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DOUBLING TROPICAL q-DIFFERENCE ANALOGUE OF THE LEMMA ON THE LOGARITHMIC DERIVATIVE

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Abstract

We present a tropical q-difference analogue of the lemma on the logarithmic derivative for doubling tropical meromorphic functions.

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1. Introduction

Following Halburd and Southall [2], tropical Nevanlinna theory is concerned with functions of a real variable which are continuous piecewise linear and meromorphic and have one-sided derivatives at each point. In the tropical framework, we start with a tropical semi-ring which endows the set $\mathbb{R} \cup \{-\infty\}$ with addition

$$x \oplus y := \max(x, y)$$

and multiplication

$$x \otimes y := x + y.$$

We also use the notation $x \oslash y := x - y$ and $x^{\otimes\alpha} := \alpha x$ for $\alpha \in \mathbb{R}$. A continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is tropical meromorphic on $\mathbb{R}$ if both one-sided derivatives are integers at each point $x \in \mathbb{R}$. Laine and Tohge [6] broadened the definition of tropical meromorphic functions by removing the condition that both one-sided derivatives of f are integers.

For each $x \in \mathbb{R}$, let

$$\omega_f(x) = \lim_{\epsilon\to 0^+}\{f'(x+\epsilon) - f'(x-\epsilon)\}.$$

If $\omega_f(x) > 0$, then x is called a zero of f with multiplicity $\omega_f(x)$. If $\omega_f(x) < 0$, then x is called a pole of f with multiplicity $-\omega_f(x)$. Observe that the multiplicity may be a nonintegral real number; it is denoted by $\tau_f(x)$ in the following. It is clear that if $f(x)$ has no zeros and no poles in $[r_1, r_2]$, then $f(x) = ax + b$ in $[r_1, r_2]$, with $a, b \in \mathbb{R}$.

The tropical version of the proximity function for a tropical meromorphic function f is defined by

$$m(r, f) := \tfrac{1}{2}\{f^+(r) + f^+(-r)\},$$

where $f^+(x) := \max\{f(x), 0\}$ (see [2]). It is well known that

$$m(r, f \oplus g) \le m(r, f) + m(r, g)$$

and

$$m(r, f \otimes g) \le m(r, f) + m(r, g).$$

Denote by $n(r, f)$ the number of distinct poles of f in the interval $(-r, r)$, each pole multiplied by its multiplicity $\tau_f(x)$. We define the tropical counting function by

$$N(r, f) := \frac{1}{2}\int_0^r n(t, f)\, dt = \frac{1}{2}\sum_{|b_\nu|<r} \tau_f(b_\nu)(r - |b_\nu|)$$

and, as usual, the tropical characteristic function $T(r, f)$ is defined by

$$T(r, f) := m(r, f) + N(r, f).$$

The characteristic function $T(r, f)$ is a positive, continuous, nondecreasing piecewise linear function of r (see [4, Theorem 3.4]).

The tropical Poisson–Jensen formula implies a weak analogue of the tropical version of Nevanlinna's first main theorem:

$$T(r, f) - T(r, -f) = f(0). \tag{1.1}$$

Tsai [8] and Laine *et al.* [5] further developed the tropical Nevanlinna theory.

We also use an analogue of the logarithmic measure (see [3]). Given any set E on a part of the positive r-axis, where $r > 1$, we define its logarithmic measure $\operatorname{lm} E$ by

$$\operatorname{lm} E := \int_E \frac{dr}{r}.$$

Writing $E(r)$ for the part of E in the interval $[1, r]$, we define the upper logarithmic density $\overline{\log\operatorname{dens}}\, E$ and the lower logarithmic density $\underline{\log\operatorname{dens}}\, E$ by

$$\overline{\log\operatorname{dens}}\, E = \limsup_{r\to\infty} \frac{\operatorname{lm} E(r)}{\log r}, \quad \underline{\log\operatorname{dens}}\, E := \liminf_{r\to\infty} \frac{\operatorname{lm} E(r)}{\log r}.$$

If $\overline{\log\operatorname{dens}}\, E = \underline{\log\operatorname{dens}}\, E = \varepsilon$, say, for a set E, we say that E has logarithmic density ε.

The doubling property [7] plays an important role in what follows.

Definition 1.1 (Doubling property). Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be a piecewise continuous increasing function and $c_1 > 1$. Then T has the doubling property if there is a constant $c_2 > 1$ such that

$$T(c_1 r) \le c_2 T(r)$$

for all r outside an exceptional set of finite logarithmic measure.

Definition 1.2. A tropical meromorphic function is called a doubling tropical meromorphic function if its tropical characteristic function $T(r, f)$ has the doubling property.

Example 1.3. The function $f(x) = x$ has no poles and its tropical characteristic function is $\frac{1}{2}r$, so it has the doubling property. From its graph, the function $f(x) = -|x|$ has a pole of multiplicity 2 at $x = 0$ and $T(r, f) = r$, so it also has the doubling property.

Example 1.4. The following example from [4, page 5] exhibits a doubling tropical meromorphic function. The tropical rational function

$$f(x) := \max\{3 - x, -1 + \tfrac{3}{2}x, 4 - \tfrac{1}{5}x\} - \max\{-2 + \tfrac{1}{2}x, 4x, 1 - \tfrac{1}{3}x\}$$

has a pole of multiplicity $\frac{13}{3}$ at $x = \frac{3}{13}$ and, for all sufficiently large r,

$$m(r, f) = 1 + \tfrac{2}{3}r, \quad N(r, f) = \tfrac{13}{6}r, \quad T(r, f) = 1 + \tfrac{17}{6}r,$$

so f has the doubling property.

2. Doubling tropical q-difference analogue of the lemma on the logarithmic derivative

The tropical Nevanlinnna theory originates from the tropical Poisson–Jensen formula. This formula implies that the values of a tropical meromorphic function f at any point x in the interval $(-r, r)$ can be expressed in terms of the zeros and poles of f and the average value of f on the boundary of the interval.

Halburd and Southall [2, Lemma 3.1] derived the tropical Poisson–Jensen formula for the case in which the multiplicities of all the zeros and poles are positive integers. Laine and Tohge [6, Theorem 2.1] gave a general form of the tropical Poisson–Jensen formula, which can be stated as follows:

$$f(x) = \frac{1}{2}\{f(r) + f(-r)\} + \frac{x}{2r}\{f(r) - f(-r)\} - \frac{1}{2r}\sum_{|a_\mu|<r} \tau_f(a_\mu)\{r^2 - |a_\mu - x|r - a_\mu x\}$$
$$+ \frac{1}{2r}\sum_{|b_\nu|<r} \tau_f(b_\nu)\{r^2 - |b_\nu - x|r - b_\nu x\}, \tag{2.1}$$

where the a_μ are the zeros and the b_ν are the poles of f in the interval $[-r, r]$ and τ_f is a positive real number which denotes the multiplicity of a zero or pole.

The lemma on the logarithmic derivative plays an important role in Nevanlinna theory; in particular, it is used in proving Nevanlinna's second main theorem. A tropical difference analogue of the lemma on the logarithmic derivative is given in [2, Lemma 3.8]. Here, we prove a doubling tropical q-difference analogue of the lemma on the logarithmic derivative in the general context of [6]. We need the following lemmas.

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Lemma 2.1. *Let f be a tropical meromorphic function and $q \in \mathbb{R}\backslash\{0\}$. Then*

$$m(r, f(x^{\otimes q}) \oslash f(x)) \leq |q-1|r\{n(\rho, f) + n(\rho, -f)\} + \frac{|q-1|r}{\rho}\{2T(\rho, f) - f(0)\},$$

where $\rho > \max\{r, |q|r\}$.

Proof. Take any $\rho > \max\{r, |q|r\}$ and $x \in [-r, r]$. Denote the zeros of f by a_μ and the poles of f by b_ν, with their corresponding multiplicities τ_f. Substitute qx for x in (2.1), giving

$$\begin{aligned} f(qx) = \frac{1}{2}\{f(\rho) + f(-\rho)\} + \frac{qx}{2\rho}\{f(\rho) - f(-\rho)\} \\ - \frac{1}{2\rho}\sum_{|a_\mu|<\rho} \tau_f(a_\mu)\{\rho^2 - |a_\mu - qx|\rho - a_\mu \cdot qx\} \\ + \frac{1}{2\rho}\sum_{|b_\nu|<\rho} \tau_f(b_\nu)\{\rho^2 - |b_\nu - qx|\rho - b_\nu \cdot qx\}. \end{aligned}$$

This implies that

$$\begin{aligned} f(qx) - f(x) &= \frac{(q-1)x}{2\rho}\{f(\rho) - f(-\rho)\} \\ &\quad + \frac{1}{2\rho}\sum_{|a_\mu|<\rho} \tau_f(a_\mu)\{(|a_\mu - qx| - |a_\mu - x|)\rho + (q-1)a_\mu x\} \\ &\quad - \frac{1}{2\rho}\sum_{|b_\nu|<\rho} \tau_f(b_\nu)\{(|b_\nu - qx| - |b_\nu - x|)\rho + (q-1)b_\nu x\} \\ &=: S_1(x) + S_2(x) - S_3(x), \text{ say.} \end{aligned}$$

Therefore,

$$m(r, f(qx) - f(x)) \leq m(r, S_1(x)) + m(r, S_2(x)) + m(r, -S_3(x)).$$

We proceed to estimate each of the three terms $m(r, S_i(x))$ separately. By using $f^+(x) \leq |f(x)|$ and $|f(x)| = f^+(x) + (-f)^+(x)$,

$$\begin{aligned} m(r, S_1(x)) &\leq \frac{1}{2\rho}|q-1|r\{|f(\rho)| + |f(-\rho)|\} \\ &= \frac{1}{2\rho}|q-1|r\{f^+(\rho) + (-f)^+(\rho) + f^+(-\rho) + (-f)^+(-\rho)\} \\ &= \frac{|q-1|r}{\rho}\{m(\rho, f) + m(\rho, -f)\}. \end{aligned}$$

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For the second term, using the triangle inequality in the form $||z_1| - |z_2|| \le |z_1 - z_2|$,

$$
\begin{aligned}
m(r, S_2(x)) &\le \sum_{|a_\mu|<\rho} \tau_f(a_\mu) \cdot m\Big(r, \frac{1}{2\rho}\{(|a_\mu - qx| - |a_\mu - x|)\rho + (q-1)a_\mu x\}\Big) \\
&\le \sum_{|a_\mu|<\rho} \tau_f(a_\mu) \cdot \frac{1}{2}\Big(\frac{1}{2\rho}\{|(|a_\mu - qr| - |a_\mu - r|)|\rho + |q-1| \cdot |a_\mu| r\} \\
&\qquad + \frac{1}{2\rho}\{|(|a_\mu + qr| - |a_\mu + r|)|\rho + |q-1| \cdot |a_\mu| r\}\Big) \\
&\le \sum_{|a_\mu|<\rho} \tau_f(a_\mu) \cdot \frac{1}{2}\Big(|q-1|r + \frac{1}{\rho}|q-1| \cdot |a_\mu| r\Big) \\
&\le \sum_{|a_\mu|<\rho} \tau_f(a_\mu) \cdot |q-1|r = |q-1|r \cdot n(\rho, -f).
\end{aligned}
$$

Similarly,

$$m(r, -S_3(x)) \le |q-1|r \cdot n(\rho, f).$$

By combining the above estimates and the tropical Nevanlinna first main theorem,

$$
\begin{aligned}
m(r, f(qx) - f(x)) &\le \frac{|q-1|r}{\rho}\{m(\rho, f) + m(\rho, -f)\} + |q-1|r\{n(\rho, f) + n(\rho, -f)\} \\
&\le \frac{|q-1|r}{\rho}\{2T(\rho, f) - f(0)\} + |q-1|r\{n(\rho, f) + n(\rho, -f)\}. \qquad \square
\end{aligned}
$$

LEMMA 2.2 [1, Lemma 5.4]. *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function and let $U : \mathbb{R}^+ \to \mathbb{R}^+$. If there exists a decreasing sequence $\{c_n\}_{n\in\mathbb{N}}$ such that $c_n \to 0$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, the set*

$$F_n := \{r \ge 1 : U(r) < c_n T(r)\}$$

has logarithmic density 1, then

$$U(r) = o(T(r))$$

on a set of logarithmic density 1.

We can now state and prove our doubling tropical q-difference analogue of the lemma on the logarithmic derivative.

THEOREM 2.3. *Let f be a doubling tropical meromorphic function and $q \in \mathbb{R}\backslash\{0\}$. Then*

$$m(r, f(x^{\otimes q}) \oslash f(x)) = o(T(r, f))$$

on a set of logarithmic density 1.

PROOF. Assume that $K \ge K_0 > \max\{|q|, 1\}$. Applying Lemma 2.1 with $\rho = Kr$,

$$m(r, f(qx) - f(x)) \le |q-1|r\{n(Kr, f) + n(Kr, -f)\} + \frac{|q-1|}{K}\{2T(Kr, f) - f(0)\}. \tag{2.2}$$

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From the definition of the tropical counting function, for any $k > 1$,

$$N(kr, f) = \frac{1}{2}\int_0^{kr} n(t, f)\, dt \geq \frac{1}{2}\int_r^{kr} n(t, f)\, dt \geq \frac{1}{2}(k-1)rn(r, f),$$

which implies that

$$n(r, f) \leq \frac{2N(kr, f)}{(k-1)r}.$$

Replacing r by Kr and taking $k = K + 1$,

$$n(Kr, f) \leq \frac{2N(K^2r + Kr, f)}{K^2r}. \tag{2.3}$$

Then, from (2.2) and (2.3),

$$\begin{aligned} m(r, f(qx) - f(x)) \leq{}& \frac{2|q-1|}{K^2}\{N(K^2r + Kr, f) + N(K^2r + Kr, -f)\} \\ &+ \frac{|q-1|}{K}\{2T(Kr, f) - f(0)\}. \end{aligned}$$

Now we use $N(r, f) \leq T(r, f)$, which is clear from the definitions, and the tropical version of Nevanlinna's first main theorem (1.1) to get

$$m(r, f(qx) - f(x)) \leq \frac{2|q-1|}{K^2}\{2T(K^2r + Kr, f) - f(0)\} + \frac{|q-1|}{K}\{2T(Kr, f) - f(0)\}.$$

By hypothesis, $T(r, f)$ satisfies the doubling property, so there is a constant $c_2 > 1$ such that

$$T(Kr, f) \leq c_2 \cdot T(r, f) \quad \text{and} \quad T(K^2r + Kr, f) \leq c_2 \cdot T(r, f)$$

for all r outside an exceptional set of finite logarithmic measure. From this, we deduce that

$$m(r, f(qx) - f(x)) \leq \frac{2|q-1|}{K^2}\{2c_2 \cdot T(r, f) - f(0)\} + \frac{|q-1|}{K}\{2c_2 \cdot T(r, f) - f(0)\}.$$

In particular, taking $K = n + 1, n \in \mathbb{N}^+$,

$$m(r, f(qx) - f(x)) \leq \left\{\frac{4|q-1|c_2}{(n+1)^2} + \frac{2|q-1|c_2}{n+1} + o\left(\frac{1}{n+1}\right)\right\}T(r, f)$$

for all r outside an exceptional set of finite logarithmic measure. Applying Lemma 2.2 with

$$U(r) = m(r, f(qx) - f(x)),$$

we can conclude that

$$m(r, f(qx) - f(x)) = o(T(r, f))$$

on a set of logarithmic density 1. This completes the proof of the theorem. □

Corollary 2.4. *Let f be a doubling tropical meromorphic function and $q_1, q_2 \in \mathbb{R}\backslash\{0\}$. Then*

$$m(r, f(x^{\otimes q_1}) \oslash f(x^{\otimes q_2})) = o(T(r, f))$$

on a set of logarithmic density 1.

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