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WIENER INDEX ON TRACEABLE AND HAMILTONIAN GRAPHS

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Abstract

We give sufficient conditions for a graph to be traceable and Hamiltonian in terms of the Wiener index and the complement of the graph, which correct and extend the result of Yang ['Wiener index and traceable graphs', *Bull. Aust. Math. Soc.* **88** (2013), 380–383]. We also present sufficient conditions for a bipartite graph to be traceable and Hamiltonian in terms of its Wiener index and quasicomplement. Finally, we give sufficient conditions for a graph or a bipartite graph to be traceable and Hamiltonian in terms of its distance spectral radius.

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1. Introduction

All graphs considered here are finite undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $N_G(v)$ denote the neighbour set of v in G. We denote the degree of a vertex v_i by d_i or $d(v_i)$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of the graph G, where $d_1 \leq d_2 \leq \cdots \leq d_n$. Then $\delta := d_1$ is called the minimum degree. We denote the distance between the vertices v_i and v_j in G by $d_G(v_i, v_j)$. The union of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a disjoint union, denoted by $G + H$. The disjoint union of k graphs G is denoted by kG. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, we obtain the join of G and H, denoted by $G \vee H$. Finally, $\overline{G}$ denotes the complement of G.

A path in a graph is called a *Hamiltonian path* if it visits every vertex precisely once. A graph containing a Hamiltonian path is said to be *traceable*. A cycle in a

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graph is called a *Hamiltonian cycle* if it contains all the vertices of a graph. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*.

The distance matrix $D = D(G)$ of G has (i, j)-entry, d_{ij}, equal to $d_G(v_i, v_j)$. The Wiener index [9], $W(G)$, of a connected graph G is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v).$$

Let $D_i(G)$ and $D_v(G)$ denote the sum of row i of $D(G)$ and the row sum of $D(G)$ corresponding to vertex v, respectively. Then

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} D_v(G) = \frac{1}{2} \sum_{i=1}^{n} D_i(G).$$

The distance spectral radius of G is the largest eigenvalue of $D(G)$, denoted by $\rho(G)$.

The problem of deciding whether a given graph is traceable or Hamiltonian is a very difficult one. Indeed, determining whether a given graph is traceable or Hamiltonian is NP-complete [3]. Many necessary or sufficient conditions have been given for a graph to be traceable or Hamiltonian. Recently, some sufficient spectral conditions involving the Wiener index and distance spectral radius for a graph to be Hamiltonian and traceable have been given in [4–6, 10].

In Sections 2–3, we give sufficient conditions for a graph to be traceable and Hamiltonian in terms of the Wiener index and the complement of the graph, which correct and extend the result of Yang [10]. In Section 4, we present sufficient conditions for a bipartite graph to be traceable and Hamiltonian in terms of its Wiener index and quasicomplement. Finally, in Sections 5–6 we give sufficient distance spectral conditions for a graph or a bipartite graph to be traceable and Hamiltonian in terms of its distance spectral radius. Our results extend and improve the results in [4–6, 10].

Notice that $\delta \geq 1$ and $\delta \geq 2$ are trivial necessary conditions on the minimum degree for a graph to be traceable and Hamiltonian, respectively. Hence we take these as standing assumptions throughout this paper.

2. Corrigendum to [10, Theorem 2.2]

Define the set of exceptional graphs

$$\begin{aligned}\mathbb{NP} = \{&K_1 \vee (K_{n-3} + 2K_1), K_1 \vee (K_{1,3} + K_1), K_{2,4}, K_2 \vee 4K_1,\\ &K_2 \vee (3K_1 + K_2), K_1 \vee K_{2,5}, K_3 \vee 5K_1, K_2 \vee (K_{1,4} + K_1), K_4 \vee 6K_1\}.\end{aligned}$$

A sufficient condition for a graph to be traceable is given in [1].

Lemma 2.1 [1, Exercise 18.3.3]. *Let G be a nontrivial graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 4$. Suppose that there is no integer $k < \frac{1}{2}(n+1)$ such that $d_k \leq k-1$ and $d_{n-k+1} \leq n-k-1$. Then G is traceable.*

Yang [10] claimed that if G is a connected graph of order $n \geq 4$ and its Wiener index satisfies $W(G) \leq \frac{1}{2}(n+5)(n-2)$, then G is traceable unless

$$G \in \{K_1 \vee (K_{n-3} + 2K_1), K_2 \vee (3K_1 + K_2), K_4 \vee 6K_1\}.$$

However, the list of exceptional graphs is incomplete. The following theorem gives the correct result.

THEOREM 2.2. *Let G be a connected graph of order $n \geq 4$. If*

$$W(G) \leq \frac{(n+5)(n-2)}{2},$$

then G is traceable unless $G \in \mathbb{NP}$.

PROOF. Suppose that G is a nontraceable connected graph. By Lemma 2.1, there exists an integer $k < \frac{1}{2}(n+1)$ such that $d_k \leq k-1$ and $d_{n-k+1} \leq n-k-1$. Since G is connected and $d_k \leq k-1$, we have $k \geq 2$. Thus

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{i=1}^{n} D_i(G) \geq \frac{1}{2}\sum_{i=1}^{n}(d_i + 2(n-1-d_i)) = n(n-1) - \frac{1}{2}\sum_{i=1}^{n} d_i \\ &\geq n(n-1) - \frac{1}{2}(k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)) \\ &= \frac{(n+5)(n-2)}{2} + \frac{(k-2)(2n-3k-5)}{2}. \end{aligned}$$

Since $W(G) \leq \frac{1}{2}(n+5)(n-2)$, we have $\frac{1}{2}(k-2)(2n-3k-5) \leq 0$. Also, if m denotes the number of edges of G, we have $W(G) \geq n(n-1) - \frac{1}{2}\sum_{i=1}^{n} d_i = n(n-1) - m$ which implies $m \geq \frac{1}{2}(n^2 - 5n + 10)$.

Case 1. $\frac{1}{2}(k-2)(2n-3k-5) = 0$, that is, $k = 2$ or $2n = 3k+5$, and all inequalities in the above argument must be equalities. If $k = 2$, then G is a graph with $d_1 = d_2 = 1$, $d_3 = d_4 = \cdots = d_{n-1} = n-3$ and $d_n = n-1$, whence $G = K_1 \vee (K_{n-3} + 2K_1)$. If $2n = 3k+5$, then $n < 13$ since $k < \frac{1}{2}(n+1)$. Hence $n = 7$, $k = 3$ or $n = 10$, $k = 5$. The corresponding permissible graphic sequences are $(2,2,2,3,3,6,6)$ and $(4,4,4,4,4,4,9,9,9,9)$, which imply $G = K_2 \vee (3K_1 + K_2)$ and $G = K_4 \vee 6K_1$, respectively.

Case 2. $\frac{1}{2}(k-2)(2n-3k-5) < 0$, that is $k \geq 3$ and $2n-3k-5 < 0$. In this case, the admissible pairs k, n satisfy $k \geq 3, n \geq 4, n \geq 2k$ and $2n-3k \leq 4$, allowing just two possibilities: $k = 3, n = 6$ and $k = 4, n = 8$.

Suppose $k = 4$. Then $d_5 \leq 3$ and $17 \leq m \leq 18$. From the inequality $d_6 + d_7 + d_8 = 2m - \sum_{1 \leq i \leq 5} d_i \geq 19$, we obtain $d_8 = 7$. Also note that $\sum d_i = 2m \geq 34$ and $\sum d_i$ is even. If $d_6 = d_7 = 6$ and $d_8 = 7$, then the permissible graphic sequence is $(3,3,3,3,3,6,6,7)$ and hence $G = K_1 \vee K_{2,5}$. If $d_6 = 5$ and $d_7 = d_8 = 7$, then the permissible graphic sequence is $(3,3,3,3,3,5,7,7)$ and hence $G = K_2 \vee (K_{1,3} + K_2)$. If $d_6 = 6$ and $d_7 = d_8 = 7$, then the permissible graphic sequence is $(2,3,3,3,3,6,7,7)$ and hence

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$G = K_2 \vee (K_{1,4} + K_1)$. If $d_6 = d_7 = d_8 = 7$, then the permissible graphic sequence is $(3, 3, 3, 3, 3, 7, 7, 7)$ and hence $G = K_3 \vee 5K_1$.

Finally, suppose $k = 3$. Then $d_4 \leq 2$ and $8 \leq m \leq 9$. From the inequality $d_5 + d_6 = 2m - \sum_{1 \leq i \leq 4} d_i \geq 8$, we obtain $4 \leq d_6 \leq 5$. Also note that $\sum d_i = 2m \geq 16$ and $\sum d_i$ is even. If $d_5 = d_6 = 4$, then the permissible graphic sequence is $(2, 2, 2, 2, 4, 4)$ and hence $G = K_{2,4}$. If $d_5 = 4$ and $d_6 = 5$, then the permissible graphic sequence is $(1, 2, 2, 2, 4, 5)$ and hence $G = K_1 \vee (K_{1,3} + K_1)$. If $d_5 = d_6 = 5$, then the permissible graphic sequence is $(2, 2, 2, 2, 5, 5)$ and hence $G = K_2 \vee 4K_1$.

Note that $K_2 \vee (K_{1,3} + K_2)$ is traceable and the other obtained graphs contain no Hamiltonian path. The proof is complete. □

3. Wiener index on traceable and Hamiltonian graphs

The following lemma allows us to give a simpler proof of Theorem 2.2.

Lemma 3.1 [8]. *Let G be a graph on $n \geq 4$ vertices and m edges with $\delta \geq 1$. If $m \geq \binom{n-2}{2} + 2$, then G is traceable unless $G \in \mathbb{NP}$.*

Second proof of Theorem 2.2. Suppose that G is nontraceable. As noted at the beginning of the proof in Section 2,

$$m \geq \frac{1}{2}(n^2 - 5n + 10) = \binom{n-2}{2} + 2.$$

By Lemma 3.1, we obtain that $G \in \mathbb{NP}$. By a direct computation, for all $G \in \mathbb{NP}$, $W(G) \leq \frac{1}{2}(n + 5)(n - 2)$. This completes the proof of Theorem 2.2. □

Theorem 3.2. *Let G be a connected graph of order $n \geq 4$. If*

$$W(\overline{G}) \geq \frac{n^3 - 6n^2 + 19n - 20}{2},$$

then G is traceable unless $G \in \mathbb{NP}$.

Proof. Suppose that G is nontraceable. Then

$$\begin{aligned} W(\overline{G}) &= \frac{1}{2}\sum_{i=1}^{n} D_i(\overline{G}) \leq \frac{1}{2}\sum_{v \in V(G)} [d_{\overline{G}}(v) + (n-1)(n-1-d_{\overline{G}}(v))] \\ &= \frac{1}{2}\sum_{v \in V(G)} [(n-1)^2 + (2-n)d_{\overline{G}}(v)] \\ &= \frac{1}{2}n(n-1)^2 - \frac{n-2}{2}\sum_{v \in V(G)} (n-1-d_G(v)) \\ &= \frac{n(n-1)}{2} + (n-2)m. \end{aligned}$$

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Since $W(\overline{G}) \geq \frac{1}{2}(n^3 - 6n^2 + 19n - 20)$,

$$m \geq \frac{n^3 - 6n^2 + 19n - 20 - n(n-1)}{2(n-2)} = \binom{n-2}{2} + 2.$$

Again, by Lemma 3.1, $G \in \mathbb{NP}$. By a direct computation, in all cases with $G \in \mathbb{NP}$, $W(\overline{G}) \geq \frac{1}{2}(n^3 - 6n^2 + 19n - 20)$. Hence G is traceable unless $G \in \mathbb{NP}$. □

Define the set of exceptional graphs

$$\mathbb{NC} = \{K_2 \vee (K_{n-4} + 2K_1), K_3 \vee 4K_1, K_2 \vee (K_{1,3} + K_1), K_1 \vee K_{2,4}, \\ K_3 \vee (K_2 + 3K_1), K_4 \vee 5K_1, K_3 \vee (K_{1,4} + K_1), K_2 \vee K_{2,5}, K_5 \vee 6K_1\}.$$

Lemma 3.3 [8]. *Let G be a graph on $n \geq 5$ vertices and m edges with $\delta \geq 2$. If $m \geq \binom{n-2}{2} + 4$, then G contains a Hamiltonian cycle unless $G \in \mathbb{NC}$.*

Theorem 3.4. *Let G be a graph with $n \geq 5$ vertices and m edges and with $\delta \geq 2$. If*

$$W(G) \leq \frac{n^2 + 3n - 14}{2},$$

then G is Hamiltonian unless $G \in \mathbb{NC}$.

Proof. Suppose that G is non-Hamiltonian. As in the second proof of Theorem 2.2,

$$W(G) = \frac{1}{2}\sum_{i=1}^{n} D_i(G) \geq n(n-1) - m.$$

Since $W(G) \leq \frac{1}{2}(n^2 + 3n - 14)$,

$$m \geq n(n-1) - \frac{n^2 + 3n - 14}{2} = \binom{n-2}{2} + 4.$$

By Lemma 3.3, we have $G \in \mathbb{NC}$. By a direct computation, $W(G) \leq \frac{1}{2}(n^2 + 3n - 14)$ for all $G \in \mathbb{NC}$. Hence G is Hamiltonian unless $G \in \mathbb{NC}$. □

Theorem 3.5. *Let G be a graph with $n \geq 5$ vertices and m edges and with $\delta \geq 2$. If*

$$W(\overline{G}) \geq \frac{n^3 - 6n^2 + 23n - 28}{2},$$

then G is Hamiltonian unless $G \in \mathbb{NC}$.

Proof. Suppose that G is non-Hamiltonian. Then

$$W(\overline{G}) = \frac{1}{2}\sum_{i=1}^{n} D_i(\overline{G}) \leq \frac{n(n-1)}{2} + (n-2)m.$$

Since $W(\overline{G}) \geq \frac{1}{2}(n^3 - 6n^2 + 23n - 28)$,

$$m \geq \frac{n^3 - 6n^2 + 23n - 28 - n(n-1)}{2(n-2)} = \binom{n-2}{2} + 4.$$

By Lemma 3.3, $G \in \mathbb{NC}$. By a direct verification, $W(\overline{G}) \geq \frac{1}{2}(n^3 - 6n^2 + 23n - 28)$ for all $G \in \mathbb{NC}$. Hence G is Hamiltonian unless $G \in \mathbb{NC}$. □

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4. Wiener index on traceable and Hamiltonian bipartite graphs

Let $G = G[X, Y]$ be a bipartite graph where $|X| = |Y| = n \geq 2$. The bipartite graph $G^* = G^*[X, Y]$, called the *quasicomplement* of G, is constructed as follows: $V(G^*) = V(G)$ and $xy \in E(G^*)$ if and only if $xy \notin E(G)$ for $x \in X$, $y \in Y$.

Let $G[X, Y]$ be a traceable bipartite graph. Then $|X| = |Y|$ or $|X| = |Y| + 1$. These two types will be discussed separately.

LEMMA 4.1 [7]. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \geq 1$ and m edges where $|X| = |Y| = n \geq 3$. If $m \geq n^2 - 2n + 3$, then G is traceable.*

THEOREM 4.2. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \geq 1$ and m edges where $|X| = |Y| = n \geq 3$. If*

$$W(G) \leq 3n^2 + 2n - 6,$$

then G is traceable.

PROOF. Let G be a graph satisfying the condition in Theorem 4.2. Then

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{i=1}^{2n} D_i(G) \geq \frac{1}{2}\sum_{i=1}^{2n}(d_i + 3(n - d_i) + 2(n-1)) \\ &= 3n^2 - \sum_{i=1}^{2n} d_i + 2n(n-1) = 3n^2 - 2m + 2n(n-1) = 5n^2 - 2n - 2m. \end{aligned}$$

Since $W(G) \leq 3n^2 + 2n - 6$, we have $m \geq \frac{1}{2}(5n^2 - 2n - (3n^2 + 2n - 6)) = n^2 - 2n + 3$ and, according to Lemma 4.1, G is traceable. □

THEOREM 4.3. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \geq 1$ and m edges, where $|X| = |Y| = n \geq 3$. If*

$$W(G^*) \geq 4n^3 - 9n^2 + 12n - 6,$$

then G is traceable.

PROOF. Let G^* be the quasicomplement of G. Then

$$\begin{aligned} W(G^*) &= \frac{1}{2}\sum_{i=1}^{2n} D_i(G^*) \leq \frac{1}{2}\sum_{v \in V(G)} [d_{G^*}(v) + (2n-1)(n - d_{G^*}(v)) + (2n-2)(n-1)] \\ &= \frac{1}{2}\sum_{v \in V(G)} [n(2n-1) + (2-2n)d_{G^*}(v)] + n(2n-2)(n-1) \\ &= n^2(2n-1) - (n-1)\sum_{v \in V(G)} (n - d_G(v)) + n(2n-2)(n-1) \\ &= n^2 + n(2n-2)(n-1) + 2(n-1)m. \end{aligned}$$

Since $W(G^*) \geq 4n^3 - 9n^2 + 12n - 6$, we have

$$m \geq \frac{2n^3 - 6n^2 + 10n - 6}{2(n-1)} = n^2 - 2n + 3.$$

By Lemma 4.1, G is traceable. □

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Let $p \geq n-1$. Let $K_{p,n-2}+4e$ be a bipartite graph obtained from $K_{p,n-2}$ by adding two vertices which are adjacent to two common vertices with degree $n-2$ in $K_{p,n-2}$.

LEMMA 4.4 [7]. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \geq 2$ and m edges where $|X| = |Y| = n \geq 4$. If $m \geq n^2 - 2n + 4$, then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.*

Next we consider the other type with $|X| = |Y| + 1$. Let $G[X, Y]$ be a bipartite graph where $|X| = n+1$ and $|Y| = n \geq 2$. Denote by δ_X and δ_Y the minimum degrees of vertices in X and Y, respectively. Note that $\delta_X \geq 1$ and $\delta_Y \geq 2$ are the trivial necessary conditions for G to be traceable. Let $G[X, Y+v]$ be the bipartite graph obtained from $G[X, Y]$ by adding a vertex v which is adjacent to every vertex in X. It is easy to see that $G[X, Y]$ is traceable if and only if $G[X, Y+v]$ is Hamiltonian.

Let $K_{n,n-1}+2e$ be a graph obtained from $K_{n,n-1}$ by adding two vertices which are adjacent to a common vertex with degree $n-1$.

THEOREM 4.5. *Let $G = G[X, Y]$ be a bipartite graph with $\delta_X \geq 1$ and $\delta_Y \geq 2$ where $|X| = n+1$ and $|Y| = n \geq 3$. If*

$$W(G) \leq 3n^2 + 5n - 4,$$

then G is traceable unless $G \in \{K_{n+1,n-2}+4e, K_{n,n-1}+2e\}$.

PROOF. Let G be a bipartite graph satisfying the conditions in Theorem 4.5.

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} D_v(G) \\ &\geq \frac{1}{2}\Big[\sum_{i=1}^{n+1}(d_i + 3(n-d_i) + 2n) + \sum_{j=1}^{n}(d_j + 3(n+1-d_j) + 2(n-1))\Big] \\ &= \frac{1}{2}\Big[5n(n+1) - 2\sum_{i=1}^{n+1} d_i + n(5n+1) - 2\sum_{j=1}^{n} d_j\Big] \\ &= 5n^2 + 3n - \sum_{v\in V(G)} d_G(v) = 5n^2 + 3n - 2m. \end{aligned}$$

From $W(G) \leq 3n^2 + 5n - 4$ we have $m \geq n^2 - n + 2$. Since $d(v) = n+1$ in $G[X, Y+v]$,

$$m(G[X, Y+v]) = m + (n+1) \geq n^2 + 3 = (n+1)^2 - 2(n+1) + 4.$$

By Lemma 4.4, $G[X, Y+v]$ is Hamiltonian or $G[X, Y+v] = K_{n+1,n-1}+4e$. Hence $G[X, Y]$ is traceable or $G \in \{K_{n+1,n-2}+4e, K_{n,n-1}+2e\}$. □

THEOREM 4.6. *Let $G = G[X, Y]$ be a bipartite graph with $\delta_X \geq 1$ and $\delta_Y \geq 2$ where $|X| = n+1$ and $|Y| = n \geq 3$. If*

$$W(G^*) \geq 4n^3 - 4n^2 + 8n - 4,$$

then G is traceable unless $G \in \{K_{n+1,n-2}+4e, K_{n,n-1}+2e\}$.

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PROOF. Let G^* be the quasicomplement of G. Then

$$\begin{aligned} W(G^*) &= \frac{1}{2}\sum_{v\in V(G)} D_v(G^*) \\ &\le \frac{1}{2}\Big[\sum_{i=1}^{n+1}(d_{G^*}(v_i) + (2n-1)(n - d_{G^*}(v_i)) + 2n^2) \\ &\quad + \sum_{j=1}^{n}(d_{G^*}(u_j) + (2n-1)(n+1-d_{G^*}(u_j)) + (2n-2)(n-1))\Big] \\ &= 4n^3 - (n-1)\sum_{i=1}^{n+1} d_{G^*}(v_i) - (n-1)\sum_{j=1}^{n} d_{G^*}(u_j) \\ &= 2n^3 + 2n + 2(n-1)m \end{aligned}$$

by substituting $d_{G^*}(v_i) = n - d_G(v_i)$ and $d_{G^*}(u_j) = n + 1 - d_G(u_j)$. By hypothesis, $W(G^*) \ge 4n^3 - 4n^2 + 8n - 4$ so $m \ge n^2 - n + 2$. Since $d(v) = n + 1$ in $G[X, Y + v]$,

$$m(G[X, Y + v]) = m + (n+1) \ge n^2 + 3 = (n+1)^2 - 2(n+1) + 4.$$

By Lemma 4.4, $G[X, Y + v]$ is Hamiltonian or $G[X, Y + v] = K_{n+1,n-1} + 4e$. Hence $G[X, Y]$ is traceable or $G \in \{K_{n+1,n-2} + 4e, K_{n,n-1} + 2e\}$. □

THEOREM 4.7. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \ge 2$ and m edges where $|X| = |Y| = n \ge 4$. If*

$$W(G) \le 3n^2 + 2n - 8,$$

then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

PROOF. Suppose that G is non-Hamiltonian. As in Theorem 4.2,

$$W(G) = \frac{1}{2}\sum_{i=1}^{2n} D_i(G) \ge 5n^2 - 2n - 2m.$$

Since $W(G) \le 3n^2 + 2n - 8$, we have $m \ge \frac{1}{2}(5n^2 - 2n - (3n^2 + 2n - 8)) = n^2 - 2n + 4$ and, from Lemma 4.4, $G = K_{n,n-2} + 4e$. □

THEOREM 4.8. *Let $G = G[X, Y]$ be a bipartite graph with $\delta \ge 2$ and m edges where $|X| = |Y| = n \ge 4$. If*

$$W(G^*) \ge 4n^3 - 9n^2 + 14n - 8,$$

then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

PROOF. Suppose that G is non-Hamiltonian. As in Theorem 4.3,

$$W(G^*) = \frac{1}{2}\sum_{v\in V(G)} D_v(G^*) \le n^2 + n(2n-2)(n-1) + 2(n-1)m.$$

Since $W(G^*) \geq 4n^3 - 9n^2 + 14n - 8$,

$$m \geq \frac{n^3 - 3n^2 + 6n - 4}{n-1} = n^2 - 2n + 4.$$

By Lemma 4.4, $G = K_{n,n-2} + 4e$. □

5. Distance spectral radius on traceable and Hamiltonian graphs

Lemma 5.1 [2]. *Let G be a graph on n vertices. Then*

$$\rho(G) \geq \frac{2W(G)}{n},$$

and the equality holds if and only if G is distance regular, that is, the row sums of $D(G)$ are all equal.

Theorem 5.2. *Let G be a connected graph of order $n \geq 4$. If*

$$\rho(G) \leq n + 3 - \frac{10}{n},$$

then G is traceable unless $G = K_2 \vee 4K_1$.

Proof. Assume that G is nontraceable. By Lemma 5.1,

$$\rho(G) \geq \frac{2W(G)}{n} \geq \frac{2}{n}[n(n-1) - m] = 2(n-1) - \frac{2}{n}m.$$

Since $\rho(G) \leq n + 3 - 10/n$, we have $m \geq \binom{n-2}{2} + 2$ and, by Lemma 3.1, $G \in \mathbb{NP}$.

The largest zero root of the equation $\lambda^3 - (n-2)\lambda^2 - (7n-17)\lambda + 10 - 4n = 0$ is $\rho(K_1 \vee (K_{n-3} + 2K_1))$. Since

$$f(n) = \left(n + 3 - \frac{10}{n}\right)^3 - (n-2)\left(n + 3 - \frac{10}{n}\right)^2 - (7n-17)\left(n + 3 - \frac{10}{n}\right) + 10 - 4n$$

is a decreasing function of n ($n \geq 5$), $f(n) \leq f(5) < 0$. Hence $\rho(K_1 \vee (K_{n-3} + 2K_1)) > n + 3 - 10/n$ for $n \geq 5$. From Table 1, we see that $G = K_2 \vee 4K_1$. □

Theorem 5.3. *Let G be a graph on $n \geq 5$ vertices and m edges with $\delta \geq 2$. If*

$$\rho(G) \leq n + 3 - \frac{14}{n},$$

then G is Hamiltonian unless $G = K_3 \vee 4K_1$.

Proof. Suppose that G is non-Hamiltonian. By Lemma 5.1, we have

$$\rho(G) \geq \frac{2W(G)}{n} \geq \frac{2}{n}[n(n-1) - m] = 2(n-1) - \frac{2}{n}m.$$

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TABLE 1. Direct computation of $\rho(G)$ for Theorem 5.2.

G	$\rho(G)$	$n+3-10/n$
$K_1 \vee (K_{1,3}+K_1)$	7.5673	7.3333
$K_{2,4}$	7.4641	7.3333
$K_2 \vee 4K_1$	7.2749	7.3333
$K_2 \vee (3K_1+K_2)$	8.8886	8.5714
$K_1 \vee K_{2,5}$	10.0401	9.75
$K_3 \vee 5K_1$	9.8990	9.75
$K_2 \vee (K_{1,4}+K_1)$	10.1205	9.75
$K_4 \vee 6K_1$	12.5208	12
$K_{1,3}$	4.6458	4.5

TABLE 2. Direct computation of $\rho(G)$ for Theorem 5.3.

G	$\rho(G)$	$n+3-14/n$
$K_3 \vee 4K_1$	8	8
$K_2 \vee (K_{1,3}+K_1)$	8.2736	8
$K_1 \vee K_{2,4}$	8.1846	8
$K_3 \vee (K_2+3K_1)$	9.5947	9.25
$K_4 \vee 5K_1$	10.6235	10.444
$K_3 \vee (K_{1,4}+K_1)$	10.8341	10.444
$K_2 \vee K_{2,5}$	10.7624	10.444
$K_5 \vee 6K_1$	13.2450	12.727

Since $\rho(G) \leq n+3-14/n$, we have $m \geq \binom{n-2}{2}+4$. By Lemma 3.3, $G \in \mathbb{NC}$. The largest zero root of $\lambda^3-(n-2)\lambda^2-(7n-23)\lambda-2n+6=0$ is $\rho(K_2 \vee (K_{n-4}+2K_1))$, and

$$f(n)=\Big(n+3-\frac{14}{n}\Big)^3-(n-2)\Big(n+3-\frac{14}{n}\Big)^2-(7n-23)\Big(n+3-\frac{14}{n}\Big)-2n+6$$

is a decreasing function on n, so $f(n) \leq f(5) < 0$ and we have $\rho(K_2 \vee (K_{n-4}+2K_1)) > n+3-14/n$. From Table 2, we see that $G = K_3 \vee 4K_1$. □

6. Distance spectral radius on traceable and Hamiltonian bipartite graphs

THEOREM 6.1. *Let $G = G[X,Y]$ be a bipartite graph with $\delta \geq 1$ and m edges where $|X|=|Y|=n\geq 3$. If*

$$\rho(G) \leq 3n+2-\frac{6}{n},$$

then G is traceable.

PROOF. According to Lemma 5.1,

$$\rho(G) \geq \frac{2W(G)}{2n} \geq \frac{5n^2-2n-2m}{n} = 5n-2-\frac{2m}{n}.$$

Since $\rho(G) \leq 3n+2-6/n$, we have $m \geq n^2-2n+3$ and, by Lemma 4.1, G is traceable. □

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