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ON THE SMOOTHNESS OF GENERIC RINGS FOR STABLY FREE MODULES

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ABSTRACT. We show certain generic rings for stably free modules are smooth. Using the theory of smooth algebras we deduce that these rings are regular when the base ring is regular. Also this enables us to calculate the dimensions of these rings. Gabel used these generic rings to determine the freeness of certain stably free modules. Our results allow a strengthening of his results when restrictions are placed on the type of stably free module—for example orthogonal stably free modules.

§1. **Introduction.** Let R be a ring (all rings are commutative with unit). We call an R-module P *stably free* if there exists positive integers $m \leq n$ such that $R^n = P \oplus R^m$. In such a case P is said to be of type (m, n). Equivalently, we can think of a stably free module of type (m, n) as the kernel of a surjective homomorphism $\alpha : R^n \to R^m$. Certainly stably free modules are projective. We say P is an *orthogonal stably free* module if $P \cong \ker \alpha$ where $\alpha : R^n \to R^m$ satisfies $\alpha\alpha^t = 1$ (where α is considered as a matrix and t denotes transpose).

Gabel [1] defined certain generic rings to study stably free modules. Theorem 6 below explains the term 'generic'. They are defined as follows:

DEFINITION. Let R be a ring. Let X_{ij}, Y_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ be a collection of commuting indeterminates over R. Let $X = (X_{ij})$ and $Y = (Y_{ij})$ denote the m by n matrices with these indeterminates as entries. Consider $R[X, Y] = R[\cdots X_{ij}, \ldots, \ldots Y_{ij} \cdots]$ and the ideal I generated by the elements of the m by m matrix $XY^t - 1$. We use the natural notation of $(XY^t - 1)$ for I. Define $R_{m,n} = R[X, Y]/(XY^t - 1)$. Similarly define $R^0_{m,n} = R[X]/(XX^t - 1)$. $R_{m,n}$ is the generic ring for stably free modules of type (m,n). $R^0_{m,n}$ is its orthogonal counterpart. We will prove the following:

THEOREM 1. *$R_{m,n}$ is smooth over R.*

THEOREM 2. *$R^0_{m,n}$ is smooth over R iff 2 is a unit in R.*

THEOREM 3. [1: p. 27] *Let R be a regular ring of dimension d such that all maximal ideals of any finitely generated polynomial algebra over R have the*

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same height (for example $R=\mathbb{Z}$ or $R=a$ field). Then $R_{m,n}$ is a regular ring of dimensionn $2\,mn+d-m^2$.

THEOREM 4. *Let R be a regular ring of dimension d such that all maximal ideals of any finitely generated polynomial algebra over R have the same height. Assume further that* 2 *is a unit in R (for example $R=\mathbb{Z}[\frac{1}{2}]$ or $R=a$ field of characteristic $\neq 2$). Then $R^0_{m,n}$ is a regular ring of dimension $mn+d-[m(m+1)/2]$.*

In section 3 we generalize Theorems 2 and 4 to cover generic rings for stably free modules of various "types". We also indicate how these results can be applied in the study of the freeness of stably free modules.

§2. What do we mean by smooth? Let A be an R-algebra via the structure map $f:R\to A$. We say A is (formally) *smooth* over R if: given a commutative diagram of rings

$$\begin{tikzcd} A \arrow[r, "v"] & C/N \\ R \arrow[u, "f"] \arrow[r, "u"'] & C \arrow[u, "q"'] \end{tikzcd}$$

with N an ideal of C such that $N^2=0$, there exists a ring homomorphism $v':A\to C$ such that $v'f=u$ and $qv'=v$. Here $q:C\to C/N$ is the quotient map. If v' exists, we call it a lifting of v. Refer to Matsumura [4; p. 198] or Grothendieck [2; #20 0 19.3].

We denote by $\bar{c}$ the image of $c\in C$ in C/N. The meaning of our matrix notation below should be evident. Now we will prove the theorems.

THEOREM 1. *$R_{m,n}$ is smooth over R.*

Proof. We write x and y for the image of X and Y respectively in $R_{m,n}$. Suppose we are given the following commutative diagram:

$$\begin{tikzcd} R_{m,n} \arrow[r, "v"] & C/N \\ R \arrow[u] \arrow[r] & C. \arrow[u] \end{tikzcd}$$

Write $v(x)=\bar{\alpha}$ with $\alpha_{ij}\in C$, and $v(y)=\bar{\beta}$ with $\beta_{ij}\in C$. Then $xy^t=1$ implies $\overline{\alpha\beta^t}=1$ and hence $\alpha\beta^t=1+\pi$ with $\pi_{ij}\in N$. Define $v':R_{m,n}\to C$ by $x\to(1-\pi)\alpha$ and $y\to\beta$. Note that this is well defined: $(1-\pi)\alpha\beta^t=(1-\pi)(1+\pi)=1$ since $N^2=0$. Since v' lifts v, the proof is complete.

THEOREM 2. *$R_{m,n}$ is smooth over R iff* 2 *is a unit in R.*

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Proof. Suppose 2 is a unit in R. Write x for the image of X in $R^0_{m,n}$. Consider

$$\begin{array}{ccc} R^0_{m,n} & \xrightarrow{v} & C/N \\ \uparrow & & \uparrow \\ R & \longrightarrow & C \end{array}$$

Write $v(x)=\bar{\alpha}$ with $\alpha_{ij}\in C$. Then $xx^t=1$ implies $\alpha\alpha^t=1+\pi$ with $\pi_{ij}\in N$. Note that π, and hence $\pi/2$, is a symmetric matrix. Put $\alpha_1=(1-\pi/2)\alpha$. Clearly $\bar{\alpha}_1=\bar{\alpha}$. Define a lifting by sending x to α_1. This is well defined: $\alpha_1\alpha_1^t=(1-\pi/2)\alpha\alpha^t(1-\pi/2)^t=(1-\pi/2)(1+\pi)(1-\pi/2)=1$ since $N^2=0$. Hence $R^0_{m,n}$ is smooth over R. Conversely, suppose 2 is *not* a unit in R. Let $C=R[T]/(2, T^4)$ and let t denote the image of T in C. Let $N=(t^2)$ be an ideal in C. Note that $N^2=0$ and $2=0$ in C. Define $v: R^0_{m,n}\to C/N$ by sending x to $\bar{\alpha}$ where $\alpha=(1+t)(I_m0)$ is an m by n matrix over C (where I_m denotes the m by m identity matrix). Note that v is well defined since $\overline{\alpha\alpha^t}=1$. Suppose v has a lifting $v': R^0_{m,n}\to C$. Write $v'(x)=\beta$ with $\beta_{ij}\in C$. Then it follows that $\beta=\alpha+\pi$ with $\pi_{ij}\in N$. Now we have $1=\beta\beta^t=(\alpha+\pi)(\alpha^t+\pi^t)=\alpha\alpha^t+(\alpha\pi^t+\pi\alpha^t)$ since $N^2=0$. Since $2=0$ in C the diagonal entries of $\alpha\pi^t+\pi\alpha^t$ are zero. Looking at the 1,1 position we obtain $1=1+t^2$ in C, a contradiction. Thus $R^0_{m,n}$ is *not* smooth over R when 2 is not a unit in R. The proof is complete.

Before we can prove Theorems 3 and 4, we need a result from the theory of smooth algebras:

THEOREM 5. *Let R be a regular ring. Set $A=R[X_1,\ldots,X_n]$. Let $I=(f_1,\ldots,f_s)$ be an ideal of A and put $B=A/I$. Let $q\in\operatorname{Spec} B$ and $p=q^c\in\operatorname{Spec} A$. Let K be the residue field at p. B_q smooth over R implies B_q is regular and*

$$\dim B_q=\dim A_p-\operatorname{rank}_K\left(\frac{\partial f_i}{\partial X_j}(p)\right).$$

NOTE. for $g\in A$, $g(p)=g$ modulo p in A/p; remember $K=$ the quotient field of A/p.

Sketch of proof. [6] has more on this, but this is sufficient for our application here. Gabel [1] noted that a proof of Matsumura's [4; p. 213] could be extended to the case when R is regular instead of just a field. This says that $ht\, IA_p=\dim A_p-\dim B_q$ and that if

$$\operatorname{rank}_K\left(\frac{\partial f_i}{\partial X_j}(p)\right)=ht\, IA_p$$

then B_q is regular. It follows from Matsumura's general Jacobian criterion [4; p. 219] that if B_q is smooth over R then

$$\operatorname{rank}_K\left(\frac{\partial f_i}{\partial X_j}(p)\right)=ht\, IA_p.$$

Proof of Theorem 3. Theorem 1 tells us that $R_{m,n}$ is smooth over R, hence so is any localization. Since R is regular, $R_{m,n}$ is regular by Theorem 5 and the local criterion for regularity. It remains to calculate the dimension of $R_{m,n}$. Let $A=R[X, Y]$ and $I=(XY^t-1)$. Hence $R_{m,n}=A/I$. Let $q\in \operatorname{Spec} R_{m,n}$ be a maximal ideal and put $p=q^c\in \operatorname{Spec} A$. Let K be the residue field at p. Let $\Delta(p)$ denote the Jacobian matrix for I at p. Note that p is maximal and the hypothesis imply that $\dim A_p=2mn+d$ (remember $d=\dim R$). Then Theorem 5 implies that $\dim(R_{m,n})_q=2mn+d-\operatorname{rank}_K \Delta(p)$. We will be done when we show $\operatorname{rank}_K \Delta(p)=m^2$ for all $p\in \operatorname{Spec} A$ with $p\supseteq I$. We leave this to the reader as we will do a similar calculation later.

Proof of Theoem 4. Theorem 2 tells us that $R^0_{m,n}$ is smooth over R. Theorem 5 gives us that $R^0_{m,n}$ is regular and allows us to compute its dimension. It remains to show the rank of the Jacobian is $m(m+1)/2$. Let $p\in \operatorname{Spec} R[X]$ be such that $p\supseteq(XX^t-1)$. We show that the Jacobian at p has rank_K equal to $m(m+1)/2$. Let $f_{ij}=\sum_{k=1}^n X_{ik}X_{jk}-\delta_{ij}=$ the i, jth entry of XX^t-1. Then

$$\frac{\partial f_{ij}}{\partial X_{st}}=\delta_{is}X_{jt}+\delta_{js}X_{it}$$

with $1\le i, j, s\le m$ and $1\le t\le n$. Note that

$$\frac{\partial f_{ij}}{\partial X_{st}}=\frac{\partial f_{ji}}{\partial X_{st}}.$$

This says that certain rows of the Jacobian are repeated. Deleting the repeated rows we have the following $m(m+1)/2$ by mn matrix:

$$\begin{bmatrix} V_1 & & & & \\ & V_2 & & ? & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & V_m \end{bmatrix}$$

where V_k is an $m-k+1$ by n matrix given by

$$V_k=\left[\frac{\partial f_{ki}}{\partial X_{kj}}\right]_{\substack{k\le i\le m\\ 1\le j\le n}}=(X_{ij}+\delta_{ik}X_{kj})=((1+\delta_{ik})X_{ij}).$$

V_k modulo p has (row) rank equal to $m-k+1$ since $XX^t=1$ modulo p and since 2 is a unit in R. It follows that the rank of the Jacobian at p over K must be

$$\sum_{k=1}^{m} (m-k+1) = \frac{m(m+1)}{2}.$$

This completes the proof.

§3. By replacing the orthogonal condition $\alpha\alpha^t = 1$ by the more general condition $\alpha A \alpha^t = B$ we can obtain similar results. This is motivated by the classical groups. Specifically, let R be a ring and $m \leq n$ be positive integers. Let A be an n by n matrix and B an m by m invertible matrix over R. To avoid degeneracy we assume there is a matrix D satisfying $DAD^t = B$. We demand that A and B be symmetric or skew-symmetric. We say a stably free R-module P is of type (A, B) if $P \cong \ker \alpha$ where $\alpha : R^n \to R^m$ satisfies $\alpha A \alpha^t = B$.

Let $X = (X_{ij})$, $(1 \leq i \leq m,\ 1 \leq j \leq n)$ be a collection of commuting indeterminates. Using our previous notation define the generic ring for stably free modules of type (A, B) over R to be $R(A, B) = R[X]/(XAX^t - B)$. Let x denote the image of X in $R(A, B)$ and think of it as defining a map $x : R(A, B)^n \to R(A, B)^m$. Put $P_R(A, B) = \ker x$. $P_R(A, B)$ is a stably free module of type (A, B) over $R(A, B)$ and it is "generic" in the sense of:

THEOREM 6. (i) *If S is an R-algebra and P is a stably free S-module of type (A, B) then there exists an R-algebra homomorphism $R(A, B) \to S$ such that $P_R(A, B) \otimes_{R(A,B)} S \cong P$ as S-modules.*

(ii) *Given a homomorphism $R(A, B) \to S$, $P_R(A, B) \otimes_{R(A,B)} S$ is a stably free S-module of type (A, B) (where A and B are considered as matrices over S via the algebra structure).*

Proof. Gabel [1; p. 39] proved this for the orthogonal case and it is easy to generalize. We leave this to the reader.

Theorems 2 and 4 are really special cases of the following theorems. We have choosen to omit the details of the proofs as the essential ideas are illustrated best for the orthogonal case.

THEOREM 7. *Assume 2 is a unit in R and $R(A, B)$ is defined as above. Then $R(A, B)$ is smooth over R.*

THEOREM 8. *Let R be a regular ring of dimension d such that all maximal ideals of any finitely generated polynomial algebra over R have the same height. Assume further that 2 is a unit in R (for example $R = \mathbb{Z}[\frac{1}{2}]$ or $R =$ a field of characteristic $\neq 2$). Then $R(A, B)$ is a regular ring of dimension*

$$\begin{cases} mn + d - \dfrac{m(m+1)}{2} & \text{if} \quad A^t = A \\[2ex] mn + d - \dfrac{m(m-1)}{2} & \text{if} \quad A^t = -A. \end{cases}$$

To prove Theorem 7, put $\alpha_1=(1-(\pi/2)B^{-1})\alpha$ in the proof of Theorem 2 and check that this works (using the fact that $A^t=\pm A$). Theorem 8 involves the calculation of the dimension of the Jacobian of (XAX^t-B) which is just slightly more complicated than the calculation in Theorem 4; note that when A is skew-symmetric the terms $\partial f_{ii}/\partial X_{st}=0$.

If 2 is not a umit in R the smoothness of $R(A,B)$ depends on A and B. If there is a non-zero diagonal element of B then $R(A,B)$ is not smooth when $\frac{1}{2}\notin R$. This can be shown using an argument similar to that found in Theorem 2. If $A^t=-A$ and A has all diagonal elements zero then $R(A,B)$ is smooth over R, see Theorem 9 below.

Gabel used his generic modules to show that certain stably free modules were free. For example, let P be a stably free R-module of type (m,n). Then $\overset{k}{\bigoplus} P$ is free if $k(n-m)>\dim \mathbb{Z}_{m,n}=2mn+1-m^2$ [or $k>m+(mn+1)/(n-m)$], Lam [3] has a better bound for this of $k\geq m+(m/n-m)$. We show by example the kind of result that can be obtained using Gabel's technique and our Theorem 8.

Let P be a stably free R-module of type (A,B) where A and B are suitable matrices defined over $\mathbb{Z}[\frac{1}{2}]$. Assume 2 is a unit in R. Then $P\underset{R}{\bigoplus}P$ is free if $n>3m$ (A is n by n and B is m by m). By Theorem 6 it suffices to show the result is true for $R=\mathbb{Z}[\frac{1}{2}]$ and $P=P_{\mathbb{Z}[1/2]}(A,B)$. By the Bass Cancellation Theorem, a stably free module is free when the rank of the module is greater than the dimension of the ring. In our case we must have $(n-m)^2>\dim \mathbb{Z}[\frac{1}{2}](A,B)=mn+1+[m(m\pm 1)/2]$. This holds when $n>3m$. The same bound works for $\mathrm{End}_R(P)$.

We remark that every stably free R-module is of type (A,B) for some suitable matrices A and B over R: $P=\ker\alpha$ for some surjective $\alpha:R^n\to R^m$, let $\beta:R^m\to R^n$ satisfy $\alpha\beta=1$ then P is of type $(\beta\beta^t,1)$ over R. Unfortunately the matrices A and B may not be definable over a nice ring so that Theorem 8 can be used.

Finally, we prove a case of the smoothness of $R(A,B)$ even when $\frac{1}{2}\notin R$.

THEOREM 9. *Suppose $A^t=-A$ and $a_{ii}=0$ for all i. Then $R(A,B)$ is smooth over R.*

Proof. We are given

$$\begin{tikzcd} R(A,B) \arrow[r, "v"] & C/N \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd} \quad \text{with} \quad N^2=0.$$

Write $v(x)=\bar{\alpha}$ with $\alpha_{ij}\in C$. Then $\alpha A\alpha^t=B+\pi$ for some π with $\pi_{ij}\in N$. The conditions on A imply that all the diagonal elements of DAD^t are zero for any matrix D. It follows that $\pi_{ii}=0$ for all i (remember we assume there exists a matrix D such that $DAD^t=B$ so $b_{ii}=0$ for all i). Let

$$\beta_{ij}=\begin{cases}\pi_{ij} & \text{if } i<j\\ 0 & \text{if } i\geq j\end{cases}.$$

Since π is skew-symmetric, $\pi=\beta-\beta^t$. A lifting is obtained by sending x to $\alpha_1=(1-\beta B^{-1})\alpha$.

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