ON AFFINE COMPLETENESS OF DISTRIBUTIVE p-ALGEBRAS

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1. Introduction. G. Grätzer in [4] proved that any Boolean algebra $\mathbf{B}$ is affine complete, i.e. for every $n \geq 1$, every function $f: B^n \to B$ preserving the congruences of $\mathbf{B}$ is algebraic. Various generalizations of this result have been obtained (see [7]–[11] and [2], [3]).

In [2], R. Beazer characterized affine complete Stone algebras having a smallest dense element and in [3] gave an analogous result for double Stone algebras with a nonempty bounded core. For both these characterizations, a result of G. Grätzer is pertinent: a bounded distributive lattice is affine complete iff it has no proper Boolean interval (see [5]).

In this paper we show that any distributive p-algebra with a finite number of dense elements is affine complete if and only if it is a Boolean algebra.

2. Preliminaries. A (distributive) p-algebra is an algebra $\mathbf{L} = \langle L: \vee, \wedge, {}^*, 0, 1\rangle$ where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and * is a unary operation of pseudocomplementation, i.e. $x \leq a^*$ iff $S \wedge a = 0$. It is well known that the class $\mathcal{B}_\omega$ of all distributive p-algebras is a variety and that the lattice of subvarieties of $\mathcal{B}_\omega$ is a chain

$$\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \quad \dots \quad \subset \mathcal{B}_n \subset \quad \dots \quad \subset \mathcal{B}_\omega$$

of type $\omega + 1$, where $\mathcal{B}_{-1}$, $\mathcal{B}_0$, $\mathcal{B}_1$ are the classes of all trivial, Boolean and Stone algebras, respectively.

In any distributive p-algebra $\mathbf{L}$, an element $a \in L$ is called *closed*, if $a = a^{**}$. The set $B(\mathbf{L}) = \{a \in L; a = a^{**}\}$ of closed elements of $\mathbf{L}$ is a Boolean algebra in which the join is defined by $a \nabla b = (a \vee b)^{**}$. Moreover, $B(\mathbf{L})$ is a Boolean subalgebra of $\mathbf{L}$ iff $\mathbf{L}$ is a Stone algebra. An element $d \in L$ is said to be *dense* if $d^* = 0$. The set $D(\mathbf{L}) = \{d \in L: d^* = 0\}$ of dense elements of $\mathbf{L}$ is a filter of L. For these and other properties of distributive p-algebras as well as the standard rules of computation we refer the reader to [1] or [6].

For a distributive p-algebra $\mathbf{L}$, the clone $A(\mathbf{L})$ of all algebraic functions of $\mathbf{L}$ is the smallest set of functions on L containing the constant functions and the projections and closed under the operations $\vee$, $\wedge$ and *. A function $f: L^n \to L$ preserves the congruences of $\mathbf{L}$ if for any congruence θ of $\mathbf{L}$ and any elements $a_1, b_1, \dots, a_n, b_n, a_i \equiv b_i(\theta)$, $i = 1, \dots, n$ yields $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)(\theta)$. Following [12], a distributive p-algebra $\mathbf{L}$ is said to be *affine complete* if all finitary functions preserving congruences of $\mathbf{L}$ are algebraic. In [2], it is proved that a Stone algebra $\mathbf{L}$ with a bounded filter $D(\mathbf{L})$ is affine complete iff $D(\mathbf{L})$ is an affine complete distributive lattice, i.e. no proper interval of $D(\mathbf{L})$ is a Boolean algebra.

3. Affine completeness. In this section we show that for any distributive p-algebra $\mathbf{L}$ with a bounded filter $D(\mathbf{L})$, the affine completeness of $\mathbf{L}$ yields the affine completeness of $D(\mathbf{L})$, partially generalizing the main result from [2]. Then the finiteness of $D(\mathbf{L})$ ensures that $\mathbf{L}$ is a Boolean algebra.

First we represent every algebraic function on a distributive p-algebra in a canonical form.

Let $\mathbf{L}$ be any distributive p-algebra. For $n \geq 1$, we define the following n-ary algebraic functions of $\mathbf{L}$:

$$A(x_1, \dots, x_n) = \bigvee_{\bar{\imath} \in \{0,2\}^n} \alpha(i_1, \dots, i_n)(x_1 \vee x_1^*)^{i_1} \dots (x_n \vee x_n^*)^{i_n}, \tag{1}$$

$$B(x_1, \dots, x_n) = \bigvee_{i=1}^{l} \Big(\nabla_{\bar{\jmath} \in \{1,2\}^n} \beta_i(j_1, \dots, j_n) x_1^{j_1} \dots x_n^{j_n} \Big) \qquad (l \in \omega), \tag{2}$$

where x^0, x^1, x^2 denote S, x^*, x^{**}, respectively, xy is an abbreviation for $x \wedge y$, ∇ denotes the join in the Boolean algebra $B(\mathbf{L})$, $\alpha(i_1, \dots, i_n)$ are coefficients equal to 0 or 1, $\beta_i(j_1, \dots, j_n)$ are elements of $B(\mathbf{L})$, and the joins $\bigvee_{\bar{\imath} \in \{0,2\}^n}$ in $\mathbf{L}$ and $\nabla_{\bar{\jmath} \in \{1,2\}^n}$ in $B(\mathbf{L})$ are taken over all n-tuples $\bar{\imath} = (i_1, \dots, i_n) \in \{0,2\}^n$ and all n-tuples $\bar{\jmath} = (j_1, \dots, j_n) \in \{1,2\}^n$ respectively. As usual, ω is the set of all nonnegative integers. We shall further denote an n-tuple $(x_1, \dots, x_n)$ by $\bar{x}$.

LEMMA. *Any n-ary algebraic function $f(x_1, \dots, x_n)$ on a distributive p-algebra $\mathbf{L}$ can be represented in the form*

$$f(\bar{x}) = \bigwedge_{i=1}^{m} (A_i(\bar{x}) \vee B_i(\bar{x}) \vee c_i) \qquad (m \in \omega), \tag{*}$$

where $A_i(\bar{x})$ and $B_i(\bar{x})$ are algebraic functions of the form (1) *and* (2), *respectively, and $c_i \in L$.*

Proof. We show that the set $\mathbf{A}$ of all n-ary functions of the form (*) contains all n-ary constant functions, projections and is closed under the operations $\vee$, $\wedge$, *and* *. From (1) and (2) we see that

$$A(\bar{x}) = B(\bar{x}) = 0, \text{ if } \alpha(\bar{\imath}) = \beta_i(\bar{\jmath}) = 0 \quad \text{for} \quad i = 1, \dots, l \quad \text{and all} \quad \bar{\imath} \in \{0,2\}^n, \bar{\jmath} \in \{1,2\}^n.$$

(i) For every constant function $c_a(x_1, \dots, x_n) = a$, $a \in L$, it is sufficient to choose a function $f(\bar{x}) \in \mathbf{A}$ in which $m = 1$, $A_1(\bar{x}) = B_1(\bar{x}) = 0$ *and* $c_1 = a$. Then $f(\bar{x}) = a = c_a(\bar{x})$.

(ii) We show that any n-ary projection $p_k^n(x_1, \dots, x_n) = x_k$ belongs to $\mathbf{A}$. Again, from (1), (2) we have:

$$A(\bar{x}) = x_k \vee x_k^*, \quad \text{if} \quad \alpha(i_1, \dots, i_n) = \begin{cases} 1 & \text{for } i_k = 0, \quad i_j = 2, \quad j \neq k \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

$$B(\bar{x}) = x_k^{**}, \quad \text{if} \quad \beta_i(j_1, \dots, j_n) = \begin{cases} 1 & \text{for } i = 1 \quad \text{and} \quad j_k = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Indeed, in the latter case $B(\bar{x}) = x_k^{**} \wedge \nabla_{\bar{\jmath} \in \{1,2\}^{n-1}} x_1^{j_1} \dots x_{k-1}^{j_{k-1}} x_{k+1}^{j_{k+1}} \dots x_n^{j_n} = x_k^{**} \wedge 1$, using the well known fact that $\nabla_{\bar{\imath} \in \{1,2\}^n} x_1^{i_1} \dots x_n^{i_n} = 1$ (cf. [**1**, Lemma 18, p. 92]). Hence, we can choose the parameters of a function $f(\bar{x})$ in $\mathbf{A}$ such that

$$m = 2, A_1(\bar{x}) = x_k \vee x_k^*, A_2(\bar{x}) = 0 = B_1(\bar{x}), B_2(\bar{x}) = x_k^{**}$$

and put $c_1 = c_2 = 0$. Then we have

$$f(\bar{x}) = (x_k \vee x_k^*) \wedge x_k^{**} = x_k = p_k^n(\bar{x}).$$

(iii) For every function $f(\bar{x}) \in A$ the function $f^*(\bar{x}) = (f(\bar{x}))^* \in \mathbf{A}$. Indeed, using the identities

$$x^* = x^{***}, (x \wedge y)^* = x^* \nabla y^*, (x \vee y)^* = x^* \wedge y^*,$$

we can represent the function $f(\bar{x})^*$ as an algebraic function of the Boolean algebra $B(\mathbf{L})$, and by a well known result in the theory of Boolean algebras such functions can be represented in the form

$$\nabla_{\bar{j} \in \{1,2\}^n} \beta(j_1, \ldots, j_n) x_1^{j_1} \ldots x_n^{j_n},$$

hence also in the form (2) and thus in the form (*).

(iv) We show that $\mathbf{A}$ is closed under $\vee$. Let

$$f_1(\bar{x}) = \bigwedge_{i=1}^{k} (A_i(\bar{x}) \vee B_i(\bar{x}) \vee c_i),$$

$$f_2(\bar{x}) = \bigwedge_{j=1}^{l} (A_j'(\bar{x}) \vee B_j'(\bar{x}) \vee c_j')$$

be any n-ary functions belonging to $\mathbf{A}$. Then

$$f_1(\bar{x}) \vee f_2(\bar{x}) = \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{l} (A_i(\bar{x}) \vee B_i(\bar{x}) \vee c_i \vee A_j'(\bar{x}) \vee B_j'(\bar{x}) \vee c_j').$$

Obviously,

$$A_i(\bar{x}) \vee A_j'(\bar{x}) = \bigvee_{\bar{i} \in \{0,2\}^n} (\alpha_i(i_1, \ldots, i_n) \vee \alpha_j'(i_1, \ldots, i_n))(x_1 \vee x_1^*)^{i_1} \ldots (x_n \vee x_n^*)^{i_n},$$

which is an algebraic function of the form (1). Evidently, $B_i(\bar{x}) \vee B_j'(\bar{x})$ is of the form (2). Hence, $f_1(\bar{x}) \vee f_2(\bar{x})$ is an algebraic function of the form (*).

(v) Clearly, $\mathbf{A}$ is closed under the operation $\wedge$.

The proof is complete.

THEOREM. *Let* $\mathbf{L}$ *be a distributive p-algebra with a smallest dense element. If* $\mathbf{L}$ *is affine complete then* $D(\mathbf{L})$ *is an affine complete distributive lattice.*

Proof. Let $\mathbf{L}$ be affine complete and d be the smallest dense element of $\mathbf{L}$. Let $f_D : D(\mathbf{L})^n \to D(\mathbf{L})$ be a function preserving the (lattice) congruences of $D(\mathbf{L})$. We define a function $f : L^n \to L$ as follows:

$$f(x_1, \ldots, x_n) = f_D(x_1 \vee d, \ldots, x_n \vee d).$$

Obviously, $f \restriction D(\mathbf{L})^n = f_D$ and f preserves the congruences of $\mathbf{L}$. By hypothesis, f is an algebraic function, so it can be represented in the form (*) by the Lemma. We show that the function f_D is an algebraic function of the lattice $D(\mathbf{L})$.

(i) $A_i(\bar{x}) \restriction D(\mathbf{L})^n = \bigvee_{\bar{j} \in \{0,2\}^n} \alpha_i(j_1, \ldots, j_n) x_1^{j_1} \ldots x_n^{j_n}$, where $x_i^0 = x_i$, $x_i^2 = 1$ and $\alpha_i(j_1, \ldots, j_n) \in \{0, 1\}^n$. Thus, $A_i(\bar{x}) \restriction D(\mathbf{L})^n = a_i(\bar{x})$ is an algebraic function of the lattice $D(\mathbf{L})$.

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(ii) $B_i(\bar{x}) \restriction D(\mathbf{L})^n = \bigvee_{i=1}^{l} \beta_i(2, \ldots, 2)$, since $x_i^1 = 0$, $x_i^2 = 1$ for $x_i \in D(\mathbf{L})$. Obviously, $B_i(\bar{x}) \restriction D(\mathbf{L})^n$ is a constant function identically equal to b_i for some $b_i \in L$. Then for $\bar{x} \in D(\mathbf{L})^n$ we get

$$f_D(\bar{x}) = \bigwedge_{i=1}^{m} (a_i(\bar{x}) \vee b_i \vee c_i).$$

As $f_D(\bar{x}) \in D(\mathbf{L})$, we have $f_D(\bar{x}) = \left[\bigwedge_{i=1}^{m} (a_i(\bar{x}) \vee b_i \vee c_i)\right] \vee d = \bigwedge_{i=1}^{m} [a_i(\bar{x}) \vee (b_i \vee c_i \vee d)]$. Hence, f_D is an algebraic function of the lattice $D(\mathbf{L})$.

Corollary. *Let* **L** *be a distributive p-algebra with a finite number of dense elements. Then* **L** *is affine complete if and only if* **L** *is a Boolean algebra.*

Proof. If **L** is affine complete then $D(\mathbf{L})$ is an affine complete distributive lattice by the Theorem. Thus $|D(\mathbf{L})| = 1$ since $D(\mathbf{L})$ is finite. Hence, **L** is a Boolean algebra. The converse is obvious.

Example. Let **B** be a Boolean algebra. We adjoin a new unit $\underline{1}$. Then we obtain a distributive p-algebra **L** having exactly two dense elements 1 and $\underline{1}$. By the previous result, **L** is not affine complete.

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