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A Note on the Vanishing of Certain Local Cohomology Modules

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Abstract. For a finite module M over a local, equicharacteristic ring (R, m), we show that the well-known formula $\operatorname{cd}(m, M) = \dim M$ becomes trivial if ones uses Matlis duals of local cohomology modules together with spectral sequences. We also prove a new ring-theoretic vanishing criterion for local cohomology modules.

1 Introduction

Let R be a noetherian ring, I an ideal of R, and M an R-module; one denotes the n-th local cohomology module of M with respect to I by $H^n_I(M)$ and the cohomological dimension of I on M by

$$\operatorname{cd}(I, M) := \sup\{l \mid H^l_I(M) \neq 0\}.$$

From now on assume that (R, m) is local and M is finitely generated. Grothendieck's Vanishing Theorem (VT) says that $\operatorname{cd}(I, M) \leq \dim M$ and Grothendieck's Non-Vanishing Theorem (NVT) says $H^{\dim M}_m(M) \neq 0$. Both are well-known theorems with various proofs. (See [1, Theorem 6.1.2], [2, Theorem 2.7] (a version for sheaves) for VT and [1, Theorem 6.1.4, Theorem 7.3.2] for NVT.) The case $I = m$ of VT and NVT *together* say that the cohomological dimension is the Krull dimension:

$$(*) \qquad \operatorname{cd}(m, M) = \dim M.$$

The first aim of this paper is to show that,using Matlis duals of local cohomology modules, formula $(*)$ become almost trivial once one knows the following facts.

(A) Local cohomology can be written as the direct limit of Koszul cohomologies. It is an easy exercise to check that the following are immediate consequences of this:

- (A_1) the base-change formula ${}_RH^i_{IS}(N) = H^i_I({}_RN)$ (S/R a noetherian algebra, N an S-module, I an ideal of R and $i \in \mathbb{N}$);
- (A_2) the formula

$$H^j_{(X_1,\dots,X_i)}(k[[X_1,\dots,X_i]]) = \begin{cases} 0 & \text{if } j > i \\ E_{k[[X_1,\dots,X_i]]}(k) = k[X_1^{-1},\dots,X_i^{-1}] & \text{if } j = i \end{cases}$$

(k a field, $X_1, \dots, X_i$ indeterminates);

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(A$_3$) the fact that each local cohomology functor of the form $H^j_{(x_1,\dots,x_i)R}$ is zero for $j > i$; in particular, $H^i_{(x_1,\dots,x_i)}R$ is right exact.

(B) Some Matlis duality theory and some spectral sequence theory. Both serve as technical tools.

Our method works only in the equicharacteristic case.

The second aim is to prove Theorem 3.1, which is a new (sufficient) criterion for the vanishing of local cohomology modules, which is of a ring-theoretic nature; the idea which is used in its proof is, to the best of our knowledge, completely new in this context.

2 The (Non-)Vanishing Theorem

Everything in this paper is based on the following easy lemma.

Lemma 2.1 *Let (R, m) be a noetherian local complete ring containing a field k, let M be an R-module, and let $x_1, \dots, x_i \in R$. Then*

$$H^i_{\underline{x}R}(M) \neq 0 \Leftrightarrow \dim(R_0) = i \text{ and } \operatorname{Hom}_{R_0}(M, R_0) \neq 0,$$

where $R_0 := k[[x_1, \dots, x_i]]$ as a subring of R and $\underline{x} := x_1, \dots, x_i$.

Proof ($\Rightarrow$): Assume $\dim(R_0) < i$. Write $R_0 = k[[X_1, \dots, X_i]]/I$, where $X_1, \dots, X_i$ are indeterminates and I is a non-zero ideal of $k[[X_1, \dots, X_i]] =: S$. Then

$$H^i_{\underline{x}R_0}(R_0) \overset{(A_1),(A_3)}{=} H^i_{\underline{X}S}(S) \otimes_S (S/I) = 0,$$

as every $0 \neq f \in I$ operates injectively on S and hence (B) surjectively on $H^i_{\underline{X}S}(S)(\overset{(A_2)}{\cong} E_S(k))$. In particular,

$$H^i_{\underline{x}R}(M) \overset{(A_3)}{=} M \otimes_{R_0} H^i_{\underline{x}R_0}(R_0) = 0,$$

which is a contradiction. Therefore, $\dim(R_0) = i$, $R_0 \cong k[[X_1, \dots, X_i]]$ with indeterminates $X_1, \dots, X_i$, and one has

$$\begin{aligned} 0 &\overset{(B)}{\neq} \operatorname{Hom}_{R_0}(H^i_{\underline{x}R}(M), E_{R_0}(k)) \\ &\overset{(A_3)}{=} \operatorname{Hom}_{R_0}(M \otimes_{R_0} H^i_{\underline{x}R_0}(R_0), E_{R_0}(k)) \\ &= \operatorname{Hom}_{R_0}(M, \operatorname{Hom}_{R_0}(H^i_{\underline{x}R_0}(R_0), E_{R_0}(k))) \\ &\overset{(A_2),(B)}{=} \operatorname{Hom}_{R_0}(M, R_0). \end{aligned}$$

($\Leftarrow$): Again, $R_0 \cong k[[X_1, \dots, X_i]]$ with indeterminates $X_1, \dots, X_i$; now

$$0 \neq \operatorname{Hom}_{R_0}(M, R_0) = \operatorname{Hom}_{R_0}(H^i_{\underline{x}R}(M), E_{R_0}(k))$$

follows as above. ∎

Theorem 2.2 (i) *If R is a noetherian ring containing a field, $\underline{x} = x_1, \dots, x_i \in R$, and M is an R-module (not necessarily finitely generated) such that $\dim_R(M) < i$, then $H^i_{\underline{x}R}(M) = 0$.*

(ii) *If (R, m) is a noetherian local ring containing a field and $\underline{x} = x_1, \dots, x_i$ is part of a system of parameters of a finitely generated R-module M, then $H^i_{\underline{x}R}(M) \neq 0$; in particular, $H^{\dim_R(M)}_m(M) \neq 0$.*

(iii) *If (R, m) is a noetherian local ring containing a field and M is a finitely generated R-module, then $\operatorname{cd}(m, M) = \dim_R(M)$.*

Proof (i) By localizing and completing, we may assume that R is local and complete. Set $R_0 := k[[x_1, \dots, x_i]]$ as a subring of R as in Lemma 2.1; we may assume that $\dim(R_0) = i$, *i.e.*, $R_0 \cong k[[X_1, \dots, X_i]]$, where $X_1, \dots, X_i$ are indeterminates. Due to dimension reasons, it is clear that $\operatorname{Hom}_{R_0}(M, R_0) = 0$ and the claim follows from Lemma 2.1.

(ii) We may assume that R is complete ($\hat{R}/R$ is faithfully flat); by base-change, we may replace R by $R/\operatorname{Ann}_R(M)$. Set $d := \dim(R)$; we choose $x_{i+1}, \dots, x_d \in R$ such that $x_1, \dots, x_d$ is a system of parameters of M. Then $R_0 := k[[x_1, \dots, x_d]] \subseteq R$ is a regular d-dimensional subring of R and, because M is module-finite over R_0, $\operatorname{Hom}_{R_0}(M, R_0) \neq 0$. Lemma 2.1 implies $H^d_{(x_1,\dots,x_d)R}(M) \neq 0$. Now a formal spectral sequence argument (namely for the spectral sequence of composed functors $E_2^{p,q} = H^p_{(x_{i+1},\dots,x_d)R}(H^q_{(x_1,\dots,x_i)R}(M)) \Rightarrow H^{p+q}_{(x_1,\dots,x_d)R}(M)$; note that $H^p_{(x_{i+1},\dots,x_d)R} = 0$ for each $p > d - i$ and that $H^q_{(x_1,\dots,x_i)R} = 0$ for each $q > i$,by (A_3)) shows that

$$0 \neq H^d_{(x_1,\dots,x_d)R}(M) = H^{d-i}_{(x_{i+1},\dots,x_d)R}(H^i_{(x_1,\dots,x_i)R}(M)).$$

(iii) follows from (i) and (ii). ∎

3 A Ring-Theoretic Vanishing Criterion

Theorem 3.1 *Let (R, m) be a noetherian local complete domain containing a field and $\underline{x} = x_1, \dots, x_i$, a sequence in R. Then the implication*

$$H^i_{\underline{x}R}(R) \neq 0 \;\Rightarrow\; \dim(R_0) = i \text{ and } R \cap Q(R_0) = R_0$$

holds, where $R_0 := k[[x_1, \dots, x_i]] \subseteq R$, $Q(R_0)$ denotes the quotient field of R_0, and the intersection is taken inside $Q(R)$.

Proof By Lemma 2.1, $R_0 \cong k[[X_1, \dots, X_i]]$, $X_1, \dots, X_i$ indeterminates, $\dim(R_0) = i$.

Let $r \in R, r_0 \in R_0$ such that $r_0 \cdot r \in R_0$. We have to show that $r \in R_0$: by Lemma 2.1, $\operatorname{Hom}_{R_0}(R, R_0) \neq 0$ and so we can choose $\varphi \in \operatorname{Hom}_{R_0}(R, R_0)$ such that $\varphi(1_R) \neq 0$ (namely, by composing a $\varphi' \in \operatorname{Hom}_{R_0}(R, R_0)$ that has $\varphi(r') \neq 0$ (for some $r' \in R$) with the multiplication map $R \xrightarrow{r'} R$). Set $r_0' := r_0 r$. One has

$$r_0 \varphi(r) = \varphi(r_0') = r_0' \varphi(1_R)$$

and then

$$\varphi(1_R) r = \varphi(1_R) \frac{r_0'}{r_0} = \varphi(r) \in R_0.$$

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On the other hand, we have $r_0'^2 = r_0^2 r^2$ and thus

$$r_0^2 \varphi(r^2) = r_0'^2 \varphi(1_R) \quad \text{and} \quad \varphi(1_R) r^2 = \varphi(1_R) \frac{r_0'^2}{r_0^2} = \varphi(r^2) \in R_0.$$

Continuing in the same way, one sees that for every $l \geq 1$, one has $\varphi(1_R) r^l \in R_0$. But this implies that the R_0-module $\varphi(1_R) \cdot \langle 1, r, r^2, \dots \rangle_{R_0}$ is finitely generated ($\langle 1, r, r^2, \dots \rangle_{R_0}$ stands for the R_0-submodule of R generated by $1, r, r^2, \dots$). But, as R is a domain, $\langle 1, r, r^2, \dots \rangle_{R_0}$ is then finitely generated, too, *i.e.*, r is necessarily contained in R_0. ∎

Remarks 3.2 (i) $H^i_{\underline{x}R}(R) \neq 0$ (and thus $R \cap Q(R_0) = R_0$) are clear if $\underline{x}$ is an R-regular sequence; but the condition $\underline{x}$ being a regular sequence is not necessary as the following example shows: $H^2_{(y_1 y_2, y_1 y_3)}(k[[y_1, y_2, y_3]])$ is non-zero (and thus $R \cap Q(R_0) = R_0$) though $y_1 y_2, y_1 y_3$ is not a regular sequence (k a field, y_1, y_2, y_3 indeterminates).

(ii) In the situation of Theorem 3.1, without the assumption $H^i_{\underline{x}R}(R) \neq 0$ the condition $R \cap Q(R_0) = R_0$ does not hold in general, *e.g.*, for

$$R_0 = k[[y_1 y_2, y_1 y_2^2]] \subseteq k[[y_1, y_2]] = R$$

(k a field, y_1, y_2 indeterminates) one has $y_2 \in (R \cap Q(R_0)) \setminus R_0$.

Remark 3.3 If R is regular, the implication from Theorem 3.1 is an equivalence for $i = 1$; while this is easy to see, the case $i = 2$ seems already unclear.

Question 3.4 Under what conditions can the implication from Theorem 3.1 be reversed?

References

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