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A round sphere theorem for positive sectional curvature

Changyu Xia

Abstract

Let M be an n-dimensional complete connected Riemannian manifold with sectional curvature $\sec(M) \geqslant 1$ and radius $\mathrm{rad}(M) > \pi/2$. In this article, we show that if $\mathrm{conj}(M)$, the conjugate radius of M, is not less than $\mathrm{rad}(M)$, then M is isometric to a round sphere of constant curvature.

1. Introduction

The structure of closed manifolds with positive sectional curvature has been an important topic in global Riemannian geometry. Let M be an n-dimensional complete connected Riemannian manifold with sectional curvature $\sec(M) \geqslant 1$. It follows from the Bonnet–Myers theorem [CE75] and Cheng's maximal diameter theorem [Che75] that if $\mathrm{diam}(M)$, the diameter of M, is bounded from below by π, then M is isometric to a unit Euclidean n-sphere. The famous classical sphere theorem tells us that M is homeomorphic to an n-sphere if in addition $\sec(M) < 4$ and M is simply connected [CE75]. In 1977, Grove and Shiohama established the critical point theory of distance functions on complete Riemannian manifolds to prove that M is homeomorphic to an n-sphere if $\mathrm{diam}(M)$ is larger than $\pi/2$ (see [GS77]). The case $\mathrm{diam}(M) = \pi/2$ (where the theorem is false, as shown by the example of real projective space) was essentially classified by Gromoll and Grove [GG87]. The above Grove–Shiohama theorem generalized the classical sphere theorem since the diameter of the manifold in the classical sphere theorem is larger than $\pi/2$ (see [CE75]). The critical point theory of distance functions has many important applications in Riemannian geometry (cf. [Che91, Gro93]). One can find other kinds of generalization of the classical sphere theorem, e.g., in [AW94, AW96, AW97, GP93, Mac93, MM88, Xia97]. It has been proven by Shiohama and Yamaguchi [SY89] that M is diffeomorphic to an n-sphere if the radius of M is close to π. Recall that for a compact metric space (X, d), the radius of X at a point $x \in X$ is defined as $\mathrm{rad}(x) = \max_{y\in X} d(x, y)$ and the radius of X is given by $\mathrm{rad}(X) = \min_{x\in X} \mathrm{rad}(x)$ (see [SY89]). The above Shiohama–Yamaguchi theorem has been strengthened by Colding to the following form: An n-dimensional complete connected Riemannian manifold with Ricci curvature larger than or equal to $n-1$ and radius close to π is diffeomorphic to S^n (see [Col96a, Col96b]). A classical result due to Toponogov states that if $n = 2$ and if M contains a closed geodesic without self-intersections of length 2π, then M is isometric to a two-dimensional unit sphere [Top59]. A partial extension of Toponogov's theorem to higher-dimensional Riemannian manifolds was given in [Xia02]. When the radius of M is larger than $\pi/2$, Grove and Petersen showed that the volume of M satisfies $C(n) \leqslant \mathrm{vol}(M) \leqslant \{\mathrm{rad}(M)/\pi\}\omega_n$, where ω_n is the volume of a unit Euclidean n-sphere and $C(n)$ is a positive constant depending only on n (see [GP92]). It has been known that when $\mathrm{rad}(M) > \pi/2$,

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any complete connected totally geodesic submanifold of M is homeomorphic to a sphere [Xia02]. Recently, a round sphere theorem for M has been proven by Wang [Wan04].

The purpose of the present article is to study the metric rigidity of complete manifolds with sectional curvature bounded below by 1 and radius larger than $\pi/2$. Before stating our main result, we fix some notation. Let M be a complete Riemannian manifold. For a point x in M, denote by S_xM the unit tangent sphere of M at x. For any $v \in S_xM$, let γ be a unit speed geodesic with $\gamma'(0) = v$. The conjugate value c_v of v is defined to be the first number $r > 0$ such that there is a nonzero Jacobi field J along γ satisfying $J(0) = J(r) = 0$. Set

$$\operatorname{conj}(x) := \inf_{v \in S_xM} c_v.$$

We call $\operatorname{conj}(x)$ the conjugate radius of M at x. The conjugate radius of M is defined as $\operatorname{conj}(M) = \inf_{p \in M} \operatorname{conj}(p)$.

Now we can state the main result in this paper as follows.

Theorem 1.1. *Let M be an n-dimensional complete connected Riemannian manifold M with $\sec(M) \geqslant 1$. If $\operatorname{conj}(M) \geqslant \operatorname{rad}(M) > \pi/2$, then M is isometric to a round sphere of constant curvature.*

We remark that Theorem 1.1 characterizes not only the unit sphere but also the family of spheres with sectional curvature bounded below by one and radius larger than $\pi/2$. It should be also mentioned that our condition '$\operatorname{rad}(M) > \pi/2$' in Theorem 1.1 is essential since the real projective space RP^n of sectional curvature 1 satisfies $\operatorname{conj}(RP^n) = \pi > \operatorname{rad}(RP^n) = \pi/2$.

2. A proof of Theorem 1.1

Let us first list some known facts that will be needed in the proof of Theorem 1.1. Let M be a complete connected Riemannian n-manifold satisfying $\sec(M) \geqslant 1$ and $\operatorname{rad}(M) > \pi/2$. It follows by using the Toponogov comparison theorem that for any $x \in M$, there exists a unique point $A(x)$ that is at the maximal distance from x. One can show that the mapping $A : M \to M$ is continuous (cf. [GP92, Xia02]). Observe that M is homeomorphic to S^n. Thus, we know from the Brouwer fixed point theorem that A is surjective.

Recall that a Wiedersehen manifold is a connected simply connected compact Riemannian n-manifold M without boundary such that for any $m \in M$ the cut locus of m is a single point [Gre63]. It is known that a Wiedersehen manifold is isometric to a round sphere of constant curvature [Gre63, Bes78, Wei74, Yan80, Yan82].

We shall assume throughout this paper that all geodesics have unit tangent vectors. Now we are ready to prove our main theorem.

Proof of Theorem 1.1. We shall show that our M is a Wiedersehen manifold. Since M is homeomorphic to S^n, it suffices to show that the cut-locus of any point in M is a single point. Let d and $\operatorname{inj}(M)$ be the distance function and the injectivity radius of M, respectively. For any $x \in M$, we denote by $\operatorname{inj}(x)$ and $\operatorname{cut}(x)$ the injectivity radius of M at x and the cut-locus of x, respectively. It is well known that $\operatorname{inj}(x) = d(x, \operatorname{cut}(x))$, that the function $\operatorname{inj} : M \to R^+$ is continuous and that $\operatorname{inj}(M) = \inf_{x \in M} \operatorname{inj}(x)$.

We *claim* that M contains a closed geodesic of length $2 \cdot \operatorname{inj}(M)$. To see this, take a point $p \in M$ such that $\operatorname{inj}(M) = \operatorname{inj}(p)$. Since $\operatorname{cut}(p)$ is a closed subset of M and so is compact, we can find a $q \in \operatorname{cut}(p)$ such that $l \equiv d(p, q) = d(p, \operatorname{cut}(p))$. By Proposition 2.12 in [Doc93, p. 274], we know that either:

(i) there exists a minimizing geodesic γ from p to q along which q is conjugate to p; or

(ii) there exist exactly two minimizing geodesics γ_1 and γ_2 from p to q satisfying $\gamma_1'(l) = -\gamma_2'(l)$.

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If statement (i) holds, then we have

$$\mathrm{inj}(M) = l \geqslant \mathrm{conj}(p) \geqslant \mathrm{conj}(M) \geqslant \mathrm{rad}(M).$$

On the other hand, it is easy to see that

$$\mathrm{rad}(M) \geqslant \mathrm{inj}(M).$$

Combining the above two inequalities, we get

$$\mathrm{inj}(M) = \mathrm{rad}(M).$$

Let us take a point $w \in M$ so that $\mathrm{rad}(w) = \mathrm{rad}(M)$, then we have

$$\mathrm{inj}(w) \geqslant \mathrm{inj}(M) = \mathrm{rad}(w) \geqslant \mathrm{inj}(w),$$

and so

$$l = \mathrm{inj}(w) = \mathrm{inj}(M).$$

It follows that any geodesic $\sigma : [0, l] \to M$ with $\sigma(0) = w$ is minimizing and satisfies $\sigma(l) = A(w)$ since $l = \mathrm{rad}(w)$ and $A(w)$ is the unique point which is at the maximal distance from w. We can see that w is at the maximal distance from Aw. In fact, suppose on the contrary that $z \equiv A(A(w)) \neq w$. Set $r = d(z, A(w)), t = d(w, z)$; then

$$r > d(A(w), w) = l > t.$$

Take a minimizing geodesic $c : [0, t] \to M$ from w to z and extend c to be a geodesic (still denoted by c) defined on $[0, l]$. From the above discussions, we know that $c(l) = A(w)$, which implies that $d(z, A(w)) = l - t < l$. This is a contradiction. Thus, we have $A(A(w)) = w$. Consequently, if $\beta : [0, 2l] \to M$ is a geodesic starting from w then it must satisfy $\beta(l) = A(w), \beta(2l) = w$. Set $z_1 = \beta(l/2)$ and $z_2 = \beta(3l/2)$; then $\beta|_{[l/2,3l/2]}$ is minimizing since $\mathrm{inj}(z_1) \geqslant l$. It then follows from

$$\mathrm{length}(\beta|_{[3l/2,2l]}) + \mathrm{length}(\beta|_{[0,l/2]}) = l = d(z_1, z_2)$$

that β is smooth at w. This shows that if statement (i) holds, then M contains a closed geodesic of length $2l = 2 \cdot \mathrm{inj}(M)$. On the other hand, if statement (ii) holds, then $\gamma_1 \cup \gamma_2$ is smooth at p since $\mathrm{inj}(M) = l$, which implies that $\gamma_1 \cup \gamma_2$ is a closed geodesic of length $2l = 2 \cdot \mathrm{inj}(M)$. Thus, our *claim* is true.

Let $\gamma : [0, 2l] \to M$ be a closed geodesic of length $2l = 2 \cdot \mathrm{inj}(M)$. Set $x = \gamma(0),\ y = \gamma(l)$. Let us prove that $l > \pi/2$. Assume on the contrary that $l \leqslant \pi/2$. We suppose that $x = A(z)$ is the unique point that is at the maximal distance from some $z \in M$. Then $z \neq y$ since $d(x, z) > \pi/2 \geqslant d(x, y)$. Set $l_1 = d(x, z)$ and $l_2 = d(y, z)$; then $l_1 > l_2$. Take a minimal geodesic β from y to z; then we have either

$$\angle(\gamma'(l), \beta'(0)) \leqslant \frac{\pi}{2},$$

or

$$\angle(-\gamma'(l), \beta'(0)) \leqslant \frac{\pi}{2}.$$

We assume without loss of generality that $\angle(-\gamma'(l), \beta'(0)) \leqslant \pi/2$. Applying the Toponogov comparison theorem to the hinge $(\gamma|_{[0,l]}, \beta)$, we get

$$\begin{aligned} 0 > \cos l_1 &\geqslant \cos l \cos l_2 + \sin l \sin l_2 \cos \angle(-\gamma'(l), \beta'(0)) \\ &\geqslant \cos l \cos l_2, \end{aligned} \tag{2.1}$$

which implies that $l \neq \pi/2$ and so we obtain from

$$\cos l_1 < \cos l_2$$

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and (2.1) that

$$\cos l_1(1-\cos l) > 0,$$

which contradicts the fact that $l_1 > \pi/2$. Thus, $l > \pi/2$.

We *claim* now that $y = A(x)$. We proceed by contradiction again. Thus suppose that $A(x) \neq y$. Set $r = d(x, A(x)),\ s = d(y, A(x))$ and take a minimal geodesic α from y to $A(x)$. Observe that either

$$\angle(\gamma'(l), \alpha'(0)) \leqslant \frac{\pi}{2},$$

or

$$\angle(-\gamma'(l), \alpha'(0)) \leqslant \frac{\pi}{2}.$$

One obtains, by using the Toponogov inequality to the hinge $(\gamma|_{[0,l]}, \alpha)$ or to the hinge $(\gamma|_{[l,2l]}, \alpha)$, that

$$0 > \cos r \geqslant \cos l \cos s. \tag{2.2}$$

Since $A(x)$ is at the maximal distance from x, it follows from Berger's lemma [CE75] that there exists a minimal geodesic δ from $A(x)$ to x satisfying

$$\angle(\delta'(0), -\alpha'(s)) \leqslant \frac{\pi}{2}.$$

Applying the Toponogov comparison theorem to the hinge (δ, α), we obtain

$$\begin{aligned}\cos l &\geqslant \cos r \cos s + \sin r \sin s \cos \angle(\delta'(0), -\alpha'(s)) \\ &\geqslant \cos r \cos s. \end{aligned} \tag{2.3}$$

It follows from $r > \pi/2,\ l > \pi/2$ and (2.2) that $\cos s > 0$, and so we have from (2.2) and (2.3) that

$$\cos r \geqslant \cos l \cos s \geqslant \cos^2 s \cos r,$$

that is

$$\cos r \sin^2 s \geqslant 0,$$

which is a contradiction. Hence, $A(x) = y$ and similarly, we have $A(y) = x$. Since $\operatorname{inj}(M) = l$ and there exists only one point of M that is at the maximal distance from x, we conclude that any geodesic $c : [0, 2l] \to M$ starting from x must satisfy $c(l) = y, c(2l) = x$ and be smooth at x. This clearly implies that $\operatorname{cut}(x) = \{Ax\}$. Similarly, we know that for any point $p \in \gamma$, $\operatorname{cut}(p) = \{Ap\}$. Now we fix a point $u \notin \gamma$ and let us prove that $\operatorname{cut}(u) = \{A(u)\}$. Set $t = d(x, u)$ and take a minimal geodesic h from x to u. From the above discussions, we know that h can be extended to a closed geodesic (still denoted by h) $h : [0, 2l] \to M$ satisfying $h(0) = h(2l) = x$ and $h(l) = y$. Since $\operatorname{inj}(M) = l = \frac{1}{2}l(h)$, we can use the same arguments as above to show that for any $q \in h, \operatorname{cut}(q) = \{A(q)\}$ and, in particular, $\operatorname{cut}(u) = \{A(u)\}$. Consequently, M is a Wiedersehen manifold and so is isometric to a round sphere of constant curvature. This completes the proof of Theorem 1.1. ☐

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Changyu Xia xia@mat.unb.br

Departamento de Matemática – IE, Universidade de Brasília, Campus Universitário, 70910-900 Brasília DF, Brazil

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