1i}. But in $\mathscr{T}$ g sends $c_{1,i+1}$ into v. Hence $d-1 = 1$ or $d = 2$. So all vertices of $\mathscr{T}$ are of degree 2 except the end vertices and so $\mathscr{T} = \mathscr{P}_n$.

Finally n is even since there are 2 vertices in its centre and an equal number of vertices in the branches at the centre vertices.

(i) Let $\mathscr{B}_1$ ($\mathscr{B}_2$) be the branches attached to c_1 (c_2) which do not include c_2 (c_1). Assume $\mathscr{B}_1 = \mathscr{B}_2$. Then $\Gamma(\mathscr{T}) = S_2[\Gamma(\underline{\mathscr{B}}_2)]$ where $\underline{\mathscr{B}}_2$ is $\mathscr{B}_2$ rooted at c_1. If $\mathscr{B}_2$ is semi-stable at v then so is $\mathscr{T}$, since c_2 is fixed in $\Gamma(\mathscr{T}_v)$ (Lemma 6 as $\mathscr{P}_n$ has no bunches) and Lemma 3 can be invoked.

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In $\mathscr{T}_v$ C_2 is fixed and either c_1 is the sole centre or it is not. We move to A in the former case and B(ii) in the latter case.

(ii) Here the branches at c_1 and c_2 are distinct. Let c_2 have the lightest branch (or equally light branch). If c_2 has degree greater than 2 then we prune branches as in A(i). If not move along the branch at c_2 which does not include c_1 to the first vertex of degree greater than 2, and apply the pruning of A(i),and then continue as in A(ii) (b). The process will stop when only one bunch remains.

COROLLARY 1. *A forest is stable if and only if each of its trees contains a bunch.*

COROLLARY 2. *Every homeomorphically irreducible tree is stable.*

4. Enumeration

Because every stable tree has a bunch we can construct stable trees with n vertices by adding a 2-bunch to the root of any rooted tree on $n-2$ vertices. If $S(n)$ is the number of stable trees with n vertices and $R(n)$ the number of rooted trees with n vertices we can see that $S(n) \leqq R(n-2)$. From the evidence of Table 1, the difference between $R(n-2)$ and $S(n)$ seems to increase as n increases.

Table 1

n	3	4	5	6	7	8	9	10
$S(n)$	1	1	2	4	8	17	37	85
$R(n)$	1	1	2	4	9	20	48	115

It would be of interest to enumerate stable trees. We suspect a cycle approach could be fruitful.

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