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ON THE SPECTRA OF UNBOUNDED SUBNORMAL OPERATORS

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1. Introduction. Putnam showed in **[5]** that the spectrum of the real part of a bounded subnormal operator on a Hilbert space is precisely the projection of the spectrum of the operator onto the real line. (In fact he proved this more generally for bounded hyponormal operators.) We will show that this result can be extended to the class of unbounded subnormal operators with bounded real parts.

Before proceeding we establish some notation. If T is a (not necessarily bounded) operator on a Hilbert space, then $D(T)$ will denote its domain, and $\sigma(T)$ its spectrum. For K a subspace of $D(T)$, $T|K$ will denote the restriction of T to K. Norms of bounded operators and elements in Hilbert spaces will be indicated by $\| \ \|$. All Hilbert space inner products will be written $\langle \ , \ \rangle$. If W is a set in $\mathbf{C}$, the closure of W will be written clos W, the topological boundary will be written bdy W, and the projection of W onto the real line will be written $\pi(W)$,

$$\pi(W) = \{\lambda \in \mathbf{R}; \lambda + i\mu \in W, \text{ for some real } \mu\}.$$

Definition 1.1. If T is an operator on a Hilbert space, we will say that T has a *Cartesian decomposition* if $T = A + iB$, where A and B are self-adjoint. In this case we set Re $T = A$ and Im $T = B$.

Normal (not necessarily bounded) operators of course always have Cartesian decompositions. An arbitrary unbounded operator need not have one.

Definition 1.2. A closed, densely defined operator S on a Hilbert space H is a *subnormal operator* if there exists a Hilbert space K containing H and a normal operator T on K such that

$$D(S) = D(T) \cap H \quad \text{and} \quad T|D(T) \cap H = S.$$

If S is bounded and has an unbounded normal extension then S also has a bounded normal extension (see for example Lemma 3.2 below), so the above definition agrees with the usual definition of bounded subnormal operators. We are now ready to make the statements in the first paragraph precise.

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THEOREM 1.3 (Putnam) *If* S *is a bounded subnormal operator then*

$$\sigma(\text{Re } S) = \pi(\sigma(S)).$$

The above result is Theorem I of [**5**]. The main purpose of this paper is to establish the next theorem.

THEOREM 1.4. *If* S *is a subnormal operator with* Re S *bounded then*

$$\sigma(\text{Re } S) = \text{clos}(\pi(\sigma(S))).$$

Both of the above results are well-known for normal operators. Each has an almost trivial proof using the appropriate functional calculus. Note that Theorem 1.4 reduces to Theorem 1.3 if S is bounded. In general, however, we cannot expect $\pi(\sigma(S))$ to be closed, even if S is itself normal.

Let us look at one situation where the consideration of bounded self-adjoint operators naturally leads to the consideration of unbounded subnormal operators. For ϕ in L^∞ of the unit circle in **C**, the Toeplitz operator T_ϕ with symbol ϕ, acting on H^2 of the circle, is defined by

$$T_\phi f = P\phi f,$$

where P is the projection of L^2 onto H^2. We can write this as

$$T_\phi = PM_\phi|H^2$$

where M_ϕ is the usual multiplication operator on L^2. The operator T_ϕ is bounded since ϕ is bounded. (See [**1**], [**2**], or [**3**] for the basic properties of bounded Toeplitz operators.) If in addition ϕ is in H^∞, then the analytic Toeplitz operator T_ϕ is subnormal, being the restriction of the normal operator M_ϕ to the now invariant subspace H^2 [**1**, p. 272].

Suppose now that ϕ is an arbitrary function in L^2. We can still define the Toeplitz operator T_ϕ, by setting

$$T_\phi = PM_\phi|D(T_\phi)$$

where

$$D(T_\phi) = \{f \in H^2 : \phi f \text{ is in } L^2\}.$$

The operator T_ϕ need not be bounded. It will be densely defined since $D(T_\phi)$ contains the analytic polynomials. In fact $D(T_\phi)$ will contain all of H^∞. If ϕ is in H^2 the resulting operator will be subnormal in the sense of Definition 1.2. (Note that T_ϕ is a closed operator since M_ϕ is closed, and $T_\phi = M_\phi|H^2$.)

Again suppose that ϕ is in L^∞. We know that $T_\phi^* = T_{\bar{\phi}}$. Thus the class of bounded self-adjoint Toeplitz operators consists precisely of operators of the form T_u, where u is an arbitrary real valued function in L^∞.

THEOREM 1.5. *Every bounded self-adjoint Toeplitz operator is the real part of a (not necessarily bounded) analytic Toeplitz operator.*

Proof. Let u be as above, and let v be the harmonic conjugate of u. In general v is in BMO but not necessarily in L^∞. The function $f = u + iv$ is in H^2, and will be in H^∞ if and only if v is in L^∞. As we have already noted, T_f is a subnormal operator. The usual algebraic properties of Toeplitz operators imply that $T_f = T_u + iT_v$. Since u is in L^∞, T_u is self-adjoint. The theorem will be established once we show that T_v is self-adjoint. Since v is real, T_v is symmetric. To show that T_v is self-adjoint, we need to show that g in $D(T_v^*)$ implies g is in $D(T_v)$. In other words, if g is in $D(T_v^*)$, we must show that vg is in L^2. Since v is in BMO we know that vg is in L^1. Let us consider the Fourier coefficients of vg. Let

$$\langle \phi, \psi \rangle = \int \phi\bar{\psi}$$

be the usual inner-product on L^2, the integral taken over the unit circle. For $n \geqq 0$ we have

$$\begin{aligned}\int vg\bar{z}^n &= \langle g, vz^n \rangle \\ &= \langle g, P(vz^n) \rangle \\ &= \langle g, T_v z^n \rangle \\ &= \langle T_v^* g, z^n \rangle,\end{aligned}$$

the last equation following from the fact that g is in $D(T_v^*)$. For $n > 0$ we also have

$$\begin{aligned}\int vgz^n &= \langle v, \bar{g}\bar{z}^n \rangle \\ &= \langle -if + iu, \bar{g}\bar{z}^n \rangle \\ &= i\langle ug, \bar{z}^n \rangle,\end{aligned}$$

since $\bar{g}\bar{z}^n$ is perpendicular to H^2. We conclude that for $n \geqq 0$ the n^{th} Fourier coefficients of vg and T_v^*g coincide, and for $n < 0$ the n^{th} Fourier coefficients of vg and iug coincide. Since T_v^*g and ug are both in L^2, we conclude that the Fourier coefficients of vg are square-summable. Thus vg is in L^2.

Since every analytic Toeplitz operator is subnormal, we see that Theorem 1.4 applies to all self-adjoint Toeplitz operators on H^2. Using the easily proved fact that the spectrum of T_f, f in H^2, is the closure of the range of the Poisson extension of f to the open disk,

$$\sigma(T_f) = \text{clos}(f(D)),$$

we easily obtain the well-known result ([**2**, p. 183])

$$\sigma(T_u) = [\inf u, \sup u].$$

This same analysis can be applied to Toeplitz operators in other contexts, see for example [**4**, p. 605].

2. Proof of theorem 1.3. As noted above, Putnam proved 1.3 for bounded hyponormal operators. By restricting the theorem to bounded subnormal operators a much more elementary proof can be constructed. Throughout this section all operators will be assumed bounded.

Let H be a Hilbert space and let S be a subnormal operator on H. Thus there exists a Hilbert space K containing H and a normal operator $T = U + iV$, U and V self-adjoint, on K such that H is an invariant subspace for T, and $S = T|H$, Let P denote the orthogonal projection of K onto H, and suppose $S = A + iB$, A and B self-adjoint. We then have

$$\text{Re } S = PUP|H = PU|H.$$

Since S is a subnormal operator on H we know that $\|S^*x\| \leqq \|Sx\|$, for x in H, and hence

$$\|(\text{Re } S)x\| \leqq \|Sx\|, \quad x \text{ in } H. \tag{1}$$

The last two inequalities are true for any hyponormal operator [**3**, p. 160]. There is, however, an inequality similar to (1) which only makes sense for subnormal operators. It is the key to simplifying Putnam's proof.

LEMMA 2.1. *Let S and T be as above. Then*

$$\|(\text{Re } T)x\|^2 \leqq \|(\text{Re } S)x\| \, \|S\| \, \|x\|, \quad x \textit{ in } H.$$

Proof. For x in H we have

$$\begin{aligned}\langle (\text{Re } S)x, Sx\rangle &= \langle PUx, Sx\rangle \\ &= \langle Ux, Sx\rangle \\ &= \langle Ux, Ux\rangle - i\langle Ux, Vx\rangle.\end{aligned}$$

Since N is normal, $\langle Ux, Vx\rangle$ is real. Thus

$$\begin{aligned}\langle Ux, Ux\rangle &\leqq |\langle (\text{Re } S)x, Sx\rangle| \\ &\leqq \|(\text{Re } S)x\| \, \|Sx\|,\end{aligned}$$

and the lemma follows.

We now begin the proof of 1.3 by showing that $\sigma(\text{Re } S)$ is contained in $\pi(\sigma(S))$. For hyponormal operators this requires hard analysis using the functional calculus [**5**, pp. 514-516].

LEMMA 2.2. *If λ is in $\sigma(\text{Re } S)$ then $\lambda + \mu i$ is in $\sigma(T)$ for some real μ.*

Proof. Without loss of generality we may assume $\lambda = 0$. Thus Re S is non-invertible. Since Re S is self-adjoint, it is not bounded below. It follows from 2.1 that Re T is not bounded below, so 0 is in $\sigma(\text{Re } T)$. Since Theorem 1.3 is true for normal operators, there exists a real μ such that μi is in $\sigma(T)$.

LEMMA 2.3. *If* λ *is in* $\sigma(\text{Re } S)$ *then* $\lambda + \mu i$ *is in* $\sigma(S)$ *for some real* μ.

Proof. Let T' be the minimal normal extension of S. Lemma 2.2 shows that $\lambda + \mu i$ is in $\sigma(T')$ for some μ. Since $\sigma(T') \subset \sigma(S)$ [**1**, p. 131], the result follows.

The last lemma shows that $\sigma(\text{Re } S)$ is contained in $\pi(\sigma(S))$.

Now we show that $\pi(\sigma(S))$ is contained in $\sigma(\text{Re } S)$. This is essentially the same proof as Putnam's, with some minor cosmetic changes. Suppose λ is in $\pi(\sigma(S))$. Since S is bounded, $\sigma(S)$ is a bounded set in **C**. Therefore

$$(2) \qquad \pi(\sigma(S)) = \pi(\text{bdy}(\sigma(S))),$$

and hence there exists a real number μ such that $\lambda + i\mu$ belongs to bdy$(\sigma(S))$. Since every point on the boundary of the spectrum of a bounded operator is in the approximate point spectrum of the operator [**1**, p. 37], $S - (\lambda + i\mu)$ is not bounded below. Thus λ is in $\sigma(\text{Re } S)$, and the theorem is established.

The above proof falls apart when S is no longer bounded, even if it has bounded real part. For example consider a conformal map ϕ of the unit disk onto an infinite vertical strip $\{z: -1 < \text{Re } z < 1\}$. Let $S = T_\phi$ and $T = M_\phi$ be as in Section 1. Then T is a normal extension of S and it is not too difficult to check that

$$\sigma(S) = \{z: -1 \leqq \text{Re } z \leqq 1\}, \quad \sigma(\text{Re } S) = [-1, 1], \quad \text{and}$$

$$\sigma(T) = \{z: \text{Re } z = \pm 1\}.$$

Hence both $\pi(\text{bdy}(\sigma(S)))$ and $\pi(\sigma(T))$ are the set $\{-1, 1\}$. Lemma 2.3 and (iv) are thus false for this case.

3. Preparatory lemmas. We now begin working towards a proof of Theorem 1.4. Operators are no longer assumed to be bounded. As noted above a bounded subnormal operator has a minimal normal extension and the spectrum of this extension is contained in the spectrum of the subnormal operator.

Definition 3.1. Let S be a subnormal operator. We say that T is a *distinguished normal extension of* S if T is a normal extension of S and $\sigma(T)$ is contained in $\sigma(S)$.

LEMMA 3.2. *Every subnormal operator has a distinguished normal extension.*

Proof. Let S be a subnormal operator on H and let T be a normal extension of S to the space K. Let E be the spectral resolution of T, so that

$$T = \int \lambda dE(\lambda).$$

We will show, using an argument of Halmos [**1**, p. 131] that

$$H \subseteq E(\sigma(S))K.$$

Let α be in the complement of $\sigma(S)$, so that $(S - \alpha)^{-1}$ is a bounded operator. Choose $\epsilon(\alpha)$ such that

$$(3) \qquad 0 < \epsilon(\alpha) < \|(S - \alpha)^{-1}\|^{-1},$$

and let $M = E(B(\alpha; \epsilon(\alpha)))K$, where $B(z; r)$ denotes the open disk of radius r about z. If y is in M then clearly y is in $D(T)$ and

$$\|(T - \alpha)^k y\| \leqq \epsilon(\alpha)^k \|y\|, \quad k = 1, 2, \ldots .$$

For x in H we have

$$\begin{aligned} |\langle x \cdot y\rangle| &= |\langle y, (S - \alpha)^k (S - \alpha)^{-k} x\rangle| \\ &= |\langle y, (T - \alpha)^k (S - \alpha)^{-k} x\rangle| \\ &= |\langle (T^* - \bar{\alpha})^k y, (S - \alpha)^{-k} x\rangle| \\ &\leqq \|(T^* - \bar{\alpha})^k y\| \, \|(S - \alpha)^{-k}\| \, \|x\| \\ &= \|(T - \alpha)^k y\| \, \|(S - \alpha)^{-k}\| \, \|x\| \\ &\leqq \epsilon(\alpha)^k \|y\| \, \|(S - \alpha)^{-1}\|^k \, \|x\|. \end{aligned}$$

It now follows from (3) that this last quantity goes to 0 as k goes to infinity. Thus $\langle x, y\rangle = 0$ for x in H and y in M. Using the notation

$$E_{x,x}(\Omega) = \langle E(\Omega)x, x\rangle,$$

we have shown that

$$E_{x,x}(B(\alpha; \epsilon(\alpha))) = 0.$$

Since $\sigma(S)^c$ can be written as the union of a countable collection of sets of the form $B(\alpha; \epsilon(\alpha))$, α in $\sigma(S)^c$, we see that

$$E_{x,x}(\sigma(S)^c) = 0,$$

for x in H. We conclude that

$$H \subset [E(\sigma(S))^c K]^{\perp} = E(\sigma(S))K.$$

The lemma is proved, since the restriction of T to $D(T) \cap E(\sigma(S))K$ is obviously a normal operator on $E(\sigma(S))K$ whose spectrum is contained in $\sigma(S)$.

For the remainder of the section we will assume that S is a subnormal operator on H with bounded real part. Let T be a distinguished normal extension of S to a space K. Let $S = A + iB$ and $T = U + iV$ be the Cartesian decompositions of S and T, with A bounded.

LEMMA 3.3. $S^* = A - iB$.

Proof. It is obvious that $S^* \supset A - iB$, so let us assume that x is in $D(S^*)$ and that $S^*x = z$. For any y in $D(S) = D(B)$ we have

$$\langle x, (A + iB)y\rangle = \langle x, Sy\rangle = \langle z, y\rangle,$$

so

$$\begin{aligned}\langle x, By\rangle &= i\langle z, y\rangle - i\langle Ax, y\rangle\\ &= \langle iz - iAx, y\rangle.\end{aligned}$$

Since B is self-adjoint, it follows that x is in $D(B) = D(A - iB)$ and that $Bx = iz - iAx$. We thus have $z = S^*x = (A - iB)x$, so $S^* \subset A - iB$.

Since A is bounded it follows that the projection of the numerical ranges of S and S^* onto the real axis are bounded. Thus $\pi(\sigma(S))$ is bounded. It follows, since $\sigma(T) \subset \sigma(S)$, that U is a bounded operator.

LEMMA 3.4.

(i) $S^* = PT^*|D(T) \cap H$.

(ii) $A = PU|H$.

Proof. It is clear that

$$S^* \supset PT^*|D(T) \cap H.$$

Since $S^* = A - iB$, we have

$$D(S^*) = D(S) = D(T) \cap H,$$

and the first statement follows. Part (ii) follows easily from (i).

LEMMA 3.5. *If x is in $D(S)$ then* $||S^*x|| \leqq ||Sx||$.

Proof. The proof proceeds as in the bounded case [**3**, p. 160], now that we know that $D(S^*) = D(S)$ and that $S^* = PT^*$ on $D(S^*)$. Recall that $||T^*x|| = ||Tx||$ since T is normal. We have for x in $D(S^*)$

$$||S^*x|| = ||PT^*x|| \leqq ||T^*x|| = ||Tx|| = ||Sx||.$$

LEMMA 3.6. *If S is not bounded below then neither is A.*

Proof. Let $\{x_i\}$ be a sequence of unit vectors in $D(S)$ such that $\|Sx_i\| \to 0$. We have

$$\begin{aligned}\|Ax_i\| &= \tfrac{1}{2}\|(S + S^*)x_i\| \\ &\leqq \tfrac{1}{2}(\|Sx_i\| + \|S^*x_i\|),\end{aligned}$$

and so the result follows from the previous lemma.

LEMMA 3.7.

(i) $\sigma(A) \subset [\min \sigma(U), \max \sigma(U)]$;

(ii) $\pi(\sigma(S)) \subset [\min \sigma(U), \max \sigma(U)]$;

(iii) $\sigma(U) \subset [\min \operatorname{clos}(\pi(\sigma(S))), \max \operatorname{clos}(\pi(\sigma(S)))]$.

Proof. (i, ii) By symmetry and translation, it is sufficient to show that if $\min \sigma(U) > 0$, then A and S are invertible. So suppose $\min \sigma(U) > 0$. Then for x in H,

$$\langle Ax, x\rangle = \langle PUx, x\rangle = \langle Ux, x\rangle \geqq \min \sigma(U) \cdot \|x\|^2.$$

Since A is self-adjoint, it follows that it is invertible. Continuing the above reasoning, for x in $D(S) = D(S^*)$,

$$\begin{aligned}\operatorname{Re}\langle Sx, x\rangle &= \operatorname{Re}\langle S^*x, x\rangle \\ &= \langle Ax, x\rangle \\ &\geqq \min \sigma(U) \cdot \|x\|^2.\end{aligned}$$

Thus

$$\|Sx\| \geqq \min \sigma(U) \cdot \|x\| \quad \text{and} \quad \|S^*x\| > \min \sigma(U) \cdot \|x\|.$$

We conclude S is invertible.

To prove (iii) note that since $\sigma(T) \subset \sigma(S)$ and 1.4 holds for normal operators, we have

$$\begin{aligned}\sigma(U) &= \operatorname{clos}(\pi(\sigma(T))) \\ &\subset \operatorname{clos}(\pi(\sigma(S))).\end{aligned}$$

LEMMA 3.8. *Let $L \subset \mathbf{C}$ be closed and let $a < b$ be the real numbers not in the closure of $\pi(L)$. Let*

$$f_N^{a,b}(\lambda) = \frac{1}{2\pi i}\int_{\Gamma(a,b,N)} (\xi - \lambda)^{-1} d\xi, \quad N > 0,$$

where $\Gamma(a, b, N)$ consists of the two line segments from $b - iN$ to $b + iN$ and from $a + iN$ to $a - iN$. Then

(i) *there exists R such that $|f_N^{a,b}(\lambda)| \leqq R$, for all λ in L and all $N > 0$,*

(ii) *if* λ *is in* L *then as* $N \to \infty$:

$$f_N^{a,b}(\lambda) \to 1, \text{ if } a < \operatorname{Re} \lambda < b,$$

$$f_N^{a,b}(\lambda) \to 0, \text{ otherwise.}$$

Proof. Fix λ in L, assume $\operatorname{Im} \lambda > 0$ for simplicity, and let

$$I^*(\alpha, \beta) = (2\pi i)^{-1} \int |(\xi - \lambda)|^{-1} |d\xi|$$

and

$$I(\alpha, \beta) = (2\pi i)^{-1} \int (\xi - \lambda)^{-1} d\xi,$$

where the integrals are line integrals along the line from α to β parametrized in the usual way. Let $I(\alpha, \beta, \gamma, \delta)$ denote the latter integral taken around the vertices $\alpha, \beta, \gamma, \delta$, and back to α. Let d be the minimum of the distances from a and b to $\operatorname{clos}(\pi(L))$.

(i) For any real N let

$$\alpha = b + Ni, \beta = a + Ni, \gamma = a + (c + \tfrac{1}{2})i, \delta = b + (c + \tfrac{1}{2})i$$

where $c = \operatorname{Im} \lambda$. Suppose first that $N > c + \frac{1}{2}$ or $N < c - \frac{1}{2}$. In either case we have

$$|I(\alpha, \beta)| \leqq I^*(\alpha, \beta) \leqq 2(b - a),$$

since no point on the line from α to β is within distance $\frac{1}{2}$ of λ. Now suppose $c - \frac{1}{2} < N < c + \frac{1}{2}$, $N \neq c$. Assume $c < N < c + \frac{1}{2}$, the calculation for $c - \frac{1}{2} < N < c$ being similar. We have

$$\begin{aligned} I(\alpha, \beta) &= I(\alpha, \beta, \gamma, \delta) - I(\beta, \gamma) - I(\gamma, \delta) - I(\delta, \alpha) \\ &= -I(\beta, \gamma) - I(\gamma, \beta) - I(\delta, \alpha), \end{aligned}$$

by Cauchy's Theorem. Thus

$$\begin{aligned} |I(\alpha, \beta)| &\leqq |I(\beta, \gamma)| + |I(\gamma, \delta)| + |I(\delta, \alpha)| \\ &\leqq I^*(\beta, \gamma) + I^*(\gamma, \delta) + I^*(\delta, \alpha) \\ &\leqq \frac{1}{d} \cdot \frac{1}{2} + 2(b - a) + \frac{1}{d} \cdot \frac{1}{2} \\ &= \frac{1}{d} + 2(b - a), \end{aligned}$$

since no point on the line from β to γ or from δ to α can be within distance d of λ, and the length of both lines is less than $\frac{1}{2}$. Thus we have shown that the integral of $(2\pi i)^{-1}(\xi - \lambda)^{-1}$ is bounded on any vertical line segment joining the lines $\operatorname{Re} z = a$ and $\operatorname{Re} z = b$, not passing through λ.

Now suppose $N > 0$, $N \neq c$. Let

$$\alpha = b - Ni, \beta = b + Ni, \gamma = a + Ni, \quad \text{and} \quad \delta = a - Ni.$$

We have

$$\begin{aligned} |f_N^{a,b}(\lambda)| &= |I(\alpha, \beta) + I(\gamma, \delta)| \\ &= |I(\alpha, \beta, \delta, \gamma) - I(\beta, \gamma) - I(\delta, \alpha)| \\ &\leqq 1 + I^*(\beta, \gamma) + I^*(\delta, \alpha), \end{aligned}$$

again by Cauchy's Theorem. It follows from the above paragraph that

$$|f_N^{a,b}(\lambda)| \leqq 1 + \frac{2}{d} + 4(b - a)$$

for $N > 0$, $N \neq c$. Since $f_N^{a,b}$ is obviously continuous in N, the above inequality holds for $N = c$ as well. Part (i) is proved since the bound is independent of λ and N.

(ii) Fix λ in L. Let α, β, γ, and δ be as in the previous paragraph. It follows from Cauchy's Theorem that $I(\alpha, \beta, \gamma, \delta) \rightarrow 1$ if $a < \operatorname{Re} \lambda < b$, and goes to 0 otherwise, as $N \rightarrow \infty$. We also know that $(\xi - \lambda)^{-1}$ goes to 0 uniformly on the vertical line segments joining β to γ and δ to α. Thus

$$I(\beta, \gamma) \rightarrow 0 \quad \text{and} \quad I(\delta, \alpha) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Since

$$f_N^{a,b}(\lambda) = I(\alpha, \beta, \gamma, \delta) - I(\beta, \gamma) - I(\delta, \alpha),$$

part (ii) follows.

Remark. If S is any closed densely defined operator and λ is on $\operatorname{bdy}(\sigma(S))$ then $S - \lambda$ is not bounded below. The proof is essentially the same as for bounded operators. Thus if S is subnormal and T is a distinguished normal extension,

$$\operatorname{bdy}(\sigma(S)) \subset \sigma(T).$$

It follows, again as in the bounded case [**1**, p. 131], that components of the complement of $\sigma(T)$ are either wholly contained in, or are disjoint from, $\sigma(S)$. If S has bounded real part then by Lemma 3.6,

$$\pi(\operatorname{bdy}(\sigma(S))) \subset \sigma(\operatorname{Re} S).$$

However as we saw in the example at the end of Section 2, this fact need not be very useful in the unbounded case.

4. Proof of theorem 1.4. Let S be a subnormal operator on H with Cartesian decomposition $S = A + iB$, A bounded. Let T be a distinguished normal extension of S to K with decomposition $T = U + iV$. We divide the proof of 1.4 into two parts.

(i) Suppose c is not in $\text{clos}(\pi(\sigma(S)))$. We will show that c is not in $\sigma(A)$. Let

$$\alpha = \min \text{clos}(\pi(\sigma(S))) \quad \text{and} \quad \beta = \max \text{clos}(\pi(\sigma(S))).$$

If c is not in $[\alpha, \beta]$ then it follows immediately from Lemma 3.7 (i) and (iii) that c is not in $\sigma(A)$. We therefore suppose that $\alpha < c < \beta$. Since c is not in $\text{clos}(\pi(\sigma(S)))$, it follows that we can find intervals $I_1 = [c_1, c_2]$ and $I_2, = [c_3, c_4]$ such that

$$\pi(\sigma(S)) \subset I_1 \cup I_2,$$

where $c_1 < c_2 < c < c_3 < c_4$. We will now show that S can be written as a direct sum of subnormal operators, $S = S_1 \oplus S_2$, such that

$$\text{Re } S = \text{Re } S_1 \oplus \text{Re } S_2,$$

and such that c is neither in $\sigma(\text{Re } S_1)$ nor $\sigma(\text{Re } S_2)$.

Let E be the spectral measure of T. Consider the projections $E(\pi^{-1}(I_1))$ and $E(\pi^{-1}(I_2))$ on K. We first show that H is an invariant subspace of both operators. Choose a and b such that

$$c_2 < a < c_3 < c_4 < b$$

and define $f_N^{a,b}$ as in Lemma 3.8, with $L = \sigma(S)$. It follows from that lemma and the fact that $\sigma(T) \subset \sigma(S)$ that

$$\|x - f_N^{a,b}(T)x\|^2 = \int |1 - f_N^{a,b}(\lambda)|^2 dE_{x,x}(\lambda) \to 0, \quad \text{as } N \to \infty,$$

if x is in $E(\pi^{-1}(I_2))K$, and

$$\|f_N^{a,b}(T)x\|^2 = \int |f_N^{a,b}(\lambda)|^2 dE_{x,x}(\lambda) \to 0, \quad \text{as } N \to \infty,$$

if x is in $E(\pi^{-1}(I_1))K$. Hence

$$f_N^{a,b}(T)x \to E(\pi^{-1}(I_2))x, \quad N \to \infty,$$

for all x in K. Now for each ξ in $\sigma(S)^c$, H is an invariant subspace for $(\xi - T)^{-1}$. Hence H is an invariant subspace for the operators

$$f_N^{a,b}(T) = (2\pi i)^{-1} \int_{\Gamma(a,b,N)} (\xi - T)^{-1} d\xi,$$

and therefore for the operator $E(\pi^{-1}(I_2))$. That H is an invariant subspace for $E(\pi^{-1}(I_1))$ follows similarly.

Define

$$K_j = E(\pi^{-1}(I_j))K \quad \text{and} \quad H_j = E(\pi^{-1}(I_j))H \text{ for } j = 1, 2.$$

Clearly for each j

$$E(\pi^{-1}(I_j))D(T) \subset D(T),$$

so we have

$$D(T) = (D(T) \cap K_1) \oplus (D(T) \cap K_2),$$

$$D(S) = (D(S) \cap H_1) \oplus (D(S) \cap H_2).$$

We also have $T(D(T) \cap H_j) \subset H_j$. Define

$$T_j = T|D(T) \cap K_j \quad \text{and} \quad S_j = S|D(S) \cap H_j.$$

Then $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$. Clearly each S_j is subnormal on H_j with distinguished normal extension T_j on K_j. By Lemma 3.7 (ii),

$$\pi(\sigma(S_j)) \subset I_j.$$

It is easy to show from the above that $S_j = A_j + iB_j$, where

$$A_j = A|H_j \quad \text{and} \quad B_j = B|D(S) \cap H_j,$$

and that A_j and B_j are self-adjoint. The argument used in the first paragraph, applied to A_1 and A_2, shows that each operator $A_j - c$ is invertible. Since $A = A_1 \oplus A_2$, it follows that $A - c$ is invertible. We have shown that

$$\sigma(A) \subset \operatorname{clos}(\pi(\sigma(S))).$$

(ii) It now remains to show

$$\sigma(A) \supset \operatorname{clos}(\pi(\sigma(S))).$$

This is equivalent to showing that if S is non-invertible then so is A. If S is in addition not bounded below then neither is A by Lemma 3.6 (ii), and hence A is not invertible.

We henceforth assume that S is non-invertible but bounded below. Thus there exists $\delta > 0$ such that

$$\|Sx\| \geqq \delta\|x\|, \quad x \text{ in } D(S).$$

As in the case of bounded operators, this implies that the range of S is closed. Since S is not invertible there exists x_0 in H such that

$$\langle S_y, x_0 \rangle = 0, \quad \text{all } y \text{ in } D(S),\ x_0 \neq 0.$$

We will show that A is not invertible by showing that x_0 is not in its range. For suppose there is an element y_0 such that $Ay_0 = x_0$. Choose $c > \|U\|$ such that

$$\operatorname{clos}(\pi(\sigma(S))) \subset (-c, c),$$

and let f be a non-constant holomorphic function on $\pi^{-1}(-c - 1, c + 1)$ such that $zf(z)$ is bounded and $\operatorname{Re} zf(z)$ has the same sign as $\operatorname{Re} z$. (For example, let $zf(z)$ be a suitable conformal map of $\pi^{-1}(-c - 1, c + 1)$ onto the open unit disk.) With $\Gamma(-c, c, N)$ as in Lemma 3.8 define

$$f_N(\lambda) = (2\pi i)^{-1} \int_{\Gamma(-c,c,N)} f(\xi)(\xi - \lambda)^{-1} d\xi.$$

As in Lemma 3.8 we can show that there exists R such that $|f_N(\lambda)| \leqq R$ for all λ in $\sigma(T)$, $N > 0$, and that $f_N(\lambda) \to f(\lambda)$ as $N \to \infty$, for all λ in $\sigma(T)$. Thus

$$f_N(T)x \to f(T)x \quad \text{for all } x \text{ in } K.$$

Since

$$f_N(T) = (2\pi i)^{-1} \int_{\Gamma(-c,c,N)} f(\xi)(\xi - T)^{-1} d\xi$$

and since $\Gamma(-c, c, N) \subset \sigma(S)^c$, we see by an argument similar to one used in part (i) that $f_N(T)H \subset H$, and hence that $f(T)H \subset H$.

With x_0 and y_0 as above, we have

$$\langle Ty, Uy_0 \rangle = \langle Sy, Ay_0 \rangle = \langle Sy, x_0 \rangle = 0$$

for all y in $D(T)H$. In particular, if $y = f(T)y_0$, then

$$0 = \langle Tf(T)y_0, Uy_0 \rangle = \langle UTf(T)y_0, y_0 \rangle$$
$$= \int \mu\lambda f(\lambda) dE_{y_0,y_0}(\lambda), \ \lambda = \mu + i\nu.$$

However f was chosen so that $\mathrm{Re}(\lambda)\mathrm{Re}(\lambda f(\lambda)) > 0$ unless $\mu = 0$. So the above equalities would imply that the measure E_{y_0,y_0} is supported in the imaginary axis. In that case

$$\|Uy_0\|^2 = \int \mu^2 dE_{y_0,y_0}(\lambda) = 0,$$

so $x_0 = PUy_0 = 0$, which is a contradiction. We conclude that A is not onto.

Remark. The restriction in Theorem 1.4 that Re S be bounded is a bit artificial and can be relaxed somewhat. However, our proof will not work for general subnormal operators possessing Cartesian decompositions. For general subnormal operators one cannot even state Theorem 1.4, as the following example shows. We again work in L^2 and H^2 of the unit circle and define

$$\phi(z) = i\frac{1 + z}{1 - z}.$$

Since the values taken by ϕ on the unit circle are real, M_ϕ is self-adjoint. Clearly H^2 is invariant for M_ϕ, so $S = M_\phi|H^2$ is subnormal by our definition. Since M_ϕ is self-adjoint S is symmetric, so the only way S could have a Cartesian decomposition would be for it to have a self-adjoint extension. However S is the Cayley transform of the unilateral shift T_z (see [**6**], Chapter 13); since T_z obviously has no unitary extension, S has no self-adjoint extension.

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