A REMARK ON THE DIVISOR FUNCTION $d(n)$

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Let $d(n)$ denote the number of positive divisors of n. A long time ago, Erdös and Mirsky [**1**] raised the question whether the equation $d(n) = d(n+1)$ holds for infinitely many n. It does not seem easy to settle this problem, and in the present note we give a partial result.

PROPOSITION. *At least one of the following two statements is valid.* (i) *For infinitely many primes p, $8p+1$ is the product of at most two distinct primes.* (ii) *For infinitely many n, $d(n) = d(n+1)$.*

Proof. Let P_3 be the set of natural numbers that are products of at most three, not necessarily distinct, primes. Denote by $\rho(n)$ the least prime divisor of n. From the work of Richert [**2**, Theorem 7], it is known that there exist positive numbers δ_1, δ_2 such that, for all sufficiently large N,

$$\sum \left\{ \begin{matrix} 8p+1 \leqq N, \\ 8p+1 \in P_3, \\ \rho(8p+1) \geqq N^{\delta_2} \end{matrix} \right\} 1 \geqq \frac{\delta_1 N}{\log^2 N}. \tag{1}$$

(Actually, the condition $\rho(8p+1) \geqq N^{\delta_2}$ is not stated in Richert's Theorem 7, but follows immediately from his appeal to Theorem 2.)

The sum on the left-hand side of (1) is equal to $\Sigma_1 + \Sigma_2$, where $\mu(8p+1) \neq 0$ in Σ_1 and $\mu(8p+1) = 0$ in Σ_2. We have

$$\Sigma_2 \leqq \sum \left\{ \begin{matrix} 8p+1 \leqq N, \\ 8p+1 \equiv 0 (\operatorname{mod} p'^2), \\ p' \geqq N^{\delta_2} \end{matrix} \right\} 1 \leqq \sum \left\{ \begin{matrix} n \leqq N, \\ n \equiv 0 (\operatorname{mod} p'^2), \\ p' \geqq N^{\delta_2} \end{matrix} \right\} 1 = O(N^{1-\delta_2})$$

and therefore, by (1),

$$\sum \left\{ \begin{matrix} 8p+1 \leqq N, \\ 8p+1 \in P_3, \\ \mu(8p+1) \neq 0 \end{matrix} \right\} 1 \geqq \frac{\delta_1 N}{\log^2 N} - \Sigma_2 \geqq \frac{\frac{1}{2}\delta_1 N}{\log^2 N}.$$

Hence, at least one of the following statements is valid. (i) For infinitely many p, $8p+1$ is the product of at most two distinct primes. (ii) For infinitely many p, $8p+1$ is the product of three distinct primes, and in that case $d(8p+1) = 8 = d(8p)$.

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